Analysis of model equations for stress-enhanced diffusion in coal layers. Part I: Existence of a weak solution.

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Abstract

This paper is motivated by the study of the sorption processes in the coal. They are modelled by a nonlinear degenerate pseudo-parabolic equation for stress enhanced diffusion of carbon dioxide in coal

$$\partial_t \phi = \partial_x \left\{ D(\phi) \partial_x \phi + \frac{D(\phi)\phi}{B} \partial_x \left(e^{-m\phi} \partial_t \phi \right) \right\},$$

where B, m are positive constants and the diffusion coefficient $D(\phi)$ has a small value when the CO_2 volume fraction ϕ is $0 \leq \phi < \phi_c$, representative of coal in the glass state and orders of magnitude higher value for $\phi > \phi_c$, when coal is in the rubber like state. These type of equations arise in a number of cases when non-equilibrium thermodynamics or extended non-equilibrium thermodynamics is used to compute the flux.

For this equation existence of the travelling wave type solutions was extensively studied. Nevertheless, the existence seems to be known only for sufficiently short time. We use the corresponding entropy functional in order to get existence, for any time interval, of an appropriate

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†The research of H.B. was supported by the Shell International Exploration and Production B.V., NWO-Novem and the Greenhouse-gas Removal Apprenticeship and Student Program (GRASP) funded by the EC, The idea of stress enhanced diffusion [30] [31] for coal, worked out in the doctoral thesis of Saikat Mazumder [24], was the inspiration to this work.

weak solution with square integrable first derivatives and satisfying uniform L^∞-bounds. Due to the degeneracy, we obtain square integrability of the mixed second order derivative only in the region where the concentration ϕ is strictly positive. In obtaining the existence result it was crucial to have the regularized entropy as unknown for the approximate problem and not the original unknown (the concentration).

Key words: degenerate pseudoparabolic equation, entropy methods, stress enhanced diffusion

AMS subject classification: 35K70, 35K65, 76R50, 80A17

1 Introduction

One of the promising methods to reduce the discharge of the "greenhouse gas" carbon dioxide (CO_2) into the atmosphere is its sequestration in unminable coal seams. A typical procedure is the injection of carbon dioxide via deviated wells drilled inside the coal seams. Carbon dioxide displaces the methane adsorbed on the internal surface of the coal. A production well gathers the methane as free gas. This process, known as carbon dioxide-enhanced coal bed methane production (CO_2-ECBM), is a producer of energy and at the same time reduces greenhouse concentrations as about two carbon dioxide molecules displace one molecule of methane. World-wide application of ECBM can reduce greenhouse gas emissions by a few percent. Coal has an extensive fracturing system called the cleat system. In actual fact it is possible to discern a number of cleat systems at different scales. In the end the matrix blocks between the smallest cleat system have diameters typically of a few tens of microns [13].

Matrix blocks consist of polymeric structure (dehydrated cellulose [21]), which provides the adsorption sites for the gases. At low temperatures or low sorption concentration the coal structure behaves like a rigid glassy polymer, in which movement is difficult. At high temperatures or high sorption concentrations the glassy structure is converted to the less rigid and open rubber like (swollen) structure [30], [31]. As coal is less dense in the rubber like state a conversion from the glassy state to the rubber like state exhibits swelling. Therefore modelling of diffusion is not only relevant for modelling transport into the matrix blocks, but also for the modelling of swelling, which affects the permeability of the coal seam.

Ritger and Peppas [30], [31] distinguish between transport by Fickean diffusion and a process that occurs on the interface between the glass state and the rubber like state. Ritger and Peppas state that the conversion process from the glass to the rubber state is controlled by a rate limiting relaxation phenomenon (see also [2]). Thomas and Windle [32] (see also [16], [17], [19]), however, suggested, in their classical paper, that the diffusion transport was enhanced by stress gradients that resulted from the accommodation of large molecules in the small cavities providing the adsorption sites. For this Alfrey et al. [1] coined the term superdiffusion or case II diffusion. At a critical concentration of the penetrants the glassy polymer transformed to a rubber state, where the diffusion coefficient is of the order of a factor 1000 larger than in the glassy state.

This paper is the first of a series, where the model equations for case II diffusion [32], [16], [17], [19] will be analyzed. Our long time interest is to investigate the 1-D sorption rate behavior, i.e., whether the equations indeed lead to a rate faster than the square root of time. In this paper we establish existence of a weak solution for all times.

Nonlinear diffusion equations with a pseudoparabolic regularizing term being the Laplacean of the time derivative are considered in [26] and in [27] . Global existence of a strong solution is proved by writing the problem as a linear elliptic operator, acting on the time derivative, equal to the nonlinear diffusion term. Then the linear elliptic operator, acting on the time derivative, was inverted and the standard geometric theory of nonlinear parabolic equations (see e.g. [15]) is applicable.

In our situation the physical model impose us a degenerate non-linear second order elliptic operator, acting on the time derivative, at the place of the Laplacean. The invertibility of this nonlinear elliptic operator is not clear anymore and it depends on the solution itself. The same type of equations can occur in models that use Classical Irreversible Thermodynamics or Extended Irreversible Thermodynamics. An important example is the model of the two-phase flow through porous media introduced in [14], where the capillary pressure relation is extended with a dynamic term, which contains the time derivative of the saturation. We refer also to [5] for the modelling. This application to multiphase and unsaturated flows through porous media motivated a number of recent papers. In paper [18] one finds a detailed study of possible travelling wave solutions and in particular of the behavior of such travelling waves near fronts where the concentration is zero. Further studies of the travelling waves are in the papers [8] and [7]. The small- and waiting time behavior of the equations was studied in [20]. Study of the viscosity limit for the linear relaxation model of the dynamic term is in [10]. Nevertheless, the study of existence of a solution to the nonlinear model from [14] was undertaken only in paper [4], where the non-degeneracy was supposed and existence is local in time. Another existence result, also local in time, is in the paper [9] , where a related pseudoparabolic equation modelling solvent uptake in polymeric solids was studied.

Our goal is to obtain a global existence of a weak solution, for any time interval, as it was obtained in [3] for a degenerate pseudoparabolic regularization of a nonlinear forward-backward heat equation. Our PDE allows a natural generalization of the classic Kullback entropy and its integrand is given by

$$\mathcal{E}(\varphi) = \int_0^{\varphi} \frac{\varphi - \xi}{\xi} \big(e^{-m\xi} \frac{1}{D(\xi)} - \frac{1}{D(0)} \big) \, d\xi + \frac{1}{D(0)} \varphi \log \varphi. \tag{1}$$

As in [25], we will use $\mathcal{E}'(\varphi)$ as a test function, with the hope to obtain a convenient a priori estimate. Presence of the initial and the boundary conditions lead to unbounded non-integrable $\mathcal{E}'$ and we do not get the entropy estimates as in [12]. We had to obtain an additional estimate for the time derivative and our calculations are more complicated than in the literature.

Our paper is organized as follows: Section 2 describes the physical model, first proposed in [32]. We repeat the derivations from [16], [17], [19] for reasons of easy reference and unified notations.

In section 3 we introduce the regularized problem and discretize it using the regularized $\mathcal{E}'(\varphi)$ as the unknown. Next the solvability of the discretized problem is proved and the uniform L^2 - a priori estimates for the first derivatives and the mixed second derivative are obtained for a small time interval $(0, T_0)$. They imply the short time existence for the regularized problem.

We continue with section 4 where we use the entropy to establish that $T = T_0$, i.e. existence of a solution for the regularized problem for all times. Next we establish L^∞ bounds independent on the regularization parameter.

The last section 5 concerns the existence for the original problem. Using again the entropy, the estimates for the time derivative and the L^∞ bounds, we are able to pass to the limit when the regularization parameter tends to zero and prove the existence of at least one solution for the original problem.

2 Model equation for stress induced diffusion

2.1 Physical model

Consider a coal particle between the fractured cleat system in coal. The matrix block can be considered as a small (30 μm diameter) cubical particle consisting of glassy coal. The coal face is exposed to the penetrant, in our case carbon dioxide. The coal face of the particle and the mechanism of the sorption process is shown schematically in Fig. 1. The coal originates from a cellulose like polymeric structure [21], with the chemical formula $C_n\left(H_2O\right)_m$, from which part of the hydrogen and oxygen have disappeared during the coalification process, which took millions of years. The remaining structure behaves like a glassy polymer, which contains holes (sites) that can accommodate CO_2, CH_4 etc. In other words, sorption of gases by coal is more a dissolution process than adsorption of gases at a coal surface. The holes receiving the CO_2 are originally too small to accommodate the molecule and need to expand. Consequently the expanded hole exerts a stress on the neighboring molecules constituting the polymeric coal. Therefore the penetration of CO_2 will both lead to a stress gradient and a concentration gradient. The concentration will be expressed as a volume fraction ϕ, i.e., $\phi = c/\Omega$, where c is the molecular concentration and Ω is the molecular volume. As the CO_2 likes to move towards a region of smaller stress, the transport of the molecule will be both caused by a concentration gradient and a stress gradient. When the stresses become too high, a deformation occurs in which the glassy polymeric structure is converted to a rubber like (swollen) structure, which is much more open. Consequently the diffusion coefficient in the rubber like structure is much higher (more than thousand times) than the diffusion coefficient in the glassy structure. The stresses are considered to depend on the CO_2 concentration in the coal and conversion to the rubber like structure occurs instantaneously when a certain critical concentration is exceeded.

