Macroscopic models for pedestrian flow and meshfree particle methods

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Content

- Hierarchy of models
- Numerical method
- Numerical results

Model hierarchies

Fine to coarse scale models

- microscopic models: interacting particles, social force models (Helbing et al.), vision-based models (Degond et. al.), ...
- kinetic models: Vlasov-type, Vlasov-McKean equations
- macroscopic models: macroscopic models with non-local terms (Colombo et. al.)
- 'localized' models (Hughes et al., ...)

Microscopic models: empirical distributions

$$\rho^N_{\delta_S}(x) = \frac{1}{N} \sum_j \delta_S(x - x_j)$$

- Function $\delta_S = \delta_S(x), x \in \mathbb{R}^d$ with $\delta_S(x) = \frac{1}{S^d}\delta_1(\frac{x}{S})$, $\int \delta_S(x)dx = 1$.
- Approximation of Dirac $\delta = \delta(x)$ as S goes to 0.
- δ_S smooth, rotationally symmetric, monotone decaying

Interaction potentials

- Function $U_R = U_R(x), x \in \mathbb{R}^d$.
- Example: U_R smooth, rotationally symmetric, monotone decaying ($\sim$ repulsive interaction potential).
- We use again $U_R(x) = \frac{1}{R^d} U_1(\frac{x}{R})$ such that $\int U_1(x)dx = 1$ and U_R approximates δ for $R \to 0$.
- Joint interaction potential:

$$\rho^N_{U_R}(x) = \frac{1}{N} \sum_j U_R(x - x_j)$$

Interaction potentials for traffic / pedestrian flow

- unsymmetric, one-sided potentials, vision cones.
- $d = 1$: potentials depending on downstream traffic density with $U(x) = 0, x > 0$. This gives $U(x_i - x_j) = 0$ if $x_i - x_j > 0$, i.e. $x_i > x_j$, i.e. interaction only with predecessor.

Microscopic equations for interacting particle system

$(x_i, v_i)(t) \in \mathbb{R}^2 \times \mathbb{R}^2,\ i = 1, \ldots, N,\ V(\rho) = 1 - \rho.$

$$
\begin{aligned}
dx_i &= v_i dt \\
dv_i &= V(\rho^N_{\delta_S}(x_i))\hat{e}(x_i)dtdt - \nabla_x \rho^N_{U_R}(x_i)dt \\
&\quad -\gamma v_i dt + AdW^i_t
\end{aligned}
$$

with

$$
\hat{e}(x) = \frac{\nabla\phi(x)}{|\nabla\phi(x)|}
$$

where ϕ is the solution of the eikonal equation

$$
|\nabla\phi(x)| = \left(V(\rho^N_{\delta_S}(x))\right)^{-1}.
$$

Remark: Reduced microscopic equations

Simplified equations neglecting time dependence of velocities for $x_i(t) \in \mathbb{R}^d$, $i = 1, \ldots, N$,

$$dx_i = V(\rho^N_{\delta_S}(x_i))\hat{e}(x_i)dt - \nabla_x \rho^N_{U_R}(x_i)dt + AdW^i_t$$

Mean field limit: Empirical measure

The empirical measures of the stochastic processes (x_i, v_i) are given by

$$\frac{1}{N}\sum_i \delta_{(x_i,v_i)}(x,v).$$

The mean field limit describes the convergence as $N \to \infty$ towards a deterministic distribution $f = f(x,v)$. This gives the convergence of

$$\rho^N_{\delta_R}(x_i) = \frac{1}{N}\sum_j \delta_R(x_i - x_j)$$

to a coarse grained density

$$\int \delta_R(x-y)\rho(y)dy = \delta_R \star \rho(x)$$

with

$$\rho(y) = \int f(y,w)dw.$$

Mean field limit

Starting from the microscopic equations for $(x_i, v_i)(t)$, $i = 1, \ldots, N$, this gives formally a limit stochastic process $(x, v) \in \mathbb{R}^2 \times \mathbb{R}^2$, the McKean -Vlasov equation

$$\begin{aligned} dx &= vdt \\ dv &= V(\delta_S \star \rho)\hat{e}dt - \nabla_x U_R \star \rho dt - \gamma vdt + AdW_t \end{aligned}$$

where $\rho(y) = \int f(y, w)dw$ and $f(x, v)$ is the distribution of the stochastic process (x, v).

