A nonlinear discrete-velocity relaxation model for traffic flow

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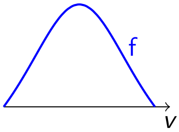
Content

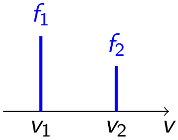
Kinetic traffic models

Let $f = f(x, v, t)$ denote the distribution of cars at time t and position x with velocity $v \in [0, 1]$.
A general evolution of f is governed by



$$\partial_t f + v \partial_x f = Q(f) \ ,$$

where $Q(f)$ is some (possibly nonlocal) interaction term.
A possible simplification is to consider discrete velocities, e.g. $v \in \{v_1, v_2\} \subset [0, 1]$. Thus the distributions f_1 and f_2 satisfy



$$\partial_t f_1 + v_1 \partial_x f_1 = Q_1(f_1, f_2)$$
$$\partial_t f_2 + v_2 \partial_x f_2 = Q_2(f_1, f_2)$$

with corresponding interaction terms Q_1 and Q_2.

Discrete velocity models

For given equilibrium functions $f_1^e(\rho), f_2^e(\rho)$ with

$$\rho = f_1^e(\rho) + f_2^e(\rho) \qquad \text{and} \qquad v_1 f_1^e(\rho) + v_2 f_2^e(\rho) = F(\rho)$$

a **discrete velocity relaxation model** is

$$\begin{aligned} \partial_t f_1 + v_1 \partial_x f_1 &= -\frac{1}{\epsilon}\left(f_1 - f_1^e(\rho)\right) \\ \partial_t f_2 + v_2 \partial_x f_2 &= -\frac{1}{\epsilon}\left(f_2 - f_2^e(\rho)\right) \ . \end{aligned}$$

Using the macroscopic variables $\rho = f_1 + f_2$ and $q = v_1 f_1 + v_2 f_2$ it reads

$$\begin{aligned} \partial_t \rho + \partial_x q &= 0 \\ \partial_t q - v_1 v_2 \partial_x \rho + (v_1 + v_2)\partial_x q &= -\frac{1}{\epsilon}\left(q - F(\rho)\right) \ . \end{aligned}$$

In the limit $\epsilon \to 0$ we want to obtain $\partial_t \rho + \partial_x F(\rho) = 0$.

Subcharacteristic condition

$$\partial_t f_i + v_i \partial_x f_i = -\frac{1}{\epsilon}\left(f_i - f_i^e(\rho)\right) \quad i = 1, 2$$

It is well known that for convergence to the conservation law $\partial_t \rho + \partial_x F(\rho) = 0$ the so called **subcharacteristic condition** is needed

$$v_1 \leq F'(\rho) \leq v_2 \ .$$

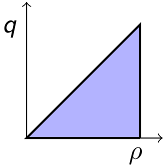
S. Jin, Z. Xin, The relaxation schemes for systems of conservation laws . . . , 1995

$v_1 \geq 0$ does not allow for situations where $F'(\rho)$ is negative (traffic jams).

Task

Develop a discrete-velocity model for traffic flow with correct invariant region and convergence to the scalar conservation law for all ranges of ρ.

The invariant region is the triangle $0 \leq \rho \leq 1$ and $0 \leq q \leq \rho$.



Deriving a new model: kinetic modeling

$$\partial_t f + v\partial_x f = J_L(f) + J_{NL}(f) \qquad v \in [0,1],\ f = f(x,v,t)$$

Local term due to acceleration and braking: Relaxation term J_L given by

$$J_L(f) = -\left(f - f_0(\rho)\right)$$

with equilibrium function $f_0(\rho)$ with $\int f_0(v)dv = \rho$ and $\int vf_0(v)dv = F(\rho)$.
Nonlocal term due to braking interactions: $J_{NL}(f)$ given by

$$J_{NL}(f) = J_{NL}(f,H) = J_B(f,H) - J_B(f,0)\ .$$

$J_B(f,H)$: braking term, H measure for the look-ahead

$$J_B(f,H) = \frac{-1}{1-\rho}\int_{\hat{v}<v}(v-\hat{v})f(x,v)f(x+H,\hat{v})d\hat{v} + \frac{1}{1-\rho}\int_{\hat{v}>v}(\hat{v}-v)f(x,\hat{v})f(x+H,v)d\hat{v}\ ,$$

a driver at x with velocity v reacts to a car at $x+H$ with velocity $\hat{v}$, if $\hat{v} < v$.
The new velocity resulting of his braking is the velocity of the leading car.