These ideas were formulated for the first time by Thomas and Windle and the derivation of the model equations will be explained below.

2.2 Derivation of model equations

The salient features of the Thomas and Windle (TW) model [32]are well summarized by Hui et al. (1987) [16], [17]. We summarize the derivation here with the help of the article by Hui et al. (1987), the book of Landau and Lifshitz, 1975, [22], i.e., the molar (diffusive) flux J is not only driven by the volume fraction (ϕ) (concentration) gradient , but also by the stress (P_{xx}) gradient, i.e.

$$J = -D\left(\frac{\partial \phi}{\partial x} + \frac{\Omega\phi}{kT}\frac{\partial P_{xx}}{\partial x}\right) \quad , \tag{2}$$

where k is the Boltzmann constant. As opposed to the equation in [22], which contains a scalar pressure gradient, the idea here is extended in [19] with the use of the stress gradient $\partial_x P_{xx}$. Note that J is the flux of a volume fraction and behaves as a velocity. The diffusion coefficient depends on the concentration. Below a critical volume fraction ϕ_c, a diffusion coefficient $D_g > 0$ characteristic of a glassy state is used, and above ϕ_c the diffusion coefficient $D_r > 0$ characteristic of the rubber (swollen) state is used. It can be expected that $D_r/D_g >> 1$. In the model an abrupt change of the diffusion coefficients at ϕ_c is used, but D_r and D_g are considered constant for $\phi > \phi_c$ and $\phi < \phi_c$ respectively:

$$D\left(\xi\right) = \begin{cases} D_g, & 0 \le \xi < \phi_c - \kappa \\ D_g + (D_r - D_g)(\xi - \phi_c + \kappa)/(2\kappa), & \phi_c - \kappa \le \xi \le \phi_c + \kappa \\ D_r & \phi_c + \kappa < \xi < +\infty, \end{cases} \tag{3}$$

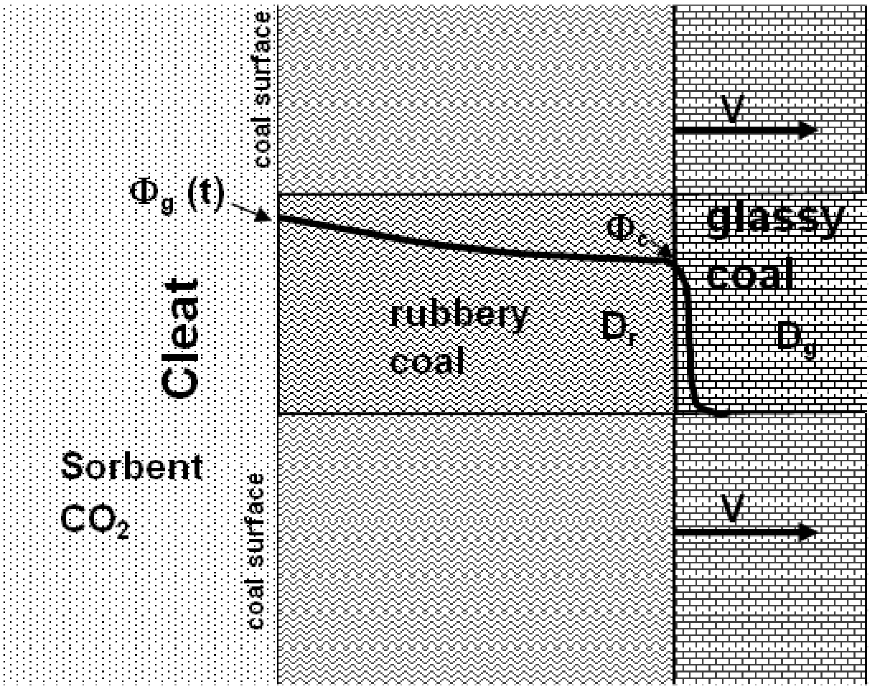

Figure 1: A coal face exposed to a sorbent (CO_2) . Far to the right the virgin coal, which behaves as a glassy polymer. As the sorbent penetrates in the coal a reorientation of the polymeric coal structure occurs and the coal becomes rubber like. The diffusion coefficient in the rubber like structure is much higher ($> 1000\ \times$) than in the glassy structure. The rubber like structure has also a lower density leading to swelling.

where $\kappa > 0$ is a small parameter. Extended non-equilibrium thermodynamics [19] suggests that vice-versa also the stress (P_{xx}) is related to the volumetric flux gradient as

$$P_{xx} = -\eta_l \frac{\partial J}{\partial x} = \eta_l \frac{\partial \phi}{\partial t} \quad , \tag{4}$$

where the second equality follows from a mass conservation law that assumes incompressible flow,

$$\frac{\partial \phi}{\partial t} + \frac{\partial J}{\partial x} = 0 \quad . \tag{5}$$

With η_l we denote the elongational velocity [6], i.e. the resistance of movement due to a velocity gradient $\frac{\partial J}{\partial x}$ in the direction of flow. The elongational viscosity η_l is supposed to depend on the volume fraction of the penetrant as

$$\eta_l = \eta_o \exp\left(-m\phi\right), \tag{6}$$

where m is a material constant and η_0 is the volumetric viscosity of the unswollen coal sample.

Substitution of expression (2) for the flux into the mass balance equation (5), where we also use Eq. (4) we obtain

$$\partial_t \phi = \partial_x \left\{ D\left(\phi\right) \partial_x \phi + \frac{D\left(\phi\right)\phi}{B} \partial_x \left(e^{-m\phi} \partial_t \phi\right) \right\}, \tag{7}$$

where the constant $B = k_B T / \left(\eta_o \Omega\right)$. This equation is defined in $Q_T = (0, L) \times (0, T)$

As initial condition we have that the concentration

$$\phi\left(x, t = 0\right) = 0 \quad \text{ on } \left(0, T\right). \tag{8}$$

The boundary condition at $x = 0$ must be derived from thermodynamic arguments. The final equilibrium concentration is reached when the coal has swollen to make the stress $P_{xx} = P^0_{xx}$ equal to zero. In this case the volume fraction of CO_2 in the coal is in equilibrium with the CO_2 in the fluid phase outside the coal. Also the CO_2 in the stressed coal is in equilibrium with the CO_2 in the fluid phase. The change in chemical potential $d\mu = \Omega dP_{xx} + k_B T d \ln \phi$. Equating the chemical potential in the unstressed and stressed state leads to:

$$\Omega P_{xx} + k_B T \ln \phi = \Omega P^0_{xx} + k_B T \ln \phi_o \quad , \tag{9}$$

where ϕ_o is the volume fraction at the coal boundary that would be in equilibrium with the carbon dioxide in the gas phase if the coal has relaxed to the rubber state with $P^0_{xx} = 0$.

Substitution of Eq. (4) and Eq.(6) into Eq. (9) leads to

$$t = -\phi_o \frac{\eta_0 \Omega}{k_B T} \int_0^{\phi/\phi_o} \frac{\exp(-m\phi_o y)}{\ln y} dy \quad , \tag{10}$$

where we use the initial condition that $\phi = 0$ at $t = 0$.

At $x = L$ we have the boundary condition on $(0, T)$

$$D\left(\phi\right) \left(\partial_x \phi + \frac{1}{B} \phi \partial_x \left(\exp\left(-m\phi\right) \partial_t \phi\right) \right)_{x=L} = 0. \tag{11}$$

In summary we have one initial condition Eq. (8), one boundary condition at $x = L$, viz. Eq. (11) and the implicit boundary condition Eq. (10), which both specifies $\phi\left(x = 0, t\right)$ and $\partial\phi/\partial t\left(x = 0, t\right)$ at $x = L$ as

$$\phi\left(0, t\right) = \phi_g\left(t\right), \tag{12}$$

which satisfies the conditions

$$0 \leq \phi_g \leq \phi_o, \tag{13}$$

$$\phi_g(0) = 0. \tag{14}$$

Remark 1 *Equations like Eq. (7) can occur in many transport problems in which the flux is calculated using classical irreversible thermodynamics (CIT) or extended irreversible thermodynamics (EIT). A well known example for CIT in porous media flow is that the deviation of the capillary pressure P_c from its equilibrium value at a given oil saturation S_o, i.e., $P_c^o = P_c^o(S_o)$ is a driving force leading to a rate of change of the saturation (scalar flux). This leads [14], [28], [29], [23] to $\partial_t S_o = L(P_c - P_c^o)$, and to the transport equation for counter current imbibition*

$$\begin{aligned} \varphi \partial_t S_o &= \partial_x (\Lambda(S_o) \partial_x P_c) = \\ &= \partial_x (\Lambda(S_o) \partial_x P_c^o(S_o)) + \partial_x \left(\Lambda(S_o) \partial_x \frac{1}{L(S_o)} \partial_t S_o \right). \end{aligned}$$

EIT [19] differs from CIT as it not only characterizes a system by its local thermodynamic variables (pressure, temperature, concentration) but also by its gradients. The explanation in reference [19] is difficult to follow by non-specialists as many thermodynamic relations are considered to be known by the reader. In isothermal systems and in the absence of other applied fields, e.g. electric fields, the volumetric flux J is, according to EIT, given by the following system of equations

$$\tau_1 \partial_t J + J = -D \left(\frac{\partial \phi}{\partial x} + \frac{\Omega \phi}{kT} \frac{\partial P_{xx}}{\partial x} \right), \tag{15}$$

$$\tau_2 \partial_t P_{xx} + P_{xx} = -\eta_l \frac{\partial J}{\partial x}. \tag{16}$$

Reference [19] uses a mass flux instead of a volumetric flux and therefore uses a factor υ_1, being the partial volume per unit mass. Here τ_1, τ_2 are time constants, which are small with respect to L^2/D. The first term on the left of Eq. (15) and Eq. (16) only appear in EIT and not in CIT. The first terms on the left are of interest for short time behavior and are omitted from the model discussed here. Another example from EIT is the Taylor dispersion problem (see Eq. 10.34 in [19]) where there is an 'xxt' derivative in the concentration, apart from many other terms. Hence EIT or CIT can lead to transport equations of the form of Eq. (7).

