# Analysis of a Roundabout Model

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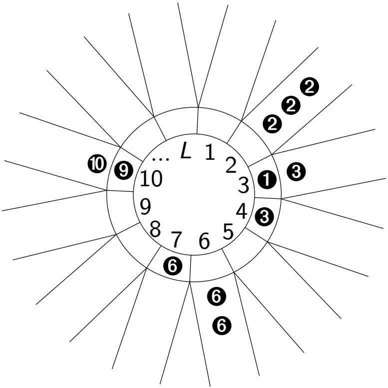
# Overview

1. The model
2. Claims and corresponding ideas

# The Model



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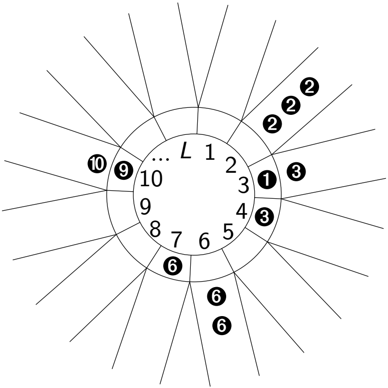


Discrete-time stochastic process

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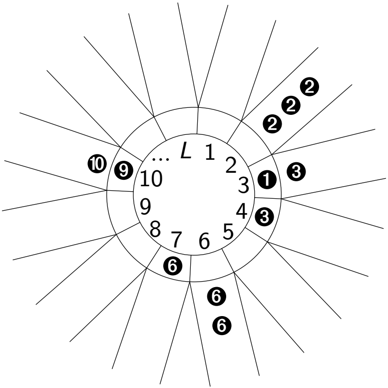


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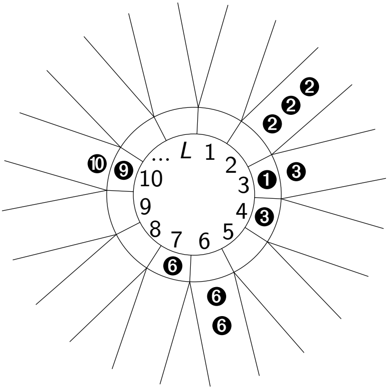
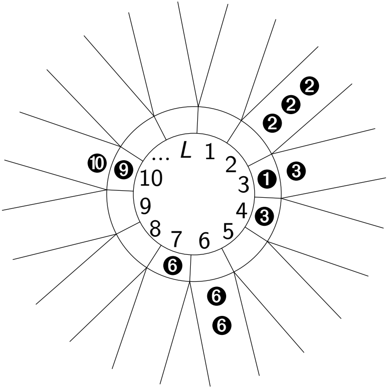



Discrete-time stochastic process

Car types

Cells in $\{0, 1, \ldots, L\}$, Queues in $\mathbb{Z}_+$

# Dynamics

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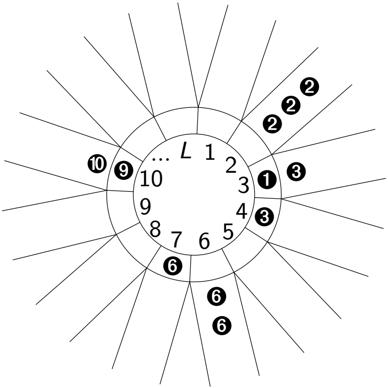


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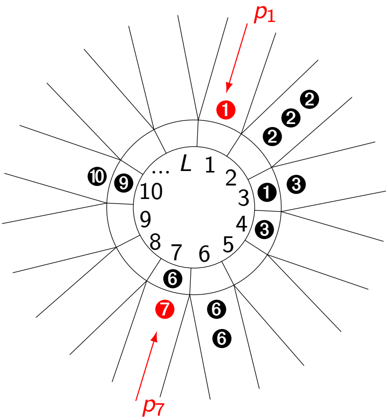


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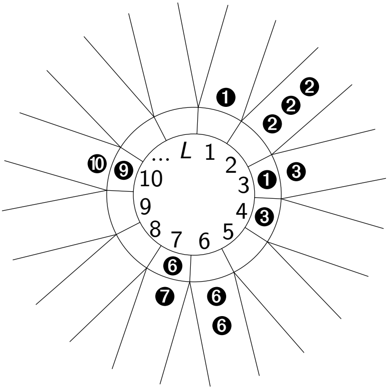


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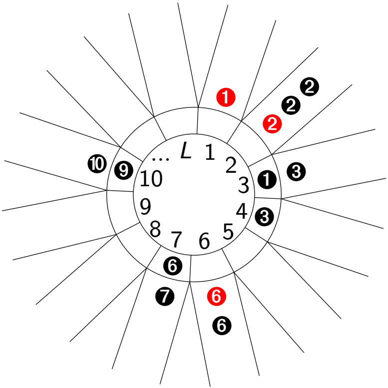


