Simulating stable, substable and weakly stable multidimensional distributions

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Introduction

In the beginning of probability theory, statistics or statistical physics, the basic model for describing events which are obtained by cumulation of very many very small independent variables was the Gaussian distribution and more generally – the Gaussian process. The Brownian random walk and Gauss theory of errors emerged in this way.

In the first half of the 20 century it turned out that the class of stable distributions, generalizing Gaussian distribution, introduced by P. Lévy and A. Ya. Khintchine can be used for more subtle description of some random events. Now the stable distributions and processes are successfully applied in financial mathematics, statistical physics, astronomy, electrical engineering and many other areas of science.

There are three main reasons why stable distributions and processes found so wide application. First, they appear in limit theorems as the only possible limit of the weighted sums of independent random variables. Thus they can be considered as a result of cumulation of very many very small independent variables. Second, they give the possibility to model variables without second moment or continuity of trajectories of the corresponding stochastic process (for example: stock market diagrams). Third, stable vectors and processes are closed under linear combinations which means that weighted sums of stable vectors are still stable, and this means also that linear combination of coordinates of a stable random vector has the same distribution as the first coordinate up to a multiplicative constant. This property seems to especially important in statistics.

On the other hand stable distributions are very difficult in calculations. The main reason is that, except for Gaussian, Cauchy and $\frac{1}{2}$-stable, any nice analytical formula for the density functions do not exist, even for one-dimensional symmetric stable distributions. Now, with a wide access to computers, this problem does not seem insurmountable. However using the inverse Fourier transform for calculating the corresponding density function still causes a lot of trouble. Thus there is a permanent need for new stochastic representations and new characterizations of stable vectors and processes.

Scientists are becoming increasingly interested in distributions and processes generated by stable distributions and processes. For example, tempered stable distributions (see e.g. [62]) are extensively used in statistical physics to model turbulence, or in mathematical finance to model stochastic volatility.

In this situation, stable, sub-stable and the most general class of weakly stable distributions and processes seem to be good candidates for use in stochastic modelling. They have nice linear properties, i.e. if $(\mathbf{X}_i)$ is a sequence of independent identically distributed random vectors with the weakly stable distribution μ then every linear combination $\sum a_i \mathbf{X}_i$ has the same distribution as $\mathbf{X}_1 \cdot \Theta$ for some random variable Θ independent of $\mathbf{X}_1$. This condition holds not only when (a_i) is a sequence of real numbers, but also when (a_i) is a sequence of random variables for $a_i, \mathbf{X}_i$, $i = 1, 2, \ldots$ mutually independent. This means that the dependence structure of the linear combination $\sum a_i \mathbf{X}_i$ and dependence structure of the random vector $\mathbf{X}_1$ are the same, and the sequence (a_i) is responsible only for the radial behavior. Moreover weak stability is preserved under taking linear operators $A(\mathbf{X}_1)$ and under taking projections or functionals $< \xi, \mathbf{X}_1 >$. On the other hand radial properties of distribution can be arbitrarily defined by choosing a proper random variable Θ independent of $\mathbf{X}_1$ and considering the distribution of $\Theta \cdot \mathbf{X}_1$.

An interested reader can find also a very rich bibliography on stable distributions and related topics on the home-page of John Nolan:
http://academic2.american.edu/~jpnolan/stable/StableBibliography.pdf

In the first chapter of this book we recall the definition and the basic properties of characteristic functions for random variables and random vectors. In the last section of this chapter we use our knowledge about norm-dependent characteristic functions to describe norm-dependent positive definite matrices, which are extensively used in stochastic modelling random vectors with fixed correlation structure.

The first section of the second chapter contains five equivalent definitions of stable random variables together with the detailed proofs of their equivalence. A part of these proofs can be found in the very well known Feller book [17], but some of them are almost forgotten. Next we give some basic properties of stable random variables and few representation involving much simpler random variables - uniformly or exponentially distributed.

Fourth chapter contains a description of stable random vectors and their spectral representation. Based on this spectral representation we define the covariation function and the covariation ratio - parameters, which in the Gaussian case, are equal to covariance and correlation. They describe the dependence structure of the corresponding symmetric α-stable random vector even though it does not have second moment. In the third section we discuss properties of symmetric α-stable random vectors, which are β-substable for some $0 < \alpha < \beta$. The last section contains one simple series representation for an $S\alpha S$ random vector.