3 Short time existence for the regularized problem

Existence is proven by showing that the 'energy' of the system remains bounded during the time evolution of the system. The 'energy' equation is derived from the differential equation by multiplying with an appropriate test function and integrating over the domain. The choice of the test function depends strongly on the choice of the coefficients. With an appropriate approximation this can also be the basis of a numerical scheme which leads to a non-linear system of ODE's, which is implicit in derivatives. The fact that the 'energy' is bounded means that the numerical scheme is stable. If convergence can be proven it shows that at least one solution exists. Solvability of the system of ODE's depends strongly on the initial conditions. One of the main problems with existence proofs is finding an appropriate test function.

In this case an appropriate test function is $\Phi'(\phi)\,\partial_t\phi$, where

$$\Phi'(\xi) = \frac{e^{-m\xi}}{\xi D(\xi)},$$

which is, however singular for $\xi = 0$. We start by extending the diffusion coefficient $D(\xi) = D(-\xi)$ for $\xi < 0$. Another problem of the test function is that for large values of ξ, $\Phi^{'}$ is exponentially small. In order to prove existence we need Φ that is bijective from $\mathbb{R}$ to $\mathbb{R}$.

We introduce Φ_δ by

$$\Phi'_\delta = \frac{e^{-m\ \min\{|\xi|,1/\delta\}}}{(|\xi|+\delta)\,D(\xi)}, \quad \delta > 0,\ \xi \in \mathbb{R}, \tag{17}$$

and

$$\Phi_\delta(\phi) = \begin{cases} \displaystyle\int_0^\phi \frac{e^{-m\ \min\{\xi,1/\delta\}}}{(\xi+\delta)D(\xi)}d\xi, & \phi > 0 \\ \displaystyle -\int_\phi^0 \frac{e^{-m\ \min\{-\xi,1/\delta\}}}{(-\xi+\delta)D(-\xi)}d\xi, & \phi < 0. \end{cases} \tag{18}$$

In this section we study the following regularized problem in Q_T

$$\partial_t\phi = \partial_x\left\{D(\phi)\,\partial_x\phi + \frac{D(\phi)\,(|\phi|+\delta)}{B}\partial_x\left(e^{-m\min\{|\phi|,1/\delta\}}\partial_t\phi\right)\right\} \tag{19}$$

with boundary condition at $x = L$

$$D(\phi)\,\partial_x\phi + \frac{D(\phi)\,(|\phi|+\delta)}{B}\partial_x\left(e^{-m\min\{|\phi|,1/\delta\}}\partial_t\phi\right)\Big|_{x=L} = 0 \tag{20}$$

and boundary condition (12) at $x = 0$ and initial condition (8) at $t = 0$.

We start by introducing a variational solution for the problem (19), (20), (12) and (8).

Definition 2 *Let*

$$\mathcal{V} = \{z \in C^\infty[0,L],\ z|_{x=0} = 0\} \quad \text{and} \quad \mathcal{H} = \{C^\infty[0,T],\ h(T) = 0\} \tag{21}$$

Then the variational formulation corresponding to the problem (12), (8), (19) and (20) is

$$-\int_0^T\int_0^L \phi(x,t)\,g(x)\,\partial_t h(t)\,dxdt + \int_0^T\int_0^L D(\phi)\,\partial_x\phi(x,t)\ \partial_x g(x)h(t)\ dxdt +$$

$$\int_0^T\int_0^L \frac{D(\phi)\,(|\phi|+\delta)}{B}\partial_x g(x)\ h(t)\ \partial_x\left(e^{-m\min\{|\phi|,1/\delta\}}\partial_t\phi\right)dxdt = 0, \quad \forall g \in \mathcal{V} \text{ and } \forall h \in \mathcal{H} \tag{22}$$

and at the boundary $x = 0$, we have

$$\phi - \phi_g = 0 \quad \text{on } \{x = 0\} \tag{23}$$

Our goal is to prove existence for (22)-(23).

In order to have the entropy estimate, we should formulate the approximate problem in terms of it. Otherwise it would not be possible to use it as a test function for the approximate problem. Getting a priori estimates without this approach is not clear.

Let $z = \Phi_\delta(\phi)$, $\phi = \Phi_\delta^{-1}(z)$, $z|_{x=0} = \Phi_\delta(\phi_g(t))$. We reformulate the problem (12), (8), (19) and (20) in terms of z:

$$\frac{1}{\Phi_\delta'\left(\Phi_\delta^{-1}(z)\right)}\partial_t z = \partial_x\left\{\frac{D\left(\Phi_\delta^{-1}(z)\right)}{\Phi_\delta'\left(\Phi_\delta^{-1}(z)\right)}\partial_x z\right.$$
$$\left.+\frac{D\left(\Phi_\delta^{-1}(z)\right)\left(\left|\Phi_\delta^{-1}(z)\right|+\delta\right)}{B}\partial_x(D\left(\Phi_\delta^{-1}(z)\right)\left(\left|\Phi_\delta^{-1}(z)\right|+\delta\right)\partial_t z)\right\} \quad \text{in } Q_T \tag{24}$$

Moreover we can express the boundary and initial conditions in z as

$$z(0,t) = \Phi_\delta(\phi_g(t)) \quad \text{on} \quad (0,T); \quad z(x,t=0) = \Phi_\delta(0) = 0 \quad \text{on} \quad (0,L), \tag{25}$$

$$\frac{1}{\Phi_\delta'\left(\Phi_\delta^{-1}(z)\right)}\partial_x z + \frac{\left(\left|\Phi_\delta^{-1}(z)\right|+\delta\right)}{B}\partial_x\left(D\left(\Phi_\delta^{-1}(z)\right)\left(\left|\Phi_\delta^{-1}(z)\right|+\delta\right)\partial_t z\right) = 0 \quad \text{on } x = L. \tag{26}$$

Let $V = \left\{g \in H^1(0,L)\,|g(0) = 0\right\}$ be the closure of $\mathcal{V}$ in $H^1(0,L)$ and let $\{\alpha_j\}_{j\in\mathcal{N}}$ be a $C^\infty-$basis for V. We set $V_N =$span$\{\alpha_1,...\alpha_N\}$ and introduce the following coefficients

$$d_1(z) = \frac{1}{\left(\Phi_\delta'\left(\Phi_\delta^{-1}(z)\right)\right)}, d_2(z) = \frac{D\left(\Phi_\delta^{-1}(z)\right)}{\left(\Phi_\delta'\left(\Phi_\delta^{-1}(z)\right)\right)} \text{ and } d(z) = D\left(\Phi_\delta^{-1}(z)\right)\left(\left|\Phi_\delta^{-1}(z)\right|+\delta\right). \tag{27}$$

We start study of the initial-boundary problem (24), (25), (26) by constructing an approximate solution for every N. It is defined as follows

Appoximate problem 3 *Find* $z_N = \sum_{j=1}^{N} c_j(t)\alpha_j(x) + \Phi_\delta(\phi_g(t)) \in W^{1,q}([0,T];V_N)$, $q \in (2,+\infty)$ *such that*

$$\int_0^L \partial_t z_N d_1(z_N)\alpha_k\, dx + \int_0^L d_2(z_N)\partial_x z_N \partial_x \alpha_k\, dx + \int_0^L \frac{1}{B} d(z_N)\partial_x(d(z_N)\partial_t z_N)\partial_x\alpha_k\, dx = 0, \tag{28}$$

$$\textit{for } k = 1,...,N \quad \textit{and} \quad z_N|_{t=0} = P_N\left(z|_{t=0} - \Phi_\delta(\phi_g(0))\right) = 0, \tag{29}$$

where $P : V \to V_N$ *is the projector* $P_N(f)(x) = \sum_{j=1}^{N}\alpha_j(x)(f,\alpha_j)_V$.

Let the vector valued function $\mathbf{F}$ be given by $F_\kappa(t,\mathbf{c},\partial_t\mathbf{c}) =$ left part of Eq. (28) and $\mathbf{c}$ is the column vector consisting of elements $(c_1(t)...c_N(t))$, then Eqs. (28), (29) are equivalent to the following Cauchy Problem in $\mathbb{R}^N$:

$$\left\{\begin{array}{c}\mathbf{F}(t,\mathbf{c},\partial_t\mathbf{c}) = 0\\ \mathbf{c}|_{t=0} = 0\end{array}\right. \tag{30}$$

The Cauchy problem (30) is difficult to solve, since the dependence of $\mathbf{F}$ on $\partial_t\mathbf{c}$ is implicit. It is crucial to reduce it to an ordinary Cauchy problem of the form $\partial_t\mathbf{c} = \varrho(t,\mathbf{c})$.