Kinetic mean field equation

The corresponding differential equation for the evolution of the distribution functions $f = f(x, v, t), x, v \in \mathbb{R}^d$ is given by

$$\partial_t f + v \cdot \nabla_x f = Sf + Lf$$

with force term

$$Sf = \nabla_v \cdot (V(\delta_S \star \rho)\hat{e} f) + \nabla_v \cdot (\nabla_x U_R \star \rho f) .$$

and diffusion term

$$Lf = \nabla_v \cdot \left(\gamma v f + \frac{A^2}{2} \nabla_v f \right) .$$

Rem.: For the reduced microscopic problems we obtain for $\rho = \rho(x, t)$

$$\partial_t \rho = \nabla_x \cdot \left(\rho \left(-V(\delta_S \star \rho)\hat{e} + \nabla_x U_R \star \rho \right) + \frac{A^2}{2} \nabla_x \rho \right) .$$

References / Lecture notes

- F. Golse, *On the Dynamics of Large Particle Systems in the mean-field limit*
- Golse, F.: *The mean field limit for the dynamics of large particle systems*. Journées équations aux dérivées partielles, **9** (2003), pp. 1–47.
- M. Hauray and P.E. Jabin, Particle approximation of Vlasov equations with singular forces: Propagation of chaos.
- P.E. Jabin, A review of the mean field limit for the Vlasov equation

Balance equations

Multiplying the mean field equation with 1 and v one obtains

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$
$$\partial_t \rho u + \nabla_x \cdot \int v \otimes v f dv = -\gamma \rho u + \rho V(\delta_S \star \rho)\hat{e} - \rho \nabla_x U_R \star \rho$$

with the momentum

$$\rho u(x,t) := \int v f(x,v,t) dv.$$

Closure problem: approximate $\int v \otimes v f dv$ using ρ and u.

The closure problem

Solution: choose ansatz function $F = F[\rho, u](v)$ with $\int F dv = \rho$ and $\int vF dv = \rho u$ such that $f \sim F$. Then, $\int v \otimes v f dv \sim \int v \otimes v F dv$

- **A monokinetic closure:** $F(v) = \rho \delta(v - u)$ such that $\int v \otimes v F dv = \rho u \otimes u$.
- **Maxwellian closure with variance θ:** $F = M[\rho, u, \theta](v)$ such that $\int v \otimes v F dv = \theta \rho I + \rho u \otimes u$.
- **A linear closure:** choose $F = \rho(1 + u \cdot v)\bar{M}$ with for example $\bar{M}(v) = M[1, 0, 1](v))$ such that $\int v \otimes v F dv = \rho I$. F might be negative!

Other closures

- Higher order expansions matching more moments:

$$F(v) = \left(\sum_{i=0}^{n} w_i v^i \right) \bar{M}$$

- **Nonlinear (Maximum-entropy) closures**: positive ansatz function

$$F(v) = \exp(\sum_{i=0} w_i v^i),$$

for example

$$F(v) = a \exp(b \cdot v)$$

- C.D. Levermore, Moment closure hierarchies for kinetic theories, J. Stat. Phys. 83 (1996) 1021–1065.
- A. M. Anile, S. Pennisi, and M. Sammartino, A thermodynamical approach to Eddington factors, J. Math. Phys. 32, 544 (1991).

Hydrodynamic macroscopic models

One obtains (maxwellian closure with variance θ)

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$
$$\partial_t \rho u + \nabla_x \cdot (\rho u \otimes u) + \theta \nabla_x \rho = -\gamma u \rho + \rho V(\delta_S \star \rho)\hat{e} - \rho \nabla_x U_R \star \rho$$

with the momentum

$$\rho u(x,t) := \int v f(x,v,t) dv.$$

$$\hat{e}(x) = \frac{\nabla \phi(x)}{|\nabla \phi(x)|}$$

where ϕ is the solution of the eikonal equation

$$|\nabla \phi(x)| = \frac{1}{V(\delta_S \star \rho(x))}.$$

Localized models

δ_S or U_R approximate for small values of S or R a δ-distribution and one obtains formally hydrodynamic equations of the form

$$\begin{aligned}\partial_t \rho + \nabla_x \cdot (\rho u) &= 0,\\ \partial_t \rho u + \nabla_x \cdot (\rho u \otimes u) &= -\gamma u \rho - \rho \nabla_x \rho + \rho V(\rho)\hat{e} - \theta \nabla_x \rho.\end{aligned}$$

