

Aide Mémoire

Subject: Useful formulas for flow in rivers and channels

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1. General basic equations

The general continuity equation for flow, following from conservation of mass, is:

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{v}) = 0$$

For an incompressible fluid $D\rho/Dt=0$. Furthermore for a homogeneous fluid also $\text{grad}(\rho)=0$, leading to the following continuity equation (conservation of volume):

$$\text{div } \vec{v} = 0$$

From the basic equations for conservation of momentum the commonly used equation of motion of *Navier-Stokes* can be derived (assuming homogeneous incompressible fluid with constant density ρ):

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \text{grad}) \vec{u} + \frac{1}{\rho} \text{grad } p + \text{grad}(gz) - \nu \nabla^2 \vec{u} = \frac{1}{\rho} \vec{K}$$

←————→	←————→	←————→	←————→	←————→
local acceleration and convection	pressure	gravity	internal friction	external forces

- The equation of *Euler* can be derived by neglecting of internal friction terms (viscous terms)
- The relation between convective term and viscous shear-stress term can be expressed in terms of a Reynolds number: $\frac{u^2 / L}{\nu u / L^2} = \frac{uL}{\nu} = \text{Re}$
 - Small Re: stable system, laminar flow, Navier-Stokes is linear.
 - Large Re ($> O(10^3)$): non-linear convection dominates, system is instable (small changes give large differences in solution), turbulent flow.
- The internal friction terms or viscous terms are diffusive terms.
- The external forces are mainly the shear stresses on the bed, walls and surface (wind-shear).
- The subdivision of units to time-average and erratic components (e.g., $u = \bar{u} + u'$) will lead to balance equation for the average components that is called the Reynold equation. It is commonly used for turbulent flows.

2. Equations for Two dimensional flow

Basic equations

The basic equations for 2DH flow follow from depth integration of the Reynolds equations. The continuity equation for flow is:

$$\frac{\partial a}{\partial t} + \frac{\partial au}{\partial x} + \frac{\partial av}{\partial y} = 0$$

The basic equations of motion (in their commonly used form):

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + g \frac{\partial(a + z_b)}{\partial x} + \frac{\tau_{bx}}{\rho a} = \frac{1}{\rho} \frac{\partial(aT_{xx})}{\partial x} + \frac{1}{\rho} \frac{\partial(aT_{xy})}{\partial y}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + g \frac{\partial(a + z_b)}{\partial y} + \frac{\tau_{by}}{\rho a} = \frac{1}{\rho} \frac{\partial(aT_{xy})}{\partial x} + \frac{1}{\rho} \frac{\partial(aT_{yy})}{\partial y}$$

in which

u	=	depth-average velocity in x-direction
v	=	depth-average velocity in y-direction
a	=	local water depth
z _b	=	local bed level
τ _{bx} , τ _{by}	=	bed-shear stress in x,y-direction
T _{xx} , T _{xy} , T _{yy}	=	horizontal exchange of momentum through viscosity, turbulence, spiral flow, and non-uniformity of velocity distribution

Shear stresses can be expressed by 2D Chézy relations:

$$\tau_{bx} = \frac{\rho g u \sqrt{u^2 + v^2}}{C^2} \quad \text{and} \quad \tau_{by} = \frac{\rho g v \sqrt{u^2 + v^2}}{C^2}$$

with C = Chézy coefficient

Spiral flow

The 2D effect of the 3D spiral flow pattern due to curvature of the streamlines can be obtained from a parametrised model. The effect of spiral flow on the direction of bottom-shear stress can be obtained from:

$$\delta = \underbrace{\arctan\left(\frac{v}{u}\right)}_{\text{regular flow}} - \underbrace{\arctan\left(A \frac{a}{R_*}\right)}_{\text{spiral flow}}$$

where δ is the angle with x-coordinate axis, R* is the effective radius of the streamline, and A is the spiral flow coefficient expressed as:

$$A = \frac{2\varepsilon}{\kappa^2} \left(1 - \frac{\sqrt{g}}{\kappa C}\right)$$

with ε is a calibration coefficient, and κ is the Von Karman coefficient (≈ 0.4)

In an equilibrium situation (at the end of a long bend) the spiral flow intensity I is expressed by

$$I = \frac{a\sqrt{u^2 + v^2}}{R_*}$$

Relaxation

For adaptation of 2D non-uniform horizontal velocity distribution to a 2D uniform equilibrium distribution a length scale is defined as (relaxation length):

$$\lambda_w = \frac{C^2 h}{2g}$$

3. Equations for one dimensional flow

Dynamic flow (Saint-Venant equations)

Integration of the continuity equation yields:

$$B \frac{\partial a}{\partial t} + \frac{\partial Q}{\partial x} - q_{in} = 0$$

in which a is water depth, B is the total width (including the storage width), and q_{in} is lateral inflow

Integration of the equation of motion yields for conservation of momentum:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left(\alpha \frac{Q^2}{A} \right) + gA_s \frac{\partial z_w}{\partial x} = -A_s \frac{\tau_b}{\rho} + q_{in} u'$$

in which A_s is cross-sectional area of flow, u' is the x-component of velocity of lateral inflow, z_w is the water surface elevation, τ_b is the bed-shear stress (hydraulic roughness).