We note that

$$F_k = \sum_{j=1}^{N} \Big\{ \int_0^L d_1(z_N)\,\alpha_k\alpha_j\ dx + \int_0^L \frac{1}{B} d(z_N)\,\partial_x\left(d(z_N)\,\alpha_j\right)\partial_x\alpha_k\ dx \Big\} \frac{dc_j}{dt} +$$

$$\sum_{j=1}^{N} \Big\{ \int_0^L d_2(z_N)\,\partial_x\alpha_j\partial_x\alpha_k\ dx \Big\} c_j + \int_0^L d_1(z_N)\,\alpha_k\partial_t\Phi_\delta\left(\phi_g(t)\right)\ dx. \tag{31}$$

Then, after introducing the matrices $\mathcal{A}(\mathbf{c})$ and $\mathcal{B}(\mathbf{c})$ and the vector $\mathbf{f}(\mathbf{c})$ by

$$\mathcal{A}_{kj}(\mathbf{c}) = \int_0^L d_1(z_N)\,\alpha_k\alpha_j\ dx + \int_0^L \frac{1}{B} d(z_N)\,\partial_x\left(d(z_N)\,\alpha_j\right)\partial_x\alpha_k\ dx, \tag{32}$$

$$\mathcal{B}_{kj}(\mathbf{c}) = \int_0^L d_2(z_N)\,\partial_x\alpha_j\partial_x\alpha_k\ dx \ \text{ and } \ f_k(\mathbf{c}) = \int_0^L d_1(z_N)\,\alpha_k\partial_t\Phi_\delta\left(\phi_g(t)\right)\ dx, \tag{33}$$

$1 \le k, j \le N$, we see that the problem (28)–(29) is equivalent to the Cauchy problem

$$\text{Find } \ \mathbf{c} \in W^{1,q}(0,T)^N \ \text{ such that}$$

$$\mathcal{A}(\mathbf{c})\frac{d\mathbf{c}}{dt} = -\mathcal{B}(\mathbf{c})\mathbf{c} - \mathbf{f}(\mathbf{c}) \ \text{ a.e. in } \ (0,T); \quad \mathbf{c}|_{t=0} = 0. \tag{34}$$

Proposition 4 *There is a $T_N > 0$ such that the problem (28)–(29) has a unique solution $z_N \in W^{1,q}(0,T_N;V_N)$, for all $q < +\infty$.*

Proof. It is enough to prove that the Cauchy problem (34) has a solution.

Obviously, $\mathcal{A}$, $\mathcal{B}$ and $\mathbf{f}$ are smooth functions of $\mathbf{c}$. Because of the singularity of $\partial_t\varphi_g$ at $t = 0$, $\mathbf{f}(\mathbf{c}) \in L^q(0,T)$, $\forall q < +\infty$, but it is not bounded. Hence, the only property to check is the invertibility of the matrix $\mathcal{A}$. Let $\mathbf{b}$ be an arbitrary vector from $\mathbb{R}^N$ and let $b_\alpha(x) = \mathbf{b}\cdot\alpha(x) = \sum_{j=1}^N b_j\alpha_j(x)$. Then we have

$$(\mathcal{A}\mathbf{b})\cdot\mathbf{b} = \sum_{k,j=1}^{N} \mathcal{A}_{k,j}b_kb_j = \int_0^L d_1(z_N)(b_\alpha)^2\ dx + \frac{1}{B}\int_0^L d(z_N)\partial_x b_\alpha\partial_x(d(z_N)b_\alpha)\ dx =$$

$$= \int_0^L d_1(z_N)(b_\alpha)^2\ dx + \frac{1}{B}\int_0^L (d(z_N)\partial_x b_\alpha)^2\ dx + \frac{1}{B}\int_0^L d(z_N)\partial_x b_\alpha b_\alpha d'(z_N)\partial_x z_N\ dx \ge$$

$$\int_0^L \big\{d_1(z_N) - \frac{1}{4B}(d'(z_N))^2(\partial_x z_N)^2\big\}(b_\alpha)^2\ dx. \tag{35}$$

Since $\partial_x z_N(x,0) = 0$ and functions $\{\alpha_j\}_{j\in\mathbb{N}}$ are linearly independent, the matrix $\mathcal{A}$ is by (35) invertible in a neighborhood of $t = 0$. Then by the classical theory, the problem (34) has a unique solution on some interval $(0,T_N)$. ∎

Next we want to prove that the existence interval does not depend on N.

Proposition 5 *We have*

$$\|\partial_x z_N\|_{L^\infty(0,T;L^2(0,L))} \le C. \tag{36}$$

Consequently, the vector valued function $\mathbf{c}$ remains bounded at $t = T_N$.

Proof. In Eq. (28) we can replace α_k by $z_N - \Phi_\delta(\phi_g)$. Then after using that $\partial_x(d(z_N)\partial_t z_N) = \partial_t(d(z_N)\partial_x z_N)$, we get

$$\int_0^L d_1(z_N) z_N \partial_t z_N dx + \int_0^L d_2(z_N)(\partial_x z_N)^2 dx + \int_0^L \frac{1}{B}\partial_t(d(z_N)\partial_x z_N) d(z_N)\partial_x z_N dx =$$

$$\int_0^L d_1(z_N)\Phi_\delta(\phi_g)\partial_t z_N dx = \partial_t \int_0^L \Phi_\delta(\phi_g)(t)\int_0^{z_N} d_1(\xi) d\xi dx - \partial_t \Phi_\delta(\phi_g)(t)\int_0^L\int_0^{z_N} d_1(\xi)\ d\xi dx. \quad (37)$$

Integrating over t leads to

$$\int_0^L (\int_0^{z_N(x,t)} d_1(\xi)\xi\ d\xi)\ dx + \int_0^t\int_0^L d_2(z_N)(\partial_x z_N)^2 dx d\tau + \frac{1}{2B}\int_0^L d(z_N)^2(\partial_x z_N)^2 dx =$$

$$\int_0^L (\int_0^{z_N(x,t)} d_1(\xi)\ d\xi)\ dx\ \Phi_\delta(\phi_g)(t) - \int_0^t \partial_\tau \Phi_\delta(\phi_g)(\tau)(\int_0^L\int_0^{z_N} d_1(\xi)\ d\xi dx)\ d\tau. \quad (38)$$

We easily find out that

$$\int_0^z d_1(\xi)\xi\ d\xi = \int_0^{\Phi_\delta^{-1}(z)} \Phi_\delta(\eta)\ d\eta \ \text{ and } \ \int_0^z d_1(\xi)\ d\xi = \Phi_\delta^{-1}(z). \quad (39)$$

The growth of the terms in (39) indicates that it will be possible to control the two terms at the right hand side of (38) by the first term at the left hand side of (38).

Let $M_\phi = \max_{0\le t\le T}|\Phi_\delta(\phi_g(t))|$. By the definition of $\Phi_\delta(\varphi)$, we have $C_0(\delta)\log\big(1+\varphi/\delta\big) \le \Phi_\delta(\varphi)$, for all $\varphi \ge 0$. Hence $\int_0^z d_1(\xi)\xi\ d\xi \ge C_0(\delta)\big((|\Phi_\delta^{-1}(z)|+\delta)\log(1+|\Phi_\delta^{-1}(z)|/\delta) - |\Phi_\delta^{-1}(z)|\big)$ and there is a constant $C_\varphi = C_\varphi(M_\phi,\delta)$ such that $g(z) = C_0(\delta)\big((|\Phi_\delta^{-1}(z)|+\delta)\log(1+|\Phi_\delta^{-1}(z)|/\delta) - |\Phi_\delta^{-1}(z)|\big) - M_\phi|\Phi_\delta^{-1}(z)| + C_\varphi > |\Phi_\delta^{-1}(z)|$, for all z. The equality (38) now implies

$$\int_0^L g(z_N(x,t))\ dx + \int_0^t\int_0^L d_2(z_N)(\partial_x z_N)^2 dx d\tau + \frac{1}{2B}\int_0^L d(z_N)^2(\partial_x z_N(t))^2 dx \le$$

$$C_\varphi(M_\phi,\delta)L + \int_0^t |\partial_\tau \Phi_\delta(\phi_g)(\tau)|(\int_0^L g(z_N(x,\tau))\ dx)\ d\tau. \quad (40)$$

Since $\partial_\tau \Phi_\delta(\phi_g) \in L^1(0,T)$, we apply Gronwall's inequality and estimate (36) follows. Hence $\mathbf{c}$ remains bounded at $t = T_N$. ■

Nevertheless, since the matrix $\mathcal{A}$ could degenerate, some components of $\frac{\partial \mathbf{c}}{\partial t}$ could blow up at $t = T_N$. In other to exclude this possibility and to prove that the maximal solution for (30) exists on $[0,T]$, we need an estimate for the time derivatives. Furthermore, if we want to pass to the limit $N \to +\infty$ in Eq. (28), estimate (36) is not sufficient. Our strategy is to obtain an estimate, uniform with respect to N, for $\partial_{xt} z_N$ in $L^2(Q_T)$.