Remark: Scalar equations

For the reduced microscopic problems we obtain Colombo-type equations

$$\partial_t \rho = \nabla_x \cdot (-V(\delta_s \star \rho)\hat{e}\rho) + \nabla_x \cdot (\rho \nabla U_R \star \rho).$$

The localized versions are Hughes-type models

$$\partial_t \rho = \nabla_x \cdot (-V(\rho)\hat{e}\rho) + \nabla_x \cdot (\rho \nabla_x \rho).$$

together with

$$|\nabla \phi| = \frac{1}{V(\rho)}.$$

Rigorous Work: Nonlocal equations to local equations

Numerical investigations:

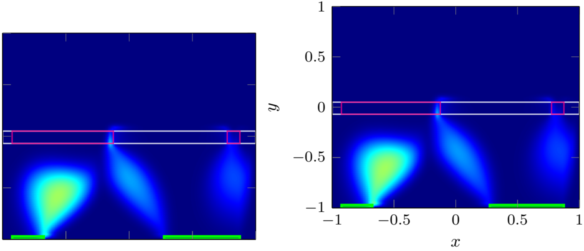
- P. Amorim, R. M. Colombo, and A. Teixeira. On the numerical integration of scalar nonlocal conservation laws. ESAIM Math. Model. Numer. Anal., 49(1):19–37, 2015.
- P. Goatin and S. Scialanga, Well-posedness and finite volume approximations of the LWR traffic flow model with non-local velocity, Netw. Heterog. Media, 11(1) (2016), 107-121.

Rigorous proof for situations with diffusion $A > 0$, several counterexample otherwise.

- Maria Colombo, Gianluca Crippa, Laura V. Spinolo, On the singular local limit for conservation laws with nonlocal fluxes, arxiv
- S. Blandin and P. Goatin. Well-posedness of a conservation law with non-local flux arising in traffic flow modeling. Numer. Math., 132(2):217–241, 2016.

Comparison of closures for hydrodynamic pedestrian flow

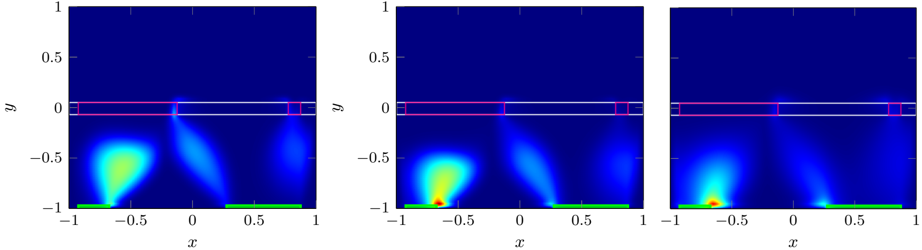
Compare microscopic/kinetic and nonlinear maximum entropy hydrodynamic



Cross-walks: densities at $t = 2.4$, $A = 5$

Comparison of closures for hydrodynamic pedestrian flow

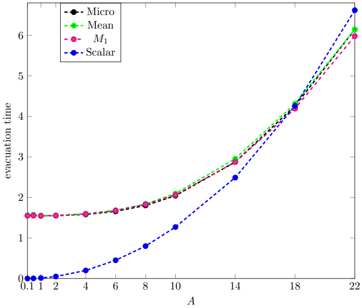
Compare microscopic and linear hydrodynamic and scalar



Cross-walks: densities at $t = 2.4$, $A = 5$.

Evacuation time

For small A the evacuation time determined from the scalar model differs strongly from the microscopic, the mesoscopic and the hydrodynamic one. For large values of A, all simulations give similar results.



The evacuation time in dependence of the parameter A for microscopic, mesoscopic, nonlinear hydrodynamic and scalar models.

Numerical methods for non-local hydrodynamic equations

- Arbitrary-Lagrangian-Eulerian particle or moving mesh methods

Procedure:

- Start from lagrangian formulation of hydrodynamic equations

$$
\begin{aligned}
\frac{dx}{dt} &= u \\
\frac{d\rho}{dt} &= -\rho \nabla_x \cdot u \\
\frac{du}{dt} &= -\gamma u - V(\delta_S \star \rho)\hat{e} - \nabla_x U_R \star \rho - \frac{\theta}{\rho} \nabla_x \rho.
\end{aligned}
$$

Remark: Lagrangian approach closer to particle idea.