Combination of the equations for motion and continuity above yields the following variant for the equation of motion:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial z_w}{\partial x} = - \frac{\tau_b}{\rho} - \frac{q_{in}}{A_s} (u - u')$$

\longleftrightarrow \longleftrightarrow \longleftrightarrow \longleftrightarrow
 local acceleration pressure roughness lateral
 and convection and gravity inflow

- Hydraulics roughness is normally expressed by the Chézy formula: $\tau_b = \frac{\rho g u |u|}{C^2 R} = \frac{\rho g Q |Q|}{C^2 A_s^2 R}$ in which R = Hydraulic radius (see uniform flow). (Alternative roughness formulations: see below)
- In a river-flood wave the 'dynamic wave' approach can be used that assumes that the inertia term (local acceleration term) can be neglected. The flood wave is then expressed by the following diffusion model (after simplification for $A_s = B_s u$, $R = a$, constant C and B):

$$\frac{\partial a}{\partial t} + \left(\frac{3}{2} \frac{Q}{Ba} \right) \frac{\partial a}{\partial x} - \left(\frac{C^2 B_s^2 a^3}{2QB} \right) \frac{\partial^2 a}{\partial x^2} = 0$$

with B is the total width (including the storage width). From the general diffusion model follows the propagation speed of the wave c :

$$c = \frac{1}{B} \frac{\partial Q}{\partial a} \cong \frac{3}{2} \frac{Q}{Ba} = \frac{3}{2} \frac{uB_s}{B}$$

Damping of the flood wave is governed by the diffusion coefficient, which is expressed as:

$$K = \frac{Q}{2Bi_b} \{1 - b \cdot Fr^2\} \quad \text{with} \quad b = B_s / B$$

Damping of the flood wave occurs proportional with $[2\sqrt{(\pi Kt)}]^{-1}$

- A more simplified approach than the dynamic-wave approach can be used for relatively 'flat' flood waves. By assuming that the flow (surface slope) behaves like a uniform flow, the 'kinematic wave' approach is obtained. Different than the dynamic wave, the kinematic wave does not

Uniform flow (Chézy formula)

The uniform-flow equations follow from the Chézy formula

$$u = C\sqrt{R \cdot i} \quad \text{with} \quad R = A/P$$

with i = local hydraulic gradient or energy gradient, and P = wetted perimeter of the cross section with flow area A .

For a rectangular channel with free surface and $B \gg a$: $R = \frac{Ba}{B + 2a} \approx a$, while for a filled circular pipe $R = D/4$ with D is the diameter of the pipe.

For the equilibrium depth follows:

$$a_e = \sqrt[3]{\frac{q^2}{C^2 i}}$$

The Froude number Fr , representing the balance between inertia and gravity, and characteristic for subdivision of sub-critical ($Fr < 1$) and supercritical flow ($Fr > 1$, torrential regime), is defined as:

$$Fr = \frac{u}{\sqrt{ga}}$$

For $Fr = 1$ the flow is critical. Then the depth equals the critical depth a_g expressed as:

$$a_g = \sqrt[3]{\frac{q^2}{g}}$$

Hydraulic roughness

Hydraulic roughness is expressed in terms of the Chézy (C), Manning-Strickler (n), Darcy-Weisbach (f). The relation between C and f is:

$$fC^2 = 8g$$

The coefficients are generally written as

$$C = 2,3 \frac{\sqrt{g}}{\kappa} \log \frac{12R}{k + \delta / 3,5} = 18 \log \frac{12R}{k + \delta / 3,5}$$

and

$$f = 0,24 \log^{-2} \left[\frac{12R}{k + \delta / 3,5} \right]$$

In these equations k is the equivalent sandroughness according to Nikuradse. For an alluvial bed the value of k varies strongly with the flow conditions. In rivers the flow regime will often be hydraulically rough ($k \gg \delta$).

