Optimal Integrated Routing and Signal Control in Simple Traffic Networks

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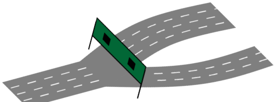
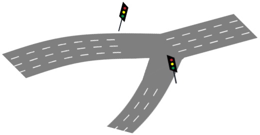
Route Guidance



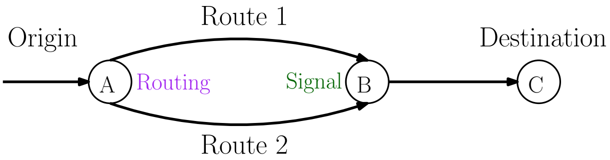
Signal Control



Main literature review

- Vuren and vliet, 1992: Investigating different traffic signal control policies with traffic routing interaction.
- Smith and Mounce, 2011: A splitting rate combined model using P_0 policy.
- Ioslovich et al., 2011: Optimal traffic control synthesis for an isolated intersection.
- Taale et al., 2015: Back-pressure control for the combined problem.

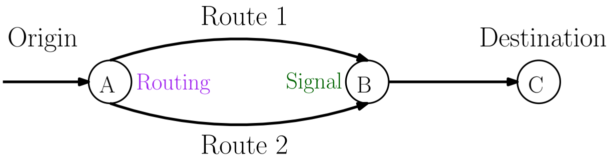
Research problem



Research goals

- Developing a continuous-time traffic model for the combined problem of routing and signal control.
- Formulating the optimal control problem and solving analytically

Research problem



Research Problem

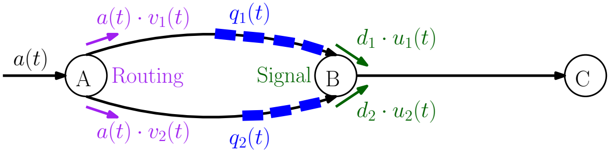
- Integrating routing and traffic signal control, while considering idealized splitting rate model.
- Optimal control synthesis for bringing initial queue lengths to a predefined steady-state or equilibrium queues, by manipulating traffic routing- and signal-control inputs.

Unlike previous works,

(i) the developed model considers dynamic transient periods

(ii) the model is solved analytically for system optimum via optimal control.

Model



Traffic terminology:

- *total inflow:* $a(t)$ $[\mathrm{veh/s}]$
- *queue lengths:* $q_1(t)$, $q_2(t)$ $[\mathrm{veh}]$
- *output saturation flows:* d_1, d_2 $[\mathrm{veh/s}]$
- *signal-control inputs:* $u_1(t)$, $u_2(t)$ $[-]$
- *routing-control inputs:* $v_1(t)$, $v_2(t)$ $[-]$
- *slack variables:* $w_1(t)$, $w_2(t)$ $[-]$

Dynamic equations

$$\frac{\mathrm{d}q_1(t)}{\mathrm{d}t} = a(t) \cdot v_1(t) - d_1 \cdot (u_1(t) - w_1(t))$$
$$\frac{\mathrm{d}q_2(t)}{\mathrm{d}t} = a(t) \cdot v_2(t) - d_2 \cdot (u_2(t) - w_2(t))$$

Model

Dynamic equations

$$\frac{\mathrm{d}q_1(t)}{\mathrm{d}t} = a(t)\cdot v_1(t) - d_1\cdot(u_1(t) - w_1(t))$$
$$\frac{\mathrm{d}q_2(t)}{\mathrm{d}t} = a(t)\cdot v_2(t) - d_2\cdot(u_2(t) - w_2(t))$$

State constraints

$$0 \le q_1(t),\quad 0 \le q_2(t)$$
$$q_1(t_\mathrm{f}) = 0\,,\;\; q_2(t_\mathrm{f}) = 0$$

Control input constraints

$$\underline{u} \le u_1(t) \le \overline{u}\,,\quad \underline{u} \le u_2(t) \le \overline{u}\,,\quad u_1(t) + u_2(t) = 1\,,\quad \underline{u} = 1 - \overline{u}$$
$$\underline{v} \le v_1(t) \le \overline{v}\,,\quad \underline{v} \le v_2(t) \le \overline{v}\,,\quad v_1(t) + v_2(t) = 1\,,\quad \underline{v} = 1 - \overline{v}$$
$$0 \le w_1(t) \le \overline{w}_1(t),\quad 0 \le w_2(t) \le \overline{w}_2(t)$$
$$\overline{w}_1(t) = \max\left\{0, u_1(t) - a(t)\cdot v_1(t))/d_1\right\},\;\; \overline{w}_2(t) = \max\left\{0, u_2(t) - (a(t)\cdot v_2(t))/d_2\right\}$$