ALE meshfree particle methods

- Use a smoothing radius ($\sim$ SPH) and determine derivatives by least square fit on a particle cloud of grid particles $\tilde{x}_i, i = 1, \cdots, \tilde{N} \leq N$!
- Add a discretization of the convolution integral

$$\nabla_x U_R \star \rho(x) = \int \nabla_x U_R(x-y)\rho(y)dy$$

- Solve resulting system of ODEs
- particle management (generate /delete grid particles)!
- Add additional procedures, for example
  - Upwinding / central procedures for hyperbolic problems
  - solution of the eikonal equation on the particle cloud $\sim$ fast marching method for unstructured grids

Macroscopic particle methods working in the localized limit

Problem: relatively small R (in the localized limit) and not very large number of macroscopic particles $\rightarrow$ underresolution
Naive/microscopic evaluation of the convolution integral leads to wrong results

$$\nabla_x U_R \star \rho(x_i) \sim \sum_{j=1, j\neq i}^{\tilde{N}} \rho_j |V_j| \nabla_x U_R(x_i - x_j) \sim 0$$

V_j: Voronoi cell around grid particle j with volume $|V_j|$
Very near to a microscopic simulation!

Macroscopic particle method and localized limit

Procedure:

- Use higher order approximation of the density in the particle method, for example

$$\rho(y) = \sum_{j=1}^{\tilde{N}} \left[\rho_j + \sigma_j \cdot (y - x_j)\right] \chi_{V_j}(y)$$

  σ_j: first order derivative approximated via least square fit from the point cloud
- plug into convolution integral
- Compute resulting integrals explicitely ($\sim$ multiscale finite elements)
- one obtains correction factors

$$\nabla_x U_R \star \rho(x_i) \sim \sum_{j \neq i} \Big(\rho_j |V_j| \nabla_x U_R(x_i - x_j) \Big) + \sigma_i \alpha_i$$

Results

- This leads to a uniform scheme for different R and to the correct localized limit for $R \to 0$ even for an underresolved situation.
- Computation times depends on the number of particles (for microscopic simulation) and grid points (for macroscopic),
- The particle method can be viewed as a numerical transition from a microscopic model if a very fine resolution is used to a macroscopic model if a coarse resolution and the above fix is used.

- A. Klar, S. Tiwari, A multi-scale particle method for mean field equations: the general case, SIAM Multiscale Mod. Sim.
- A. Klar, S. Tiwari, A multi-scale meshfree particle method for macroscopic mean field interacting particle models, SIAM Multiscale Mod. Sim. 2014

Numerical Results

Test-cases:

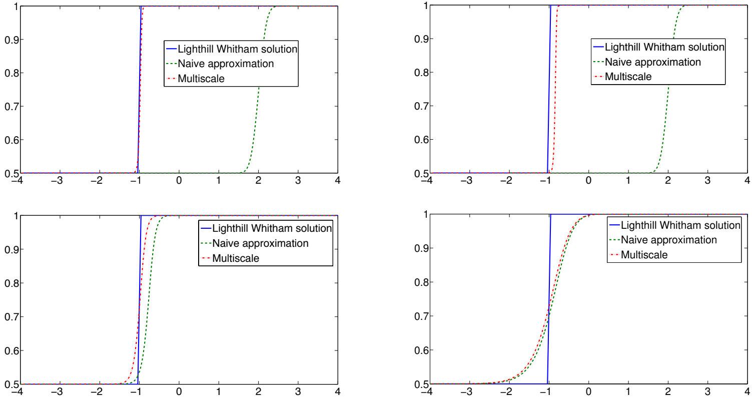
I: Conservation laws, shock solutions, Lighthill-Whitham