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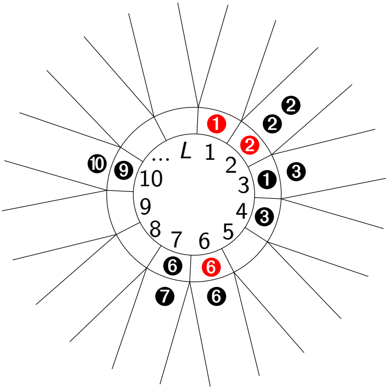


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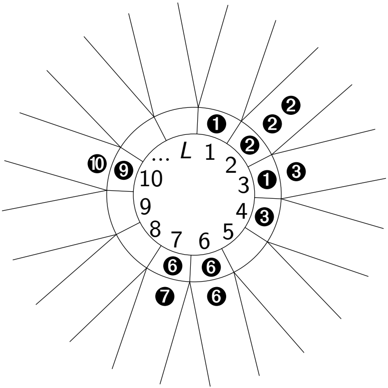


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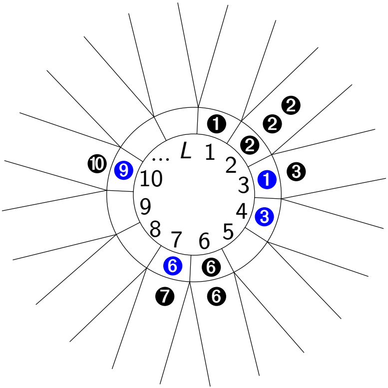


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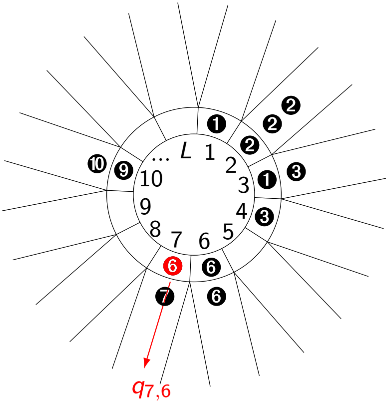


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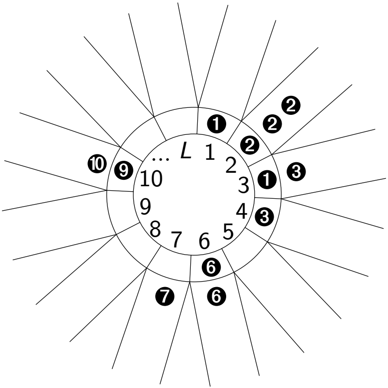


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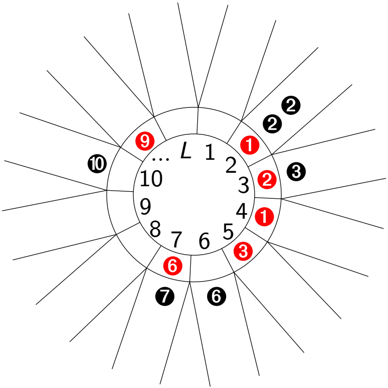


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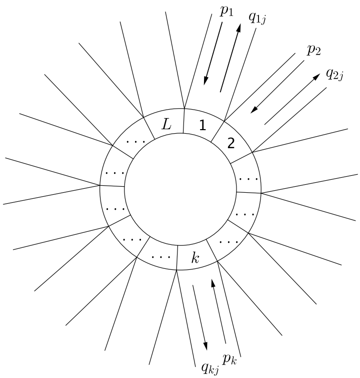


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- Discrete-time Markov chain
  - Stability conditions
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- Queueing Network
  - Performance measures
  - Practical results for application

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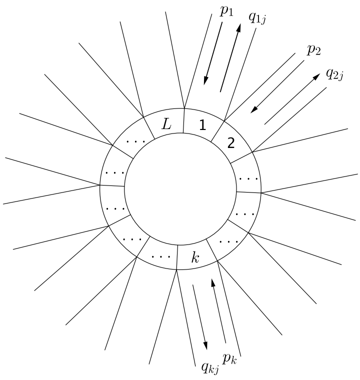


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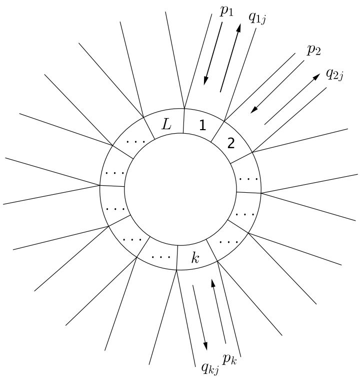


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# What Do We Have?