Optimal control problem definition

Given:

- dynamic equations
- initial queue lengths: $q_1(0)$, $q_2(0)$
- constant total inflow: a
- output saturation flows: d_1, d_2
- control and state constraints

manipulate $u_1(t)$ (or $u_2(t)$) and $v_1(t)$ (or $v_2(t)$) to minimize the total delay

$$J = \int_0^{t_\mathrm{f}} \left[q_1(t) + q_2(t)\right] \mathrm{d}t$$

the final time t_f $[\mathrm{s}]$ is free (not fixed), when both queues are dissolved.

Pontryagins maximum principle

optimal control problem (OCP)

$$\int_0^T f_0(x,u)\mathrm{d}t \to \min \tag{1}$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x,u) \tag{2}$$

$$x(0) = x_0,\ x(T) = x_T \tag{3}$$

$$u_{\min} \le u(t) \le u_{\max} \tag{4}$$

where the *control variables* $u(t) \in \mathbf{R}^m$, the *state variables* $x(t) \in \mathbf{R}^n$, $f(x,u) \in \mathbf{R}^n$, with $m \le n$.

According to PMP:

$$H = p^T \cdot f(x,u) - f_0(x,u)\,. \tag{5}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial x}^T = -\frac{\partial f}{\partial x}^T \cdot p + \frac{\partial f_0}{\partial x}^T\,. \tag{6}$$

The column vector $p(t) \in \mathbf{R}^n$ is the vector of *costate variables*. If exists an optimal solution (x^*, u^*), then, there exists p^* such that the following conditions are satisfied:

(a) $H(x^*, u^*, p^*) \ge H(x^*, u, p^*)$, i.e. H attains maximum over the control u,

(b) the variables x^*, p^* satisfy (2) and (6),

(c) the variable u^* satisfy (4),

(d) the end conditions in (3) must be hold.

Conditions of optimality

According to the Pontryagin maximum principle (PMP), the Hamiltonian function, H(t), is formed as:

$$H(t) = p_1(t) \cdot \left[a \cdot v_1(t) - d_1 \cdot (u_1(t) - w_1(t))\right] + p_2(t) \cdot \left[a \cdot v_2(t) - d_2 \cdot (u_2(t) - w_2(t))\right] - q_1(t) - q_2(t)$$

The differential costate equation:

$$\frac{\mathrm{d}p_1(t)}{\mathrm{d}t} = -\frac{\partial H}{\partial q_1} = 1\,, \quad \frac{\mathrm{d}p_2(t)}{\mathrm{d}t} = -\frac{\partial H}{\partial q_2} = 1$$

Switching functions:

$$S_u(t) = -d_1 \cdot p_1(t) + d_2 \cdot p_2(t)\,, \quad S_v(t) = p_1(t) - p_2(t)$$

From $H(u_1, u_2, v_1, v_2, w_1, w_2) \to \max$ follows

$$\forall S_u(t) > 0:\ u_1(t) = \overline{u}\,,\ u_2(t) = \underline{u}\,; \qquad \forall S_u(t) < 0:\ u_1(t) = \underline{u}\,,\ u_2(t) = \overline{u}\,;$$
$$\forall S_v(t) > 0:\ v_1(t) = \overline{v}\,,\ v_2(t) = \underline{v}\,; \qquad \forall S_v(t) < 0:\ v_1(t) = \underline{v}\,,\ v_2(t) = \overline{v}\,.$$

Note that:

$$\forall p_1(t) < 0: \; w_1(t) = 0\,; \qquad \forall p_1(t) > 0: \; w_1(t) = \overline{w}_1(t)\,;$$
$$\forall p_2(t) < 0: \; w_2(t) = 0\,; \qquad \forall p_2(t) > 0: \; w_2(t) = \overline{w}_2(t)\,.$$

We shall look for the solutions that have the following properties:

$$p_1(t) = 0 \quad \text{only if} \quad q_1(t) = 0\,, \qquad p_2(t) = 0 \quad \text{only if} \quad q_2(t) = 0$$

We also assume that the initial costate variables are negative:

$$p_1(0) < 0\,, \qquad p_2(0) < 0$$

- We shall find such solutions that satisfy these assumptions.
- It may be shown that these solutions are optimal with sufficiency (Krotov 1973, Ioslovich and Gutman 2016).