$$\partial_t \rho + \partial_x((1 - U_R \star \rho)\rho) = 0$$

II: 2D pedestrian dynamics

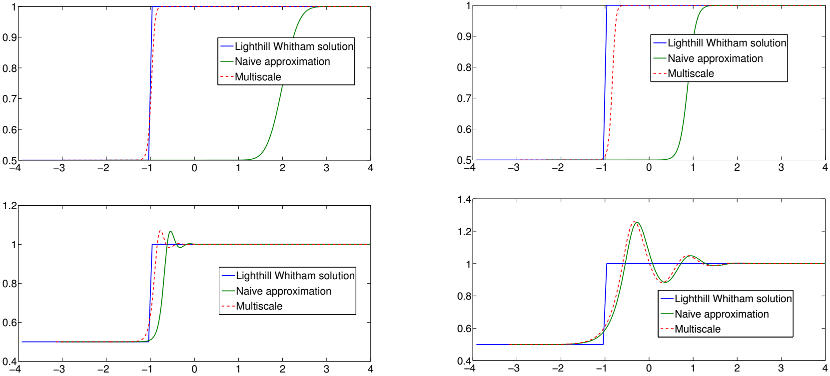
$$\begin{aligned} \partial_t \rho + \nabla_x(\rho u) &= 0 \\ \partial_t u + u \nabla_x u + \frac{\theta}{\rho} \nabla_x \rho &= -\gamma \left((1 - U_R \star \rho)\hat{e}(x) + \nabla_x U_R \star \rho - u \right) \end{aligned}$$

Test case I: LWR model



Shock solution for $N = 800$ particles and $R = 0.002$ to $R = 0.8$ for local and non-local model with microscopic and multi-scale approximation and downwind potential.

Test case I: LWR model

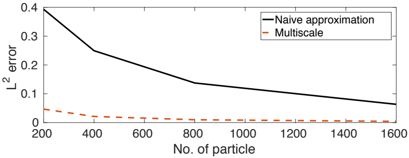


Shock solution for $N = 800$ particles and $R = 0.002$ to $R = 0.8$ for local and non-local version with symmetric potential and $\delta > 0$

Convergence error

# particles	naive error	multi-scale error	CPU time in seconds
200	0.40	0.05	8
400	0.26	0.02	17
800	0.15	0.01	36
1600	0.08	0.002	77

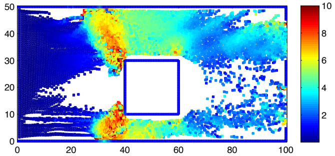
Convergence study for nonlocal Lighthill-Whitham equations with downwind interaction potential and $R = 0.2$.



$\mathcal{L}^2$-error plot

Test case II: Pedestrian dynamics

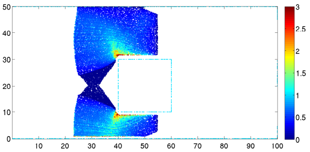
repulsive interaction potential, Lighthill Whitham type velocity function, coupling to eikonal equation



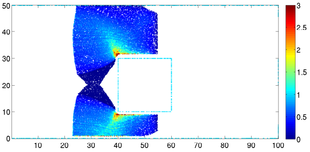
- N.K. Mahato, A. Klar, S. Tiwari, Particle methods for multi-group pedestrian flow, Appl. Math. Modeling 53, 447-461, 2018
- R. Etikyala, S. Göttlich, A. Klar, S. Tiwari, Particle methods for pedestrian flow models: from microscopic to non-local continuum models, Mathematical Methods and Models in Applied Sciences 24 (12), 2503-2523, 2014

Pedestrian dynamics: fine resolution $N \sim \tilde{N}$

repulsive potential, coupling to eikonal equation, meshfree solution of the eikonal equation



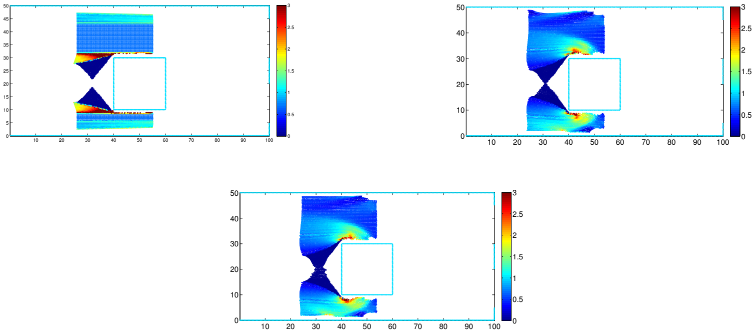
macroscopic



microscopic

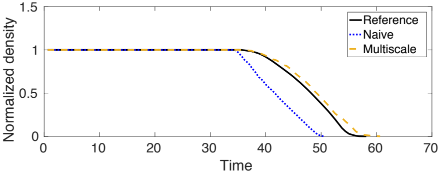
Density plot determined from local limit equation and nonlocal equations (microscopic or multi-scale approximation) for inital spacing $\Delta x = 0.2$ and $R = 0.4$ (fine resolved situation near local limit)

Pedestrian dynamics: coarse resolution $N >> \tilde{N}$



Density for nonlocal equations with microscopic and multi-scale approximation for $\Delta x = 0.5$, $R = 0.2$ and local limit.(coarsely resolved situation near local limit)