The last chapter in this book contains a description of weakly stable random vectors. As this kind of vectors is much less known, we will give here detailed proofs of their basic properties as well as the basic properties of a generalized convolution defined by weak stability. In section 6 we give a description of a very well known example of weakly stable random vectors, which are the extreme points for the set of elliptically contoured or rotationally invariant vectors. Much less known are random vectors discovered by S. Cambanis, R. Keener and G. Simons (*see* [**?**], which are the extreme points of the set of ℓ_1-symmetric (or ℓ_1-pseudo-isotropic) random vectors in $\mathbb{R}^n$. Here a vector $\mathbf{X} \in \mathbb{R}^n$ is ℓ_1-symmetric if its characteristic function is of the form $\varphi(\|\xi\|_1)$, $\xi \in \mathbb{R}^n$. The section 7 contains the description of Cambanis, Keener and Simons distributions and their computer simulation.

Basically, we have not given in this book any explicitly written problems to be solved. Students attending the course were supposed to use the presented stochastic representations for stable and weakly stable random variables and vectors in computer simulation. As an exercise they were writing computer programs for calculating and drawing densities graphs, scatter plots and histograms, so all the figures contained in this book were done by Leszek Grzelak, Sebastian Kuniewski, Dorota Kurowicka and Daniel Lewandowski. The figures ilustrating applications of copulae in the spectral representation of symmetric α-stable vectors come from the paper [7] written together with Jacek Bojarski. The density plots for Cambanis, Keener and Simons distribution were made by Grazyna Mazurkiewicz (in [49]), who kindly allowed us to present them here.

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2

Characteristic functions

Throughout this book we will use the following notation:
$\mathcal{L}(\mathbf{X})$ is the distribution of the random vector $\mathbf{X}$,
$\mathbf{X} \stackrel{d}{=} \mathbf{Y}$ means that random vectors $\mathbf{X}$ and $\mathbf{Y}$ have the same distribution,
$\mathcal{P}^n$ is the set of all probability distributions on $\mathbb{R}^n$,
$\mathcal{P}$ the set of probability measures on $\mathbb{R}$,
$\mathcal{P}^+$, probability measures on $[0, +\infty)$.

For every $a \in I\!\!R$ and every probability measure μ we define the rescaling operator $T_a : \mathcal{P} \to \mathcal{P}$ by the formula:

$$T_a\mu(A) = \begin{cases} \mu(A/a) & \text{for} \quad a \neq 0; \\ \delta_0(A) & \text{for} \quad a = 0, \end{cases}$$

for every Borel set $A \in \mathbb{R}^n$. Equivalently

$$T_a\mathcal{L}(X) = \mathcal{L}(aX).$$

The scale mixture $\mu \circ \lambda$ of a measure $\mu \in \mathcal{P}^n$ with respect to the measure $\lambda \in \mathcal{P}$ is defined by the formula:

$$\mu \circ \lambda(A) \stackrel{def}{=} \int_{I\!\!R} T_s\mu\,(A)\,\lambda(ds).$$

It is easy to see that for X and Θ independent we have

$$\mathcal{L}(X) \circ \mathcal{L}(\Theta) = \mathcal{L}(X \cdot \Theta).$$

2.1 Characteristic functions for random variables

Definition 2.1.1. *Let X be a random variable with the distribution μ. Then the function φ on $\mathbb{R}$ with values in the complex numbers, defined as*

$$\varphi_X(t) = \mathbf{E}\exp\{itX\} = \int_{\mathbb{R}} e^{itx}\mu(dx),$$

is called the characteristic function of the random variable X, or of the distribution μ. Whenever it is not misleading we will simply write $\varphi(t)$ instead of $\varphi_X(t)$.

The main properties of characteristic functions are described in the following theorem.

Theorem 2.1.1. *The characteristic function $\varphi(t)$ of the random variable X has the following properties:*

1) $|\varphi(t)| \le \varphi(0) = 1$;
2) $\varphi_X(-t) = \overline{\varphi_X(t)} = \varphi_{-X}(t)$;
3) $\varphi_{aX+b}(t) = e^{itb}\varphi_X(at)$;
4) variables X, Y are independent if and only if $\varphi_{aX+bY}(t) = \varphi_X(at)\varphi_Y(bt)$ for every $a, b \in \mathbb{R}$.
5) φ is a positive definite function, i.e. for every $n \in \mathbb{N}$, every choice of $t_1, \dots, t_n \in \mathbb{R}$ and every choice of complex numbers $c_1, \dots, c_n$ we have

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \varphi(t_i - t_j) \ge 0;$$

6) φ is uniformly continuous on $\mathbb{R}$.