I. Prigogine and R. Herman, Kinetic Theory of Vehicular Traffic, 1971.

The kinetic equation is now scaled with $t \to \frac{t}{\epsilon}$ and $x \to \frac{x}{\epsilon}$ (zoom out)

$$\partial_t f + v\partial_x f = \frac{1}{\epsilon} J_L(f) + \frac{1}{\epsilon} J_{NL}(f, \epsilon H) \ .$$

Computing the Taylor expansion of J_{NL} with respect to ϵ gives to first order

$$\partial_t f + v\partial_x f - J^A_{NL}(f, \partial_x f) = \frac{1}{\epsilon} J_L(f)$$

with the following approximation of J_{NL}

$$\begin{aligned} J^A_{NL}(f, \partial_x f, v) = & \frac{H}{1-\rho} \int_{\hat{v}>v} (\hat{v} - v) f(\hat{v}) \partial_x f(v) d\hat{v} \\ & - \frac{H}{1-\rho} \int_{\hat{v}<v} (v - \hat{v}) f(v) \partial_x f(\hat{v}) d\hat{v} \ . \end{aligned}$$

For a two velocity model with the velocities $0 \leq v_1 < v_2 \leq 1$ we obtain

$$J^A_{NL}(v_1) = \frac{H}{1-\rho}(v_2 - v_1) f_2 \partial_x f_1 \ , \quad J^A_{NL}(v_2) = -\frac{H}{1-\rho}(v_2 - v_1) f_2 \partial_x f_1 \ .$$

Consider the choice $v_1 = 0, v_2 = 1$. Then the discrete velocity model is

$$\partial_t f_1 - \frac{H}{1-\rho} f_2 \partial_x f_1 = -\frac{1}{\epsilon}\left(f_1 - \rho + F(\rho)\right)$$
$$\partial_t f_2 + \partial_x f_2 + \frac{H}{1-\rho} f_2 \partial_x f_1 = -\frac{1}{\epsilon}\left(f_2 - F(\rho)\right) \ .$$

With $f_1 = \rho - q, f_2 = q$ the associated macroscopic equation is

$$\partial_t \rho + \partial_x q = 0$$
$$\partial_t q + \frac{Hq}{1-\rho}\partial_x \rho + (1 - \frac{Hq}{1-\rho})\partial_x q = -\frac{1}{\epsilon}\left(q - F(\rho)\right) \ .$$

The subcharacteristic condition is

$$-\frac{HF(\rho)}{1-\rho} \leq F'(\rho) \leq 1 \quad \text{for} \quad 0 \leq \rho \leq 1$$

since the eigenvalues of the hyperbolic part are $\lambda_1 = -\frac{Hq}{1-\rho}$, $\lambda_2 = 1$.
For the LWR model the subcharacteristic condition is satisfied if $H \geq 1$.

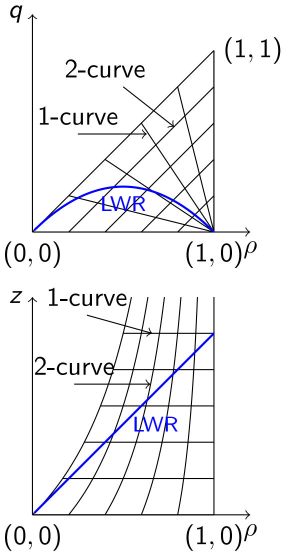
The system of conservation laws The system can be reformulated in conservative form using $z = \frac{Hq}{(1-\rho)^H}$

$$\partial_t \rho + \partial_x q = 0$$
$$\partial_t z + \partial_x z = -\frac{1}{\epsilon} \frac{H}{(1-\rho)^H} \left(q - F(\rho) \right) \ .$$

Hyperbolic classification:

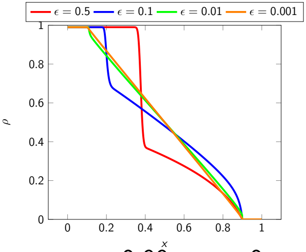
- Shock and integral curves coincide (Temple systems), 2-field is linearly degenerate.
- In the special case $H = 1$ the system is **totally linear degenerate**.
- The value of z remains bounded.

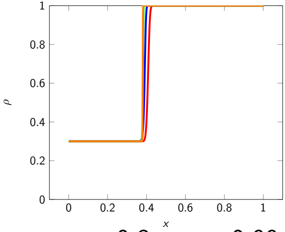
The region $0 \leq \rho \leq 1$, $0 \leq q \leq \rho$ is an invariant for the system for all $H > 0$.