Irreducibility and aperiodicity

Marginal stationary distributions cells (we can prove this assuming stability)

$$\pi_{ij} = \frac{p_j \prod_{l=j+1}^{i-1}(1 - q_{lj})}{1 - \prod_{l=1}^{L}(1 - q_{lj})} \text{ stationary probability } \mathbb{P}(C_i = j)$$

$$\pi_{i0} := 1 - \sum_{j=1}^{L} \pi_{ij} \text{ stationary probability } \mathbb{P}(C_i = 0)$$

# Stability

Conjecture for stability region

$$\pi_{i0} > p_i, \quad \text{for each } i = 1, \dots, L$$

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Problem is: we cannot use marginal stationary distribution

We are trying to use fluid limits instead.

# Correlations

Weak dependency between segments on the roundabout

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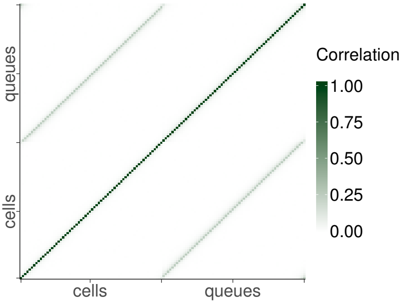

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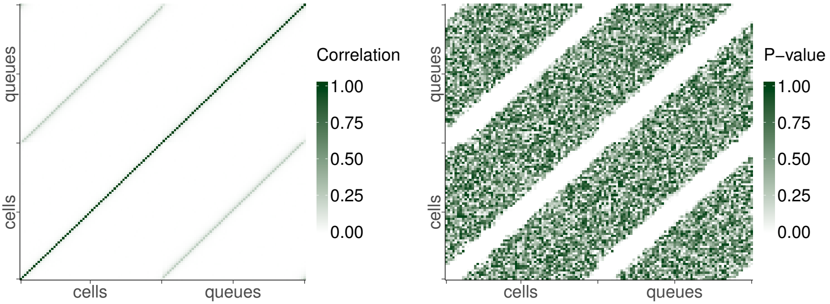
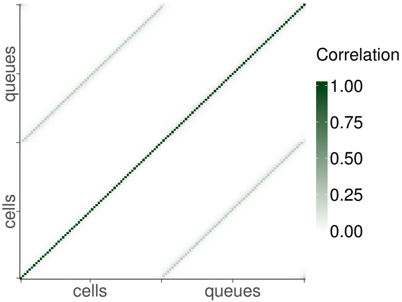
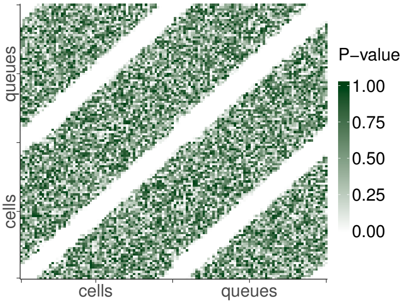


# Correlations

Weak dependency between segments on the roundabout

Actually: correlations decrease geometrically in distance between objects



# Scaling limits

**Scaling parameters** Idea: map roundabout to $[0, 1]$ and let $L \to \infty$.

Scale $p$ and $q$ s.t. arrivals and departures remain invariant over each segment.

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**Invariance arrival density**

$$p_i \equiv p_i(L) = \theta \int_{i/L}^{(i+1)/L} \varrho(x)\,\mathrm{d}x, \text{ for some periodic-1 } \varrho : \mathbb{R} \to \mathbb{R}_+ \text{ and } \theta \in \mathbb{R}_+$$

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**Invariance driving distance**

Introduce $(X_x)_{x\in(0,1]}$, representing driving distance for a car entered at $x$.

$$q_{ij} \equiv q_{ij}(L) = 1 - \Pr\left(X_{j/L} > \tfrac{i-j+1}{L} \mid X_{j/L} > \tfrac{i-j}{L}\right), \qquad i \geq j$$

# Scaling limits (FCLT for spatial occupation cells)

Define $T_k^L := \sum_{\ell=1}^{k} 1_{\{C_\ell = 0\}}$ $\qquad \mathbb{E}(T_k^L) = \sum_{\ell=1}^{k} \pi_{l0}$

Denote $s_L^2 := \frac{1}{L} \sum_{\ell=1}^{L} \pi_{l0}(1 - \pi_{l0})$ $\qquad t_k^L := \frac{1}{L s_L^2} \sum_{\ell=1}^{k} \pi_{l0}(1 - \pi_{l0})$

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**Claim:** as $L \to \infty$, $T^L(\cdot) \Rightarrow \int_0^{\cdot} \eta_s \,\mathrm{d}B_s$, with $B$ a standard Brownian motion and $\eta$ a continuous deterministic function.

# Scaling limits (cumulative occupation queues)

Define $P_i^L := Q_1 + \ldots + Q_i$, with $Q_i :=$ length of queue $i$, $P_0^L = 0$.

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**Claim:** as $L \to \infty$, $P^L(\cdot) \Rightarrow P$, where $P$ is an time-inhomogeneous

Poisson process.

# Summary and Conclusion

Model given

Analysis difficult

Interesting ideas still possible and further research (maybe) necessary

**Thank you**