Proof. **Ad. 1)** It is evident that $\varphi(0) = \mathbf{E}e^0 = \mathbf{E}1 = 1$. Thus we have $|\varphi(t)| = |\mathbf{E}e^{itX}| \le \mathbf{E}|e^{itx}| = \mathbf{E}1 = \varphi(0)$.
Ad. 2) It is enough to apply the formula $e^{iu} = \cos u + i\sin u$:

$$\varphi(-t) = \mathbf{E}\cos(-tX) + i\mathbf{E}\sin(-tX) = \overline{\mathbf{E}\cos(tX) + i\mathbf{E}\sin(tX)} = \overline{\varphi(t)}.$$

It is easy to see also that $\varphi(-t) = \mathbf{E}e^{it(-X)} = \varphi_{-X}(t)$.
Ad. 3) $\varphi_{aX+b}(t) = \mathbf{E}e^{itaX+itb} = e^{itb}\mathbf{E}e^{itaX} = e^{itb}\varphi_X(at)$.
Ad 4) If the random variables X, Y are independent, then also the variables e^{itX} and e^{itY} are independent thus

$$\mathbf{E}\exp\{it(aX + bY)\} = \mathbf{E}e^{itaX} \cdot \mathbf{E}e^{itbY} = \varphi_X(at)\varphi_Y(bt).$$

Ad. 5) Let $n \in \mathbb{N}$, $t_1, \dots, t_n \in \mathbb{R}$ and let $c_1, \dots, c_n$ be complex numbers. Then we have

$$\sum_{k,j=1}^{n} c_k\overline{c_j}\varphi(t_k - t_j) = \sum_{k,j=1}^{n} c_k\overline{c_j}\mathbf{E}e^{i(t_k-t_j)X}$$

$$= \mathbf{E}\sum_{k,j=1}^{n}\left(c_k e^{it_kX}\right)\overline{\left(c_j e^{it_jX}\right)} = \mathbf{E}\Big|\sum_{k=1}^{n} c_k e^{it_kX}\Big|^2 \geq 0.$$

Ad. 6) Notice first that

$$|\varphi(t+h) - \varphi(t)| = |\mathbf{E}e^{itX}(e^{ihX} - 1)| \leq \mathbf{E}|e^{ihX} - 1|.$$

The function $g_h(x) = |e^{ihx} - 1|$ is bounded by 2 on the whole real line. Notice also that for each $m > 0$ the function g_h converges uniformly to zero on the interval $[-m, m]$ for $h \to 0$. For $\varepsilon > 0$ we choose m large enough to have $\mathbf{P}(|X(\omega)| > m) < \varepsilon/4$. Next we choose h_0 small enough such that for every $h \in (0, h_0)$ and every $x \in [-m, m]$ we have $g_h(x) < \varepsilon/2$. Now for $h \in (0, h_0)$ we obtain

$$\begin{aligned} |\varphi(t+h) - \varphi(t)| &\leq \int_{-m}^{m} |e^{ihx} - 1| dF(x) + \int_{|x|>m} |e^{ihX} - 1| dF(x) \\ &\leq \varepsilon/2\mathbf{P}(|X(\omega)| \leq m) + 2\mathbf{P}(|X(\omega)| > m) < \varepsilon. \end{aligned}$$

□

Remark 2.1.1. As an easy consequence of the proof of property 2) in Theorem 2.1.1 we obtain that the characteristic function $\varphi_X(t)$ of the random variable X is real if and only if X is symmetric, i.e. $\mathcal{L}(X) = \mathcal{L}(-X)$.