Theorem 6 *There exists $T_0 > 0$, independent of N, such that*

$$\|\partial_x z_N\|_{L^\infty(0,T_0;L^2(0,L))} \le C \tag{41}$$

$$\|\partial_t z_N\|_{L^2(0,T_0;L^2(0,L))} \le C \tag{42}$$

$$\|\partial_{xt} z_N\|_{L^2(0,T_0;L^2(0,L))} \le C \tag{43}$$

$$\left\| \partial_{xt} \int_0^{z_N} d(\xi)\, d\xi \right\|_{L^2((0,T_0)\times(0,L))} \le C, \tag{44}$$

with constants independent of N. Consequently, the maximal solution for (30) exists on $[0, T_0]$.

Proof. We replace α_k in Eq. (28) by $\partial_t z_N - \partial_t \Phi_\delta(\phi_g)$. This yields

$$\int_0^L d_1(z_N)(\partial_t z_N)^2 dx + \int_0^L d_2(z_N)\,\partial_x z_N \partial_{xt} z_N dx +$$
$$\frac{1}{B}\int_0^L d(z_N)\,\partial_t(d(z_N)\,\partial_x z_N)\,\partial_{xt} z_N dx = \int_0^L d_1(z_N)\,\partial_t z_N \partial_t \Phi_\delta(\phi_g)\,dx. \tag{45}$$

In the estimates which follow we will use the fact that integrability of higher order derivatives implies continuity and boundedness in x or in t. We recall that for one dimensional Sobolev embeddings Morrey's theorem applies and $H^1(0,t)$ (respectively $H^1(0,L)$) is continuously embedded into the Hölder space $C^{0,1/2}[0,t]$ (respectively into $C^{0,1/2}[0,L]$). See e.g. [11] for more details. In our particular situation we use the explicit dependence of the embedding constant on the length of the time interval and we prefer to derive the estimates directly.

First, as $\partial_x z_N \in L^2(0,L;H^1(0,t))$ and $\partial_x z_N|_{\tau=0} = 0$, we have for a.e. $x \in (0,L)$ and for every $\tau \in (0,t)$

$$|\partial_x z_N(x,\tau)| = |\int_0^\tau \partial_\xi \partial_x z_N(x,\xi)\ d\xi| \le \sqrt{\tau}\sqrt{\int_0^\tau |\partial_\xi \partial_x z_N(x,\xi)|^2\ d\xi}. \tag{46}$$

Next, as $\partial_\tau z_N \in L^2(0,t;H^1(0,L))$ and $\partial_\tau z_N|_{\tau=0} = \partial_\tau \Phi(\phi_g)$, we have for a.e. $\tau \in (0,t)$ and for every $x \in (0,L)$:

$$|\partial_\tau z_N(x,\tau)| \le |\partial_\tau \Phi(\phi_g(\tau))| + |\int_0^x \partial_\xi \partial_\tau z_N(\xi,\tau)\ d\xi| \le$$
$$|\partial_\tau \Phi(\phi_g(\tau))| + \sqrt{L}\sqrt{\int_0^L |\partial_\xi \partial_\tau z_N(\xi,\tau)|^2\ d\xi}. \tag{47}$$

Estimates (46)-(47) imply

$$\int_0^L \int_0^t |\partial_\tau z_N(x,\tau)|^2 |\partial_x z_N(x,\tau)|^2\ dx d\tau \le 2\int_0^L \int_0^t \tau\Big(\int_0^t |\partial_\xi \partial_x z_N(x,\xi)|^2\ d\xi\Big)\big(|\partial_\tau \Phi(\phi_g(\tau))|^2 +$$
$$L\int_0^L |\partial_\xi \partial_\tau z_N(\xi,\tau)|^2\ d\xi\big)\ d\tau dx \le 2Lt\,\|\partial_{x\tau} z_N\|^4_{L^2((0,t)\times(0,L))} +$$
$$2\,\|\partial_{x\tau} z_N\|^2_{L^2((0,t)\times(0,L))}\int_0^t \tau |\Phi'_\delta(\phi_g)|^2 |\partial_\tau \phi_g|^2\ d\tau. \tag{48}$$

Now we integrate Eq. (45) with respect to time, over $(0,t)$, and estimate the obtained terms. The second term is estimated as follows:

$$|\int_0^t\int_0^L d_2\left(z_N\right)\partial_x z_N \partial_{x\tau} z_N\ dxd\tau| \le C\int_0^t \left\|\partial_{x\tau} z_N\left(\tau\right)\right\|_{L^2(0,L)}\left\|\partial_x z_N\left(\tau\right)\right\|_{L^2(0,L)} d\tau$$

$$\le C\sqrt{\int_0^t \left\|\partial_{x\tau} z_N\left(\tau\right)\right\|^2_{L^2(0,L)} d\tau}\sqrt{\int_0^t\int_0^L \left(\partial_x z_N\right)^2 dxd\tau} \le C\left\|\partial_{x\tau} z_N\right\|_{L^2((0,t)\times(0,L))}, \tag{49}$$

where we have used the estimate (36). We rewrite the third term of (45) omitting the $1/B$ factor as

$$\int_0^t\int_0^L d\partial_\tau\left(d\partial_x z_N\right)\partial_{x\tau} z_N dxd\tau = \int_0^t\int_0^L d\left(\partial_{x\tau} z_N\right)^2 dxd\tau + \int_0^t\int_0^L dd'\partial_\tau z_N \partial_x z_N \partial_{x\tau} z_N\ dxd\tau. \tag{50}$$

The last term in (50) is cubic in derivatives of z_N. Our idea is to use the estimate (48), showing that for small times it enters with small coefficient and then controlling it using other terms. Using the estimate (48), we find out that it satisfies the following inequality:

$$\left|\int_0^t\int_0^L d\partial_{x\tau} z_N\ d'\partial_\tau z_N \partial_x z_N\ dxd\tau\right| \le C\left\|\partial_{x\tau} z_N\right\|_{L^2((0,t)\times(0,L))}\left\|\partial_\tau z_N \partial_x z_N\right\|_{L^2((0,t)\times(0,L))} \le$$

$$\le C\sqrt{t}\Big(\left\|\partial_{x\tau} z_N\right\|^3_{L^2((0,t)\times(0,L))} + \left\|\Phi'_\delta(\phi_g)\partial_\tau\phi_g\right\|^3_{L^2(0,t)}\Big) \tag{51}$$

Let $X^2\left(t\right) = \int_0^t\int_0^L \left|\partial_{x\tau} z_N\right|^2\ dxd\tau$. Since $\partial_\tau \Phi_g\left(\phi_g\right) \in L^2\left(0,t\right)$, then estimates (36), (49), (50) and (51) imply

$$\left\|\sqrt{d_1}\partial_\tau z_N\right\|^2_{L^2((0,t)\times(0,L))} + X^2\left(t\right) - C_1\sqrt{t}X^3\left(t\right) \le C_o, \tag{52}$$

where C_o depends on $\left\|\partial_\tau \Phi_\delta\left(\phi_g\right)\right\|_{L^2(0,t)}$ and on the constant from estimate (36). We note that the last term on the left hand side correspond to the lower bound for the cubic term, corresponding to the stress gradient part of the diffusive flux. Inequality (52) is satisfied for $t=0$. The function $\varrho\left(X\right) = X^2 - C_1\sqrt{t}X^3$ has its maximum on $(0,+\infty)$ in the point $X_o = 3/\left(2C_1\sqrt{t}\right)$. If $C_o < \varrho\left(X_o\right)$ then inequality (52) gives an estimate for $X\left(t\right)$. We note that $C_o < \varrho\left(X_o\right)$ if $t < \dfrac{4}{27C_1^2C_o}$. Hence for $T \le \dfrac{4}{27C_1^2C_o} = T_0$ we have estimates (41)-(43).

From (41)-(43) it follows that $\partial_x z_N \partial_t z_N \in L^2\left(0,T_0;L^2\left(0,L\right)\right) \le C$ and we have (44) as well. ■

The estimates (41)-(44) allow us to pass to the limit $N\to+\infty$. Using classical compactness and weak compactness arguments and due to the a priori estimates (41)-(44) we can extract a subsequence of z_N, denoted by the same subscripts, which converges to an element $z \in H^1\left((0,T_0)\times(0,L)\right)$, $\partial_{xt} z \in L^2\left((0,T_0)\times(0,L)\right)$, in the following sense

$$z_N \to z \quad \text{strongly in} \quad L^2\left((0,T_0)\times(0,L)\right) \quad \text{and a.e. on}\,(0,T_0)\times(0,L) \tag{53}$$

$$\partial_x z_N \rightharpoonup \partial_x z \quad \text{weakly in} \quad L^2((0,T_0)\times(0,L)) \tag{54}$$

$$\partial_t z_N \rightharpoonup \partial_t z \quad \text{weakly in} \quad L^2((0,T_0)\times(0,L)) \tag{55}$$

$$\partial_{xt} z_N \rightharpoonup \partial_{xt} z \quad \text{weakly in} \quad L^2((0,T_0)\times(0,L)) \tag{56}$$

$$\partial_{xt}\int_0^{z_N} d(\xi)\,d\xi \rightharpoonup \partial_{xt}\int_0^{z} d(\xi)\,d\xi \quad \text{weakly in} \quad L^2((0,T_0)\times(0,L)) \tag{57}$$

Now passing to the limit $N\to\infty$ in Eq. (28) does not pose problems and we conclude that z satisfies (24)-(26).