Optimal solutions

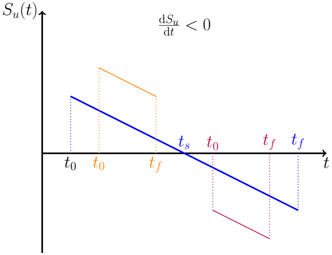
Without loss of generality, it is also assumed that:

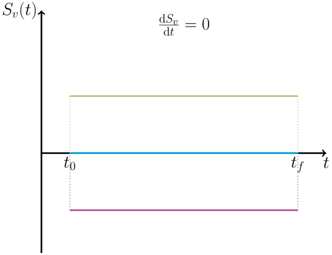
$$d_1 > d_2$$

Recall: $S_u(t) = -d_1 \cdot p_1(t) + d_2 \cdot p_2(t)$, $S_v(t) = p_1(t) - p_2(t)$, the derivatives of the switching functions:

$$\frac{\mathrm{d}S_u}{\mathrm{d}t} = -d_1 + d_2 < 0 \Rightarrow S_u(t) \quad \text{decreasing}$$

$$\frac{\mathrm{d}S_v}{\mathrm{d}t} = 0 \Rightarrow S_v(t) \quad \text{constant}$$





Optimal solutions

- Following the transversality condition for free final time t_f, and since the system does not depend on t, then $H(t) = 0\,, \quad \forall t \in [0, t_\mathrm{f}]$.

In the following, three different feasible cases of the optimal solutions depending on the initial values of non-positive costates $p_1(0)$, $p_2(0)$ are considered:

Case 1	**Case 2**	**Case 3**
$p_1(0) = p_2(0)$	$p_1(0) > p_2(0)$	$p_1(0) < p_2(0)$

Case 2: $p_1(0) > p_2(0)$

Recall: $\frac{\mathrm{d}p_1(t)}{\mathrm{d}t} = \frac{\mathrm{d}p_2(t)}{\mathrm{d}t}$ and $d_1 > d_2$.

$S_v(t) = p_1(t) - p_2(t)\,, \quad S_u(t) = -d_1 \cdot p_1(t) + d_2 \cdot p_2(t)$

if $p_1(0) > p_2(0)$ then $S_v(t) > 0\,, \forall t \in [0, t_\mathrm{f})$

Hence, the optimal routing control inputs are:

$$v_1(t) = \overline{v} \quad v_2(t) = \underline{v}\,, \quad \forall t \in [0, t_\mathrm{f}]$$

In case 2, $S_u(0)$ can be negative or positive:

Case 2a: $S_u(0) > 0$ and a switching point occurs.

Case 2b: $S_u(0) < 0$ and a switching point does not occur.

Case 2c: $S_u(0) > 0$ and a switching point does not occur.

Case 2a: $p_1(0) > p_2(0)$ and a switching point occurs

Recall: t_{s} - switching time, t_{f} - final time. and the instant time t_{q_1} when $q_1(t) = 0$.
The optimal signal control inputs by time are:

$$S_u(t) > 0 : \quad u_1(t) = \overline{u}\,,\; u_2(t) = \underline{u}\,,\; w_1(t) = 0\,,\; w_2(t) = 0 \quad \forall t \in [0, t_{\mathrm{s}}]$$

$$S_u(t) < 0 : \quad u_1(t) = \underline{u}\,,\; u_2(t) = \overline{u}\,,\; w_1(t) = 0\,,\; w_2(t) = 0 \quad \forall t \in [t_{\mathrm{s}}, t_{q_1}]$$

$$S_u(t) < 0 : \quad u_1(t) = \underline{u}\,,\; u_2(t) = \overline{u}\,,\; w_1(t) = \underline{u} - (a \cdot \overline{v}/d_1)\,,\; w_2(t) = 0 \quad \forall t \in [t_{q_1}, t_{\mathrm{f}}]$$

Queue feedback control policy

$$R_i = \frac{\mathrm{d}q_1}{\mathrm{d}q_2} = \frac{a \cdot v_1(t) - d_1 \cdot u_1(t)}{a \cdot v_2(t) - d_2 \cdot u_2(t)} \qquad R_2 = \frac{a \cdot \overline{v} - d_1 \cdot \overline{u}}{a \cdot \underline{v} - d_2 \cdot \underline{u}} \qquad R_3 = \frac{a \cdot \overline{v} - d_1 \cdot \underline{u}}{a \cdot \underline{v} - d_2 \cdot \overline{u}}$$