The value of C then becomes according to White-Colebrook:

$$\text{White - Colebrook: } C = 18 \log \frac{12R}{k}$$

or according to Strickler:

$$\text{Strickler: } C = 25 \left[\frac{R}{k} \right]^{1/6}$$

Most often used, and linked with Strickler's equation, is the Manning roughness formula (or Manning-Strickler roughness formula). The relation between Manning's roughness coefficient n and the Chézy coefficient C is (with R in meters):

$$C = \frac{R^{1/6}}{n}$$

Manning n-value	Channels			Minor natural streams (Width at flood < 30 m)														Major stream													
	Lined channels		Excavated dredged	Natural stream on plain				Mountain streams		Flood plains																					
	Smooth metal	Corrugated metal	Wood	Concrete	Brick	Asphalt	Straight earth, clean	Straight, gravel	Rock-cut	Clean, straight, full stage, no pools	Idem, more stones and weeds	Clean, winding, some pools and shoals	Idem, more stones and weeds	Idem, lower stages. More ineffective slopes and sections	Sluggish reaches; weedy, deep pools	Very weedy reach, deep pools, floodways with timber and underbrush	Vegetated steep banks, gravel, cobbles and few boulders		Vegetated steep banks, cobbles with large boulders	Pasture, no brush, grass	Cultivated areas	Scattered brush, heavy weeds	Light brush and trees	Medium to dense brush	dense willows, summer, straight	cleared land with tree stumps and sprouts	heavy stand of timber, little undergrowth	Regular section, no boulders of brush	Irregular and rough section		
0.25																															
0.2																															
0.15																															
0.1																															
0.09																															
0.08																															
0.07																															
0.06																															
0.05																															
0.04																															
0.03																															
0.02																															
0.01																															

Table Range of values of the roughness coefficient *n* for different types of channels

Surface profiles

The *equation of Bélanger* holds for a channel with constant width and bed slope i_b (i.e., following from one-dimensional-flow basic equations or Saint-Venant equations simplified for stationary flow):

$$\frac{dh}{dx} = i_b \frac{h^3 - h_e^3}{h^3 - h_g^3}$$

With $h = \eta h_e$ and $\beta = 1 - \{h_g / h_e\}^3 = 1 - Fr^2$ follows:

$$x = \frac{h_e}{i_b} \left[\eta - \beta \int \frac{d\eta}{1 - \eta^3} \right] + \text{constante}$$

to be solved using the *Bresse method*:

$$\begin{aligned} \psi(\eta) &= \int \frac{d\eta}{1 - \eta^3} = \\ &= \frac{1}{6} \ln \left[\frac{\eta^2 + \eta + 1}{(\eta - 1)^2} \right] + \frac{1}{\sqrt{3}} \left[\arctan \left(\frac{2\eta + 1}{\sqrt{3}} \right) - \arctan \left(\frac{1}{\sqrt{3}} \right) \right] \end{aligned}$$

After introduction of a dimensionless length scale

$$\Lambda = \frac{x i_b}{h_e}$$

follows from equations above:

$$\Lambda_2 - \Lambda_1 = \{ \eta_2 - \eta_1 \} - \beta \{ \psi(\eta_2) - \psi(\eta_1) \}$$

If at $x = 0$ holds that $h_0 = \eta_0 h_e$ then the Bresse method can be used to determine (if $Fr^2 \ll 1$, hence $\beta \rightarrow 1$) for which x still half (50%) of the induced backwater effect remains. By fitting of Bresse function through a power function the following approximation can be obtained:

$$\Lambda_{1/2} = 0,24 \eta_0^{4/3}$$

For relatively small backwater effects ($\eta_0 \rightarrow 1$) the value $\Lambda_{1/2} = 1/4$ is obtained. These approximations are useful for sketching surface profiles.

Flow over weirs, spillways and other structures

Flow over a broad-crested weir can be submerged (drowned flow) or free (overflow or modular

flow). The discharge through the weir q can be expressed as function of the upstream velocity head H_u (energy level) and the water depth downstream h_d , both relative to the crest level.

Broad-crested weir, submerged flow

$$q = mh\sqrt{2g(H-h)}$$

with: $0,9 < m < 1,3$ (average $m \approx 1,1$)

Broad-crested weir, free flow

$$q = m\frac{2}{3}\sqrt{\frac{2}{3}g} \cdot H^{3/2}$$

These formulas are used for sharp-crested weirs as well, but usually with a higher discharge coefficient (m).

Weirs with oblique flow:

If in a stream channel with discharge q_0 flow passes a weir with an angle α , then q_0 follows from

$$q_0 = q \cdot \sin^{-1} \alpha$$

in which q follows from the regular discharge formula (for weirs in perpendicular flow)