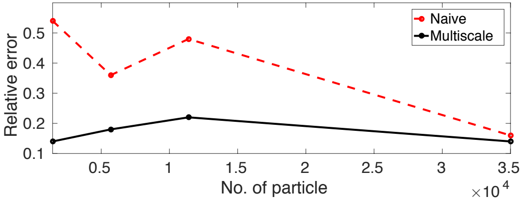
Evacuation times



Time development of the normalized total mass in the computational domain determined from the different models
$R = 0.2$ and coarse initial spacing $\Delta x = 1$ with $N = 1400$ grid particles.

Pedestrian dynamics: CPU time

initial spacing	# particles	naive error	multi-scale error	CPU time
1	1400	0.54	0.14	8 min
0.5	5700	0.36	0.18	23 min
0.35	11500	0.48	0.22	52 min
0.2	35200	0.16	0.14	223 min



Comparison of CPU times between microscopic and multiscale simulations

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Extensions: Pedestrian dynamics with groups

Extension using multiphase / mixture models

$$
\begin{aligned}
\frac{dx_i^{(k)}}{dt} &= v_i^{(k)} \\
\frac{dv_i^{(k)}}{dt} &= -\sum_{l=1}^{M} \sum_{j \in \text{group}^{(l)}} \nabla_{x_i} U^{(k,l)}(\mid x_i^{(k)} - x_j^{(l)} \mid) + G^{(k)}(x_i^{(k)}, v_i^{(k)}, \rho^N(x_i^{(k)})),
\end{aligned}
$$

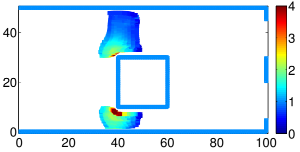
where $U^{(k,l)}$ is an interaction potential denoting the interaction between members of groups k and l. We choose the Morse potential

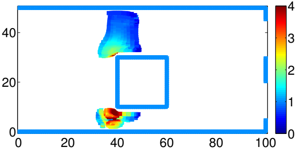
$$
U^{(k,l)}(r) = -C_a e^{-r/l_a} + C_r e^{-r/l_r}.
$$

C_a, C_r are attractive and repulsive strengths and l_a, l_r are length scales.

Pedestrian dynamics with groups

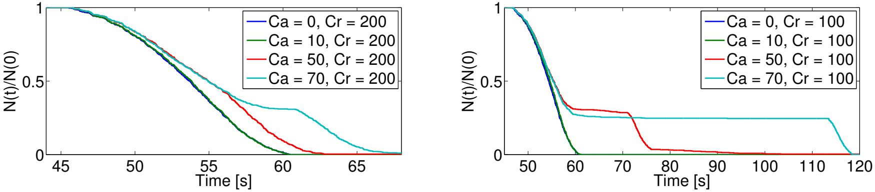
Repulsive potential and Morse potential for attraction in groups.





Density of pedestrians for single and multi-group hydrodynamic model

Multi-group evacuation times



Ratio of initial and actual grid particles over time in single $C_a = 0$ and multi-group $C_a = 10, 50, 70$ hydrodynamic model

The evacuation time is larger in the case of grouped pedestrians.

Compare experimental results in

- J. Xi, X. Zou, Z. Chen, J. Huang, Multi-pattern of Complex Social Pedestrian Groups Transportation Research Procedia Volume 2, 2014
- C. Kruchten, A. Schadschneider, Empirical study on social groups in pedestrian evacuation dynamics, Physica A, 2017

Conclusions

- Derivation of a hierachy of models for pedestrian flow.
- The particle method can be viewed as a uniform numerical transition from a microscopic model if a very fine resolution is used to a macroscopic model if a coarse resolution is used.
- Other interaction models can be included:
  attraction with center of mass of the group or vision based models

  M. Moussaid et al., Walking Behaviour of Pedestrian Social Groups and Its Impact on Crowd Dynamics, PLoS ONE

  P. Degond et al., Vision-based macroscopic pedestrian models, KRM
- A review of the mathematical aspects:

  R. Borsche, A. Klar, F. Schneider, Numerical methods for mean field and moment models for pedestrian flow, to appear in Crowd Dynamics Volume 1 - Theory, Models, and Safety Problems, N. Bellomo and L. Gibelli Eds., Birkhäuser-Springer, MSSET Series.