The feasible state region for (q_2, q_1)-plane is:

$$\frac{q_1(t)}{q_2(t)} \leq R_2$$

Case 2a: $p_1(0) > p_2(0)$ and a switching point occurs

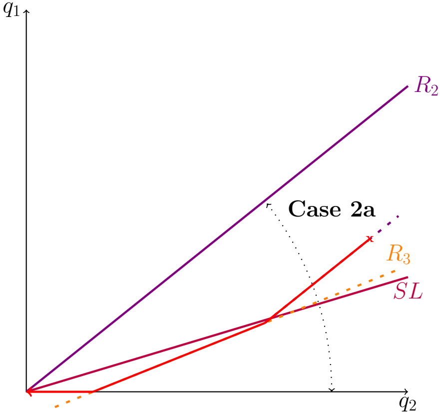
From $S_u(t_s) = 0$; $H(t_s) = 0$; $p_1(t_{q_1}) = 0$ one get $q_1(t_s), q_2(t_s)$ and the switching line in $(q_2(t), q_1(t))$ plane will be:

$$SL = \frac{q_1(t_s)}{q_2(t_s)} = \frac{d_2 \cdot [d_1 \cdot \underline{u} - a \cdot \overline{v}]}{d_1 \cdot [d_2 \cdot \overline{u} - a \cdot \underline{v}]} \geq 0$$

The feasibility conditions for case 2a are:

$$0 < d_1 \cdot \underline{u} - a \cdot \overline{v}$$
$$0 < d_2 \cdot \overline{u} - a \cdot \underline{v}$$

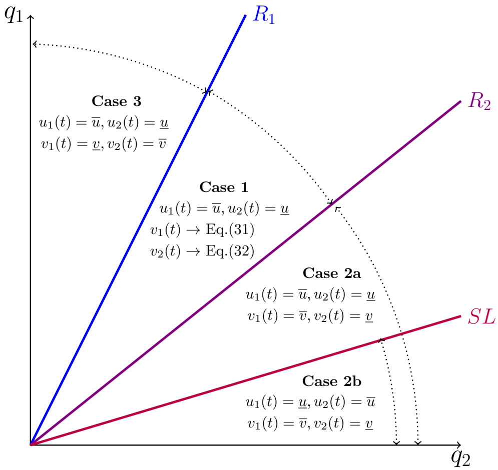
Optimal control solution - case 2a



Synthesis of the optimal control

$$\begin{bmatrix} u_1(t), u_2(t) \\ v_1(t), v_2(t) \\ w_1(t), w_2(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} \overline{u}, \underline{u} \\ \overline{v}, \underline{v} \\ 0, 0 \end{bmatrix} & \text{if} \quad SL \leq \frac{q_1(t)}{q_2(t)} \leq R_2 \\ \begin{bmatrix} \underline{u}, \overline{u} \\ \overline{v}, \underline{v} \\ 0, 0 \end{bmatrix} & \text{if} \quad 0 \leq \frac{q_1(t)}{q_2(t)} \leq SL \\ \begin{bmatrix} \underline{u}, \overline{u} \\ \overline{v}, \underline{v} \\ \overline{w}_1(t), 0 \end{bmatrix} & \begin{cases} \text{if} \quad 0 \leq \frac{q_1(t)}{q_2(t)} \leq SL, \\ \text{when } q_1(t) = 0 \end{cases} \end{cases}$$

Queue feedback control policy



Solution verification

- Verification by a numerical solver: the optimal control software DIDO (Ross 2015).
- Numerical example parameters:
  - The arrival and departure rates: $a = 0.4$ [veh/s] and $d_1 = 2$ [veh/s], $d_2 = 1.5$ [veh/s].
  - The control input bounds: $\underline{u} = 0.3$, $\overline{u} = 0.7$, $\underline{v} = 0.2$, $\overline{v} = 0.8$ [-].
  - Different initial values of the queue lengths for each case.