We summarize the results in the following theorem

Theorem 7 *Let $\phi_g \in H^1(0,T)$. Then there exists $T_0>0$ such that problem (24)-(26) has at least one variational solution $z\in H^1((0,T_0)\times(0,L))$, $\partial_{xt}z\in L^2((0,T_0)\times(0,L))$.*

Corollary 8 *Let $\phi_g \in H^1(0,T)$. Then there exists $T_0>0$ such that the variational formulation (22) of the problem ((8), 12), (19) and (20) has at least one solution $\phi=\Phi_\delta^{-1}(z)\in H^1((0,T_0)\times(0,L))$, $\partial_{xt}\phi\in L^2((0,T_0)\times(0,L))$.*

4 Existence of a solution for the regularized problem (24)-(26) for all times and uniform L^∞ bounds

Let us prove that $T_0=T$. First we test (22) by $\Phi_\delta(\phi)-\Phi_\delta(\phi_g(t))$. We have

$$\int_0^t\int_0^L \partial_\tau\phi\Phi_\delta(\phi)\,dxd\tau+\int_0^t\int_0^L D(\phi)\,\partial_x\phi\partial_x\Phi_\delta(\phi)\,dxd\tau+$$

$$\int_0^t\int_0^L \frac{D(\phi)(|\phi|+\delta)}{B}\partial_x\left(e^{-m\min\{|\phi|,1/\delta\}}\partial_\tau\phi\right)\partial_x\Phi_\delta(\phi)\,dxd\tau=\int_0^t\int_0^L \partial_\tau\phi\Phi_\delta(\phi_g)\,dxd\tau$$

and it follows that

$$\int_0^L\left(\int_0^{\phi(t)}\Phi_\delta(\xi)\,d\xi\right)dx+\int_0^t\int_0^L \frac{1}{2B}\partial_\tau\left(e^{-m\min\{|\phi|,1/\delta\}}\partial_x\phi\right)^2 dxd\tau+$$

$$\int_0^t\int_0^L D(\phi)\,\Phi_\delta'(\phi)(\partial_x\phi)^2\ dxd\tau=\int_0^L \phi(t)\,\Phi_\delta(\phi_g(t))\,dx-\int_0^t\int_0^L \phi\partial_\tau\Phi_\delta(\phi_g)\,dxd\tau$$

and we get as before

$$\|\partial_x\phi\|_{L^\infty(0,t;L^2(0,L))}\le C \tag{58}$$

This estimate implies the boundedness of ϕ. We note that C does not depend on the smoothing of D at $\phi=\phi_c$.

Next we test (22) by

$$e^{-m\ \min\{|\phi|,1/\delta\}}\partial_t\phi - e^{-m\ \min\{|\phi_g|,1/\delta\}}\partial_t\phi_g$$

and we get

$$\int_0^t\int_0^L (\partial_\tau\phi)^2 e^{-m\ \min\{|\phi|,1/\delta\}}dxd\tau + \int_0^t\int_0^L D(\phi)\,\partial_x\phi\partial_x\left(e^{-m\ \min\{|\phi|,1/\delta\}}\partial_t\phi\right)dxd\tau +$$

$$\int_0^t\int_0^L \frac{D(\phi)(|\phi|+\delta)}{2B}\left(\partial_{x\tau}\int_0^\phi e^{-m\min\{|\xi|,1/\delta\}}d\xi\right)^2 dxd\tau = \int_0^t\int_0^L \partial_\tau\phi\ e^{-m\ \min\{|\phi_g|,1/\delta\}}\partial_\tau\phi_g dxd\tau$$

and as before, by estimating the second and the fourth term and after using (58), we conclude that

$$\|\partial_\tau\phi\|_{L^2((0,t)\times(0,L))} \le C \tag{59}$$

$$\left\|\partial_{x\tau}\int_0^\phi e^{-m\min\{|\xi|,1/\delta\}}d\xi\right\|_{L^2((0,t)\times(0,L))} \le C \tag{60}$$

and from this it follows that

$$\|\partial_{x\tau}\phi\|_{L^2((0,t)\times(0,L))} \le C \tag{61}$$

Therefore we arrive at the following theorem

Theorem 9 *Let $\phi_g \in H^1(0,T)$. Then for all $T>0$ there exists a weak solution $\phi \in H^1((0,T)\times(0,L))$, $\partial_{xt}\phi \in L^2((0,T)\times(0,L))$ for the variational formulation (22) of the problem (8), (12), (19) and (20).*

We conclude this section by establishing uniform L^∞-bounds for ϕ. we have

Proposition 10 *Let $\phi_g \in H^1(0,T)$ and $\phi_g \ge 0$. Then any weak solution ϕ of the problem (12), (8), (19) and (20), obtained in Theorem 9, satisfies $\phi(x,t) \ge 0$, a.e. on Q_T.*

Proof. Let $a_- = -\inf\{a,0\}$ and $a_+ = \sup\{a,0\}$. Then $a = a_+ - a_-$ and $\Phi_\delta\left((\phi_g)_-\right) = \Phi_\delta(0) = 0$. We test (22) by $\Phi_\delta(\phi_-)$. Note that $\Phi_\delta(\phi_-)|_{x=0} = 0$ and $\Phi_\delta(\phi_-) \ge 0$. Then we have

$$\int_0^t\int_0^L (\partial_\tau\phi)\,\Phi_\delta(\phi_-)\,dxd\tau + \int_0^t\int_0^L D(\phi)\,\partial_x\phi\partial_x\Phi_\delta(\phi_-)\,dxd\tau +$$

$$\int_0^t\int_0^L \frac{D(\phi)(|\phi|+\delta)}{B}\partial_x\left(e^{-m\min\{|\phi|,1/\delta\}}\partial_t\phi\right)\Phi'_\delta(\phi_-)\,\partial_x\phi_- dxd\tau = 0.$$

Since $\phi_-|_{t=0} = 0$, $\phi_+\phi_- = 0$ and $|\phi|\,\phi_- = \phi_-^2$ we get

$$\int_0^L \left(\int_0^{\phi_-(x,t)} \Phi_\delta(\xi)\, d\xi \right) dx + \int_0^t \int_0^L D(\phi_-)\, \Phi'_\delta(\phi_-)\, (\partial_x \phi_-)^2\, dx d\tau +$$

$$\int_0^L \frac{D(\phi)(|\phi|+\delta)}{2B} \left(e^{-m \min\{|\phi|, 1/\delta\}} \partial_x \phi_- \right)^2 (t)\, dx = 0$$

It follows that $\partial_x \phi_- = 0$ and $\phi_-|_{x=0} = 0$ and therefore $\phi_- = 0$ and consequently $\phi = \phi_+ \geq 0$. ■

Proposition 11 *Let $\phi_g \in H^1(0,T)$, $\phi_g \geq 0$ and $\partial_t \phi_g \geq 0$ a.e. on $(0,T)$. Then any weak solution ϕ of the problem ((8), 12), (19) and (20), obtained in Theorem 9, satisfies $\phi_g(t) \geq \phi(x,t)$, a.e. on Q_T.*

Proof. Let $G(z) = \int_0^z \exp\{-m \min\{\xi, 1/\delta\}\}\, d\xi$, $z \geq 0$. We test (22) by $(G(\phi) - G(\phi_g))_+$. Note that $(G(\phi) - G(\phi_g))_+\Big|_{x=0} = 0$. Then we have

$$\int_0^t \int_0^L \partial_\tau \phi\, (G(\phi) - G(\phi_g))_+\; dx d\tau + \int_0^t \int_0^L D(\phi)\, \partial_x \phi \partial_x (G(\phi) - G(\phi_g))_+\, dx d\tau +$$

$$\int_0^t \int_0^L \frac{D(\phi)(|\phi|+\delta)}{B} \partial_x \left(e^{-m \min\{|\phi|, 1/\delta\}} \partial_t \phi \right) \partial_x (G(\phi) - G(\phi_g))_+\, dx d\tau = 0. \tag{62}$$

Note that

$$\partial_\tau \phi\, (G(\phi) - G(\phi_g))_+ = \partial_\tau \Big(\int_0^\phi (G(\xi) - G(\phi_g))_+\; d\xi \Big) + G'(\phi_g) \partial_t \phi_g\, (\phi - \phi_g)_+ \tag{63}$$

and $$\frac{D(\phi)(|\phi|+\delta)}{B} \partial_t \partial_x G(\phi) \partial_x (G(\phi) - G(\phi_g))_+ =$$

$$\partial_t \Big(\frac{D(\phi)(|\phi|+\delta)}{2B} (\partial_x (G(\phi) - G(\phi_g))_+)^2 \Big) - (\partial_x (G(\phi) - G(\phi_g))_+)^2 \partial_t \Big(\frac{D(\phi)(|\phi|+\delta)}{2B} \Big) \tag{64}$$

Then using the monotonicity of ϕ_g and G we obtain from (62),(63) and (64), the following inequality is found

$$\int_0^L \Big(\int_0^{\phi(x,t)} (G(\xi) - G(\phi_g))_+\; d\xi \Big)\, dx + \int_0^t \int_0^L \frac{D(\phi)}{G'(\phi)} \left(\partial_x (G(\phi) - G(\phi_g))_+ \right)^2 dx d\tau +$$

$$\int_0^L \frac{D(\phi)(|\phi|+\delta)}{2B} (\partial_x (G(\phi) - G(\phi_g))_+)^2\, dx \leq \int_0^t \int_0^L (\partial_x (G(\phi) - G(\phi_g))_+)^2 \partial_t \Big(\frac{D(\phi)(|\phi|+\delta)}{2B} \Big)\, dx d\tau.$$

Since $\partial_t \Big(\frac{D(\phi)(|\phi|+\delta)}{2B} \Big) \in L^2(0,T; L^\infty(0,L))$, we apply Gronwall's lemma and conclude that $(G(\phi) - G(\phi_g))_+ = 0$, from which it follows that $G(\phi) \leq G(\phi_g)$. Inversion of this equation leads to $\phi(x,t) \leq \phi_g(t)$ a.e. on Q_T. ■

Proposition 12 *Let $\phi_g \in H^1(0,T)$ and let us suppose in addition that there are constants $\alpha > 0$ and $C_0 > 0$ such that*

$$\phi_g(t) \geq C_0 t^\alpha, \quad \forall t \in [0,T]. \tag{65}$$

Then any weak solution ϕ of the problem (12), (8), (19) and (20), obtained in Theorem 9, satisfies $\phi(x,t) \geq C_0 t^\alpha$, a.e. on Q_T.