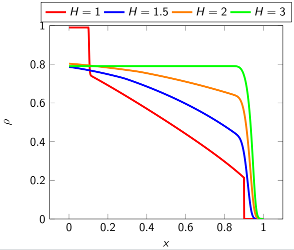
Numerical results

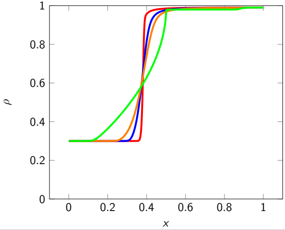
$H = 1$ $\quad \rho_L = 0.99,\ \rho_R = 0,\ q = 0,$ $\quad \rho_L = 0.3,\ \rho_R = 0.99,\ q = 0.$





$\epsilon = 0.1$ $\quad \rho_L = 0.99,\ \rho_R = 0,\ q = 0,$ $\quad \rho_L = 0.3,\ \rho_R = 0.99,\ q = 0.$





Coupling conditions

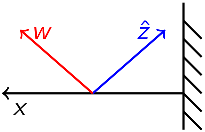
Boundary values

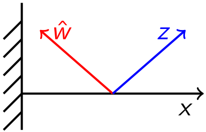
Advantage: The eigenvalues do not change sign

$$\lambda_1 = -\frac{q}{1-\rho} < 0 < \lambda_2 = 1 \ .$$

The Riemann Invariants of the system are

$$w = \rho - q \qquad z = \frac{q}{(1-\rho)}$$

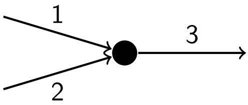




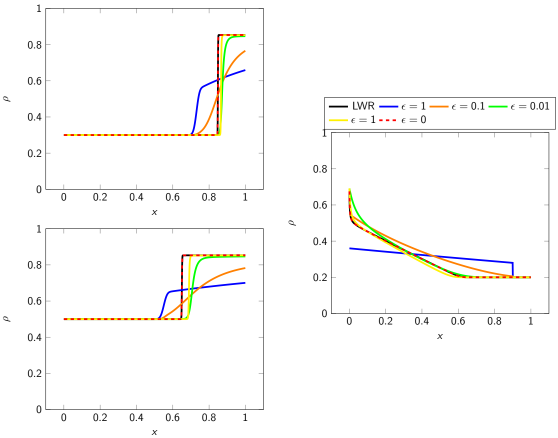
Notation: The $\hat{\cdot}$ values are the known states at the node (entering characteristics)

Coupling Conditions: Numerical results

Merge



Initial Conditions $\rho_1 = 0.3,\ \rho_2 = 0.5,\ \rho_1 = 0.2,$



Stopped cars $w = \rho - q$

Free space $1 - w$

Free space of 3 split in half

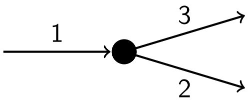
$$1 - w_1 = \frac{1}{2}(1 - \hat{w}_3)$$

$$1 - w_2 = \frac{1}{2}(1 - \hat{w}_3)$$

$$q_3 = q_1 + q_2$$

Coupling Conditions: Numerical results

Split: First In First Out



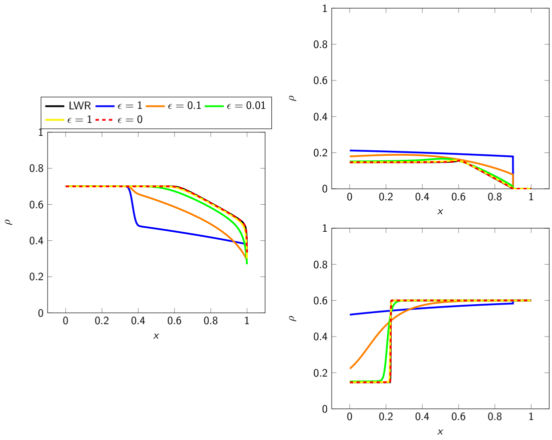
Distribution parameter

α_2 and $\alpha_3 = (1 - \alpha_2)$

$$q_1 = \min\left(\hat{q}_1^{max}, \frac{\hat{q}_2^{max}}{\alpha_2}, \frac{\hat{q}_3^{max}}{\alpha_3}\right)$$

$$q_2 = \alpha_2 q_1$$

$$q_3 = \alpha_3 q_1$$



Outlook

Features

- Boundary layers can be solved explicitly.
- The model can be extended to describe cluster dynamics.

Upcoming investigations

- Other choices of coupling conditions.
- Coupling conditions for general values of H.
- Analysis of coupling conditions in the limit $\epsilon \to 0$.

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Thank you for your attention.