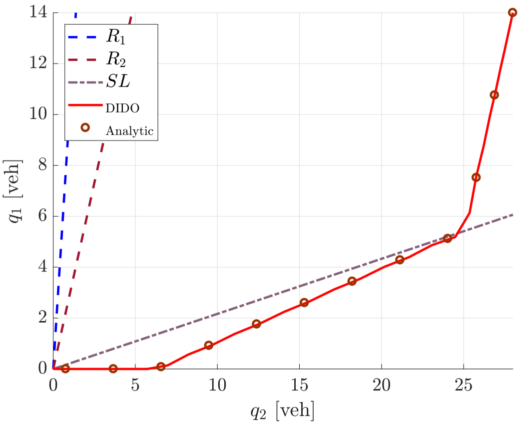
Solution verification - case 2a

Case 2a

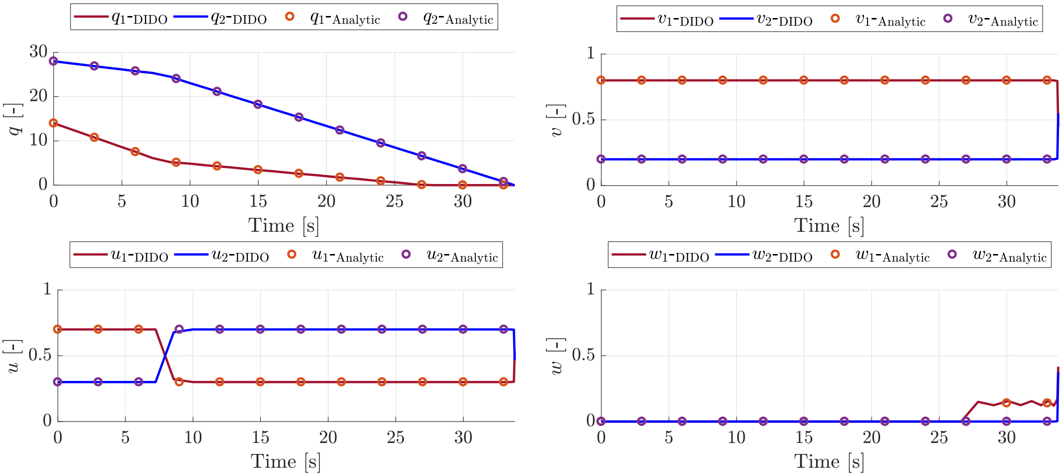
$$(q_1(0), q_2(0)) = (14, 28) \quad [\text{veh}]$$
$$t_f = 34.3 \quad [\text{s}]$$
$$t_s = 8.4 \quad [\text{s}]$$
$$J_{\text{Analytic}} = 707.1442 \quad [\text{veh} \cdot \text{s}]$$
$$J_{\text{DIDO}} = 664.0782 \quad [\text{veh} \cdot \text{s}]$$

DIDO

Analytic



Solution verification - case 2a



Solution verification

-	Case 1	Case 2		Case 3
Costate	$p_1(0) = p_2(0)$	$p_1(0) > p_2(0)$		$p_1(0) < p_2(0)$
Routing control inputs (v_1, v_2)	$\left(\frac{d_1 \cdot \overline{u} \cdot t_f - q_1(t_0)}{a \cdot t_f}, \frac{d_2 \cdot \overline{u} \cdot t_f - q_2(t_0)}{a \cdot t_f}\right)$	$(\overline{v}, \underline{v})$	$(\overline{v}, \underline{v})$	$(\underline{v}, \overline{v})$
Signal control inputs (u_1, u_2)	$(\overline{u}, \underline{u})$	$(\overline{u}, \underline{u}) \rightarrow (\underline{u}, \overline{u})$	$(\underline{u}, \overline{u})$	$(\overline{u}, \underline{u})$
Switching point (V-yes / X-no)	X	V	X	X
$(q_1(0), q_2(0))$	(50,10)	(14,28)	(4,20)	(35,3)
t_f [s]	41.4	34.3	20.7	26.6
t_s [s]	-	8.4	-	-
J-Analytic [veh·s]	1.3029×10^3	707.1442	259.3573	537.5808
J-DIDO [veh·s]	1.2414×10^3	664.0782	234.7583	498.6357

Table: Analytical and numerical results for the different cases.

Summary

- A continuous-time model is developed to integrate both traffic routing and signal control.
- Optimal control analytical solutions, based on PMP, were found under the objective function of minimizing total delay.
- The derived optimal analytical solutions were verified via numerical solutions.
- A feedback routing and signal control policy based on link queue lengths is presented.

Future work

- Extending the model to handle upper bound constraints on queue lengths.
- Testing the problem with different objective functions.
- Studying the problem with more complex structure of network or model.

The most important question is the one nobody ever asked

Thank you!

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