Proof. The proof follows the lines of Proposition 11. Here we test (22) by $\left(G\left(C_0 t^\alpha\right) - G\left(\phi\right)\right)_-$. Note that $\left(G\left(C_0 t^\alpha\right) - G\left(\phi\right)\right)_- |_{x=0} = 0$. Then as in the proof of Proposition 11 we have

$$\int_0^t \int_0^L \partial_\tau \phi \left(G\left(C_0 t^\alpha\right) - G\left(\phi\right)\right)_- \, dx d\tau + \int_0^t \int_0^L D\left(\phi\right) \partial_x \phi \partial_x \left(G\left(C_0 t^\alpha\right) - G\left(\phi\right)\right)_- dx d\tau +$$
$$\int_0^t \int_0^L \frac{D\left(\phi\right)\left(|\phi| + \delta\right)}{B} \partial_x \left(e^{-m \min\{|\phi|, 1/\delta\}} \partial_t \phi\right) \partial_x \left(G\left(C_0 t^\alpha\right) - G\left(\phi\right)\right)_- dx d\tau = 0. \tag{66}$$

Note that

$$G(\phi) = G(C_0 t^\alpha) - \left(G\left(C_0 t^\alpha\right) - G\left(\phi\right)\right)_+ + \left(G\left(C_0 t^\alpha\right) - G\left(\phi\right)\right)_-; \tag{67}$$

$$\partial_\tau \phi \left(G\left(C_0 \tau^\alpha\right) - G\left(\phi\right)\right)_- = \frac{\partial_\tau G(C_0 \tau^\alpha)}{G'(\phi)} \left(G\left(C_0 \tau^\alpha\right) - G\left(\phi\right)\right)_- +$$
$$\frac{1}{2G'(\phi)} \partial_\tau \left(G\left(C_0 \tau^\alpha\right) - G\left(\phi\right)\right)_-^2 \geq \frac{1}{2} \, e^{m \min\{\phi, 1/\delta\}} \, \partial_\tau \left(G\left(C_0 \tau^\alpha\right) - G\left(\phi\right)\right)_-^2 \tag{68}$$

and $$\frac{D\left(\phi\right)\left(|\phi| + \delta\right)}{B} \, \partial_t \partial_x G(\phi) \, \partial_x \left(G\left(C_0 t^\alpha\right) - G\left(\phi\right)\right)_- =$$
$$\partial_t \Big(\frac{D\left(\phi\right)\left(|\phi| + \delta\right)}{2B} \left(\partial_x \left(G\left(C_0 t^\alpha\right) - G\left(\phi\right)\right)_-\right)^2 \Big) - \left(\partial_x \left(G\left(C_0 t^\alpha\right) - G\left(\phi\right)\right)_-\right)^2 \partial_t \Big(\frac{D\left(\phi\right)\left(|\phi| + \delta\right)}{2B} \Big) \tag{69}$$

Then using the monotonicity of G we obtain from (66),(68) and (69), the following inequality

$$\int_0^L \frac{\left(G\left(C_0 t^\alpha\right) - G\left(\phi\right)\right)_-^2}{2G'(\phi)} \, dx + \int_0^t \int_0^L \frac{D\left(\phi\right)}{G'\left(\phi\right)} \left(\partial_x \left(G\left(C_0 \tau^\alpha\right) - G\left(\phi\right)\right)_-\right)^2 dx d\tau +$$
$$\int_0^L \frac{D\left(\phi\right)\left(|\phi| + \delta\right)}{2B} \left(\partial_x \left(G\left(C_0 t^\alpha\right) - G\left(\phi\right)\right)_-\right)^2 \, dx \leq \int_0^t \int_0^L \left(\partial_x \left(G\left(C_0 \tau^\alpha\right) - G\left(\phi\right)\right)_-\right)^2 \partial_\tau \frac{1}{2G'(\phi)} \, dx d\tau$$
$$+ \int_0^t \int_0^L \left(\partial_x \left(G\left(C_0 \tau^\alpha\right) - G\left(\phi\right)\right)_-\right)^2 \partial_\tau \Big(\frac{D\left(\phi\right)\left(|\phi| + \delta\right)}{2B} \Big) \, dx d\tau$$

Since $\partial_t \Big(\frac{D\left(\phi\right)\left(|\phi| + \delta\right)}{2B} \Big) \in L^2(0,T; L^\infty(0,L))$, we apply Gronwall's lemma and conclude that $\big(G\left(C_0 \tau^\alpha\right) - G\left(\phi\right)\big)_- = 0$, from which it follows that $G\left(\phi\right) \geq G(C_0 t^\alpha)$. Inversion of G leads to $\phi(x,t) \geq C_0 t^\alpha$ a.e. on Q_T. ■

Theorem 13 *Let $\phi_g \in H^1\left(0,T\right)$, $A_0 = \max_{0 \leq t \leq T} \phi_g(t)$, $A_0 \geq \phi_g \geq C_0 t^\alpha$, $\alpha > 1$ and $\partial_t \phi_g \geq 0$. Then there exists a weak solution ϕ, $C_0 t^\alpha \leq \phi\left(x,t\right) \leq \phi_g(t)$, $\partial_{xt} \phi \in L^2\left((0,T) \times (0,L)\right)$, $\phi \in$*

$H^1((0,T)\times(0,L))$, *for problem (and (8, 19), (20) and (12)).By choosing* $\delta < 1/A_0$, *we can replace* $e^{-m\min\{|\phi|,1/\delta\}}$ *by* $e^{-m\phi}$ *and* $|\phi|+\delta$ *by* $\phi+\delta$.

If we drop the monotonicity of ϕ_g, *than we have a weaker result: there exists a weak solution* ϕ, $C_0t^\alpha \le \phi(x,t) \le A_0$, $\partial_{xt}\phi \in L^2((0,T)\times(0,L))$, $\phi \in H^1((0,T)\times(0,L))$, *for problem (19), (20), (12) and (8).*

5 Existence for the problem (7), (8), (11), and (12)

It remains to pass to the limit $\delta \to 0$. Let $h(\xi) = e^{-m\ \min\{\xi,A_o\}}$, $\xi \ge 0$. We have existence for the system (22)-(23), i.e. for every $g \in L^2(0,T;V)$, $V = \{g \in H^1(0,L)\,|g(0)=0\}$, we have

$$\int_0^T\int_0^L \partial_t\phi_\delta\ gdxdt + \int_0^T\int_0^L D(\phi_\delta)\left\{\partial_x\phi_\delta + \frac{1}{B}(\phi+\delta)\partial_x(h(\phi_\delta)\partial_t\phi_\delta)\right\}\partial_x g dxdt = 0 \tag{70}$$

$$\phi_\delta|_{x=0} = \phi_g(t) \qquad \text{and} \qquad \phi_\delta|_{t=0} = 0 \tag{71}$$

and we want to pass to the limit $\delta \to 0$.

Let

$$\Psi'_\delta(\xi) = \frac{h(\xi)}{D(\xi)(\xi+\delta)}, \quad \xi \ge 0 \tag{72}$$

and

$$\Psi_\delta(\phi) = \begin{cases} \displaystyle\int_0^\phi \frac{1}{\xi+\delta}\left(\frac{h(\xi)}{D(\xi)} - \frac{h(0)}{D(0)}\right)d\xi + \frac{h(0)}{D(0)}\log(\phi+\delta) & \text{for} \quad \phi \le \phi_c \\ \displaystyle\Psi_\delta(\phi_c) + \int_{\phi_c}^\phi \frac{h(\xi)}{D(\xi)}\frac{1}{\xi+\delta}d\xi & \text{for} \quad \phi > \phi_c \end{cases} \tag{73}$$

It should be noted that $\Psi_\delta(0) = \frac{h(0)}{D(0)}\log\delta < 0$ which would cause some complications.

Theorem 14 *Let* $\alpha > 1$, C_0, A_0 *positive constants and*

$$\phi_g \in H^1(0,T), \quad C_0t^\alpha \le \phi_g \le A_o \text{ and } \quad \log\phi_g \in L^2(0,T). \tag{74}$$

Then problem (7), (8), (11) and (12) has at least one weak solution $\phi \in H^1((0,T)\times(0,L))$, *such that* $\sqrt{\phi}\ \partial_x(e^{-m\phi}\partial_t\phi) \in L^2((0,T)\times(0,L))$ *and* $C_0t^\alpha \le \phi \le A_o$.

Proof.

1. STEP. (A priori estimates uniform in δ.) We test (70) by $\Psi_\delta(\phi_\delta) - \Psi_\delta(\phi_g)$ and get

$$\int_0^t\int_0^L \partial_t\phi_\delta\ \Psi_\delta(\phi_\delta)dxd\tau + \int_0^t\int_0^L \frac{h(\phi_\delta)}{\phi_\delta+\delta}(\partial_x\phi_\delta)^2 dxd\tau +$$

$$+\frac{1}{B}\int_0^t\int_0^L D(\phi_\delta)(\phi_\delta+\delta)\partial_t(h(\phi_\delta)\partial_x\phi_\delta)\frac{h(\phi_\delta)\partial_x\phi_\delta}{D(\phi_\delta)(\phi_\delta+\delta)}dxd\tau = \int_0^t\int_0^L \partial_t\phi_\delta\ \Psi_\delta(\phi_g)dxd\tau$$

and from this

$$\int_0^L \Big(\int_0^{\phi_\delta(t)} \Psi_\delta(\xi)\, d\xi + \frac{1}{2B}\left(h(\phi_\delta)\,\partial_x\phi_\delta\right)^2 \Big) dx + \int_0^t\int_0^L \frac{h(\phi_\delta)}{\phi_\delta+\delta}\left(\partial_x\phi_\delta\right)^2 dx d\tau = \int_0^t\int_0^L \partial_t\phi_\delta\ \Psi_\delta(\phi_g)\, dx d\tau. \quad (75)$$

In order to get an useful estimate we should find a bound for the first term on the left hand side of (75). First we note that $\int_0^{\phi_\delta}\int_0^{\xi} \frac{1}{\eta+\delta}\left(\frac{h(\eta)}{D(\eta)} - \frac{h(0)}{D(0)}\right) d\eta d\xi$ defines a continuous function of ϕ_δ. Since ϕ_δ takes values between 0 and A_o, it is bounded independently of δ. Hence

$$|\int_0^L \int_0^{\phi_\delta(t)} \Psi_\delta(\xi)\, d\xi dx| \le \int_0^L \frac{h(0)}{D(0)} |\{\phi_\delta+\delta\}\log\{\phi_\delta+\delta\} - \phi_\delta - \delta\log\delta|\ dx + C \quad (76)$$

Next $(\phi_\delta(t)+\delta)\log(\phi_\delta(t)+\delta) - \phi(t) - \delta\log\delta$ takes value zero at $t=0$. It is a continuous function of ϕ_δ. Obviously $|(\phi(t)+\delta)\log(\phi(t)+\delta) - \phi(t) - \delta\log\delta| \le \max\Big\{1-\delta+\delta\log\delta, (A_o+\delta)\log(A_o+\delta)$ $-A_o - \delta\log\delta\Big\}$ and it is uniformly bounded with respect to δ.

With (76), (75) leads to

$$\int_0^t\int_0^L \frac{h(\phi_\delta)}{\phi_\delta+\delta}\left(\partial_x\phi_\delta\right)^2 dx d\tau \le C + \left| \int_0^t\int_0^L \partial_t\phi_\delta\ \Psi_\delta(\phi_g)\, dx d\tau \right|, \quad (77)$$

Next we test (70) by $h(\phi_\delta)\,\partial_t\phi_\delta - h(\phi_g)\,\partial_t\phi_g$ and get

$$\int_0^t\int_0^L h(\phi_\delta)\left(\partial_t\phi_\delta\right)^2\ dx d\tau + \int_0^t\int_0^L D(\phi_\delta)\,\partial_x\phi_\delta\partial_x\left(h(\phi_\delta)\,\partial_x\phi_\delta\right) dx d\tau +$$

$$+\frac{1}{B}\int_0^t\int_0^L D(\phi_\delta)\left(\phi_\delta+\delta\right)\left(\partial_x\left(h(\phi_\delta)\,\partial_x\phi_\delta\right)\right)^2 dx d\tau = \int_0^t\int_0^L \partial_t\phi_\delta\ h(\phi_g)\,\partial_t\phi_g dx d\tau$$

and from this

$$\int_0^t\int_0^L h(\phi_\delta)\left(\partial_t\phi_\delta\right)^2\ dx d\tau + \frac{1}{B}\int_0^t\int_0^L D(\phi_\delta)\left(\phi_\delta+\delta\right)\left(\partial_x\left(h(\phi_\delta)\,\partial_x\phi_\delta\right)\right)^2 dx d\tau \le$$

$$B\int_0^t\int_0^L \frac{D(\phi_\delta)}{\phi_\delta+\delta}\left(\partial_x\phi_\delta\right)^2 dx d\tau + \int_0^t\int_0^L \frac{h^2(\phi_g)}{h(\phi_\delta)}\left(\partial_t\phi_g\right)^2 dx d\tau. \quad (78)$$

Let $h_{min} = e^{-mA_o}$. Then inserting (77) into (78) yields

$$\int_0^t \int_0^L h(\phi_\delta) (\partial_t \phi_\delta)^2 \ dxd\tau + \frac{1}{B} \int_0^t \int_0^L D(\phi_\delta)(\phi_\delta + \delta)(\partial_x (h(\phi_\delta) \partial_t \phi_\delta))^2 dxd\tau \leq C +$$

$$\frac{BD_r}{h_{min}} \left| \int_0^t \int_0^L \partial_t \phi_\delta \Psi_\delta (\phi_g) dxd\tau \right| + \int_0^t \int_0^L \frac{h^2(\phi_g)}{h(\phi_\delta)} (\partial_t \phi_g)^2 dxd\tau \leq C +$$

$$\frac{1}{2} \int_0^t \int_0^L h(\phi_\delta)(\partial_t \phi_\delta)^2 \ dxd\tau + \frac{B^2 (D_r)^2}{2h_{min}^3} \|\Psi_\delta (\phi_g)\|^2_{L^2((0,t)\times(0,L))} + \frac{1}{h_{min}} \int_0^t \int_0^L (\partial_t \phi_g)^2 \ dxd\tau.$$

2. STEP. (Weak and strong compactness) From the above a priori estimate and assumptions (74) on ϕ_g, we conclude that

$$\|\partial_t \phi_\delta\|_{L^2((0,T)\times(0,L))} + \left\| \frac{1}{\sqrt{\phi_\delta + \delta}} \ \partial_x \phi_\delta \right\|_{L^2((0,T)\times(0,L))} \leq C \tag{79}$$

$$\left\| \sqrt{\phi_\delta + \delta} \partial_x (h(\phi_\delta) \partial_t \phi_\delta) \right\|_{L^2((0,T)\times(0,L))} \leq C \tag{80}$$

Hence there is a $\phi \in H^1((0,T)\times(0,L))$ and a subsequence $\{\phi_\delta\}$, denoted by the same subscripts, such that

$$\phi_\delta \to \phi \quad \text{strongly in } L^2((0,T)\times(0,L)) \text{ and a.e. on} \quad (0,T)\times(0,L) \tag{81}$$

$$\partial_t \phi_\delta \rightharpoonup \partial_t \phi \quad \text{weakly in} \quad L^2((0,T)\times(0,L)) \tag{82}$$

$$\partial_x \phi_\delta \rightharpoonup \partial_x \phi \quad \text{weakly in} \quad L^2((0,T)\times(0,L)) \tag{83}$$

With the part of the flux containing the second order operator situation is more complicated. Obviously, there is $F \in L^2((0,T)\times(0,L))$ such that

$$\sqrt{\phi_\delta + \delta} \ \partial_{xt} \int_0^{\phi_\delta} h(\xi) d\xi \rightharpoonup F \quad \text{weakly in} \quad L^2((0,T)\times(0,L)) \tag{84}$$

Using the lower bound $\phi_\delta \geq C_0 t^\alpha$, we get from the estimate (80) and convergence (81)

$$\partial_{xt} \int_0^{\phi_\delta} h(\xi) d\xi \rightharpoonup \partial_{xt} \int_0^{\phi} h(\xi) d\xi, \quad \text{weakly in} \quad L^2((0,T)\times(0,L)) \tag{85}$$

The convergences (81) and (85) imply that F in (84) is given by $F = \sqrt{\phi} \ \partial_{xt} \int_0^{\phi} h(\xi) d\xi$.

3. STEP. (passing to the limit) Consequently for every $g \in L^2(0,T;V)$ we have

$$\int_0^T\int_0^L \partial_t \phi_\delta \; g \; dxdt \to \int_0^T\int_0^L \partial_t \phi \; g \; dxdt \quad \text{for } \delta \to 0 \tag{86}$$

$$\int_0^T\int_0^L D(\phi_\delta)\,\partial_x \phi_\delta \; \partial_x g dxdt \to \int_0^T\int_0^L D(\phi)\,\partial_x \phi \; \partial_x g \; dxdt \quad \text{for } \delta \to 0 \tag{87}$$

$$\int_0^T\int_0^L \frac{1}{B} D(\phi_\delta)(\phi_\delta + \delta)\,\partial_x (h(\phi_\delta)\,\partial_t \phi_\delta) \; \partial_x g \; dxdt \to \int_0^T\int_0^L \frac{1}{B} D(\phi) \; \phi \; \partial_x (h(\phi)\,\partial_t \phi) \; \partial_x g \; dxdt. \tag{88}$$

Hence we conclude that ϕ satisfies the system (7), (8), (11) and (12). ■

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