Examples of characteristic functions of discrete distributions.

$$\mathbf{P}\{X = x_i\} = p_i, \ \sum_i p_i = 1 \qquad \varphi(t) = \sum_i e^{itx_i}p_i.$$

name	distribution	characteristic function
one point	$\mathbf{P}\{X = a\} = 1$	e^{ita}
two points	$\mathbf{P}\{X = 1\} = \mathbf{P}\{X = -1\} = \frac{1}{2}$	$\cos t$
discrete uniform	$\mathbf{P}\{X = \frac{k}{n}\} = \frac{1}{n}, n \leq n-1$	$\frac{(e^{it} - 1)}{n(e^{it/n} - 1)}$
Bernoulli	$\mathbf{P}\{X = k\} = \binom{n}{k}p^k q^{n-k}$	$\left(pe^{it} + q\right)^n$
Poisson	$\mathbf{P}\{X = k\} = e^{-\lambda}\frac{\lambda^k}{k!}$	$\exp\{-\lambda(1 - e^{it})\}$
geometrical	$\mathbf{P}\{X = k\} = pq^{k-1}$	$\frac{pe^{it}}{1 - qe^{it}}$
negative binomial	$\mathbf{P}\{X = k\} = \binom{k}{r-1}p^r q^{k-r}$	$\left(\frac{pe^{it}}{1 - qe^{it}}\right)^r$

Examples of characteristic functions of continuous distributions.

$$X \text{ has density } f(x) \qquad \varphi(t) = \int_{\mathbb{R}} e^{itx} f(x)\,dx.$$

name	density function	characteristic function
uniform	$\frac{1}{b-a} I(a < x < b)$	$\frac{e^{itb} - e^{ita}}{it(b-a)}.$
	$\frac{1}{2} I(-1 < x < 1)$	$\frac{\sin t}{t}$
exponential	$\lambda e^{-\lambda x}$	$\frac{\lambda}{\lambda - it}$
Gamma	$\frac{\lambda^p}{\Gamma(p)} x^{p-1} e^{-\lambda x}$	$\left(\frac{\lambda}{\lambda - it}\right)^p$
Gaussian	$\frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\}$	$\exp\left\{itm - \frac{1}{2}\sigma^2 t^2\right\}$
Cauchy	$\frac{a}{\pi(a^2 + (x-m)^2)}$	$\exp\{itm - a\|t\|\}$

For symmetric random variables the characteristic functions take values in $\mathbb{R}$ so we can simply visualize such functions drawing their graphs in $\mathbb{R}^2$. Notice that the characteristic function for Gaussian distribution is equal to its density function multiplied by a normalizing constant.

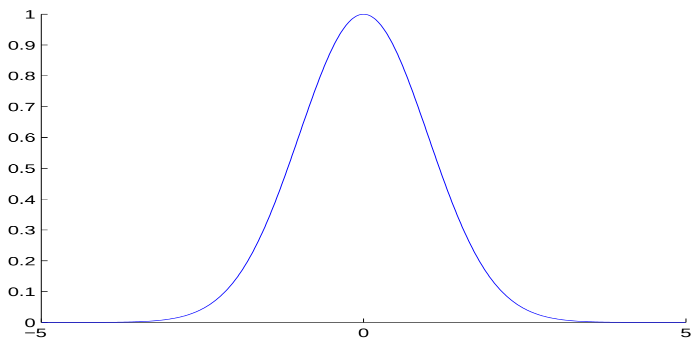

Figure 2.1: The characteristic function of Gaussian $N(0,1)$ distribution

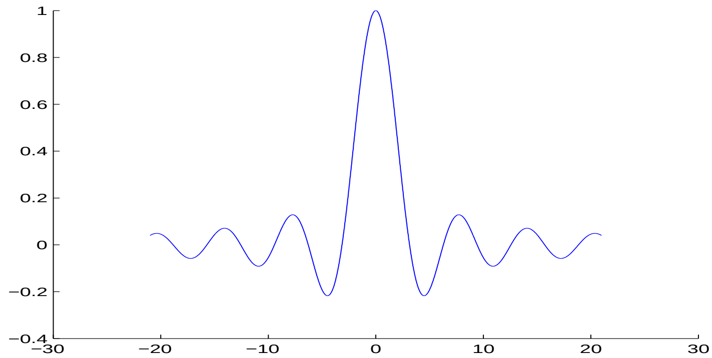

Figure 2.2: The characteristic function of uniform distribution on $-1, 1]$

If the random variable is not symmetric then its characteristic function takes values in the complex plane, thus it is better to make drawings in $\mathbb{R}^3$. Sometimes we can get very interesting graphical effects by drawing the projection of this 3-dimensional diagram into the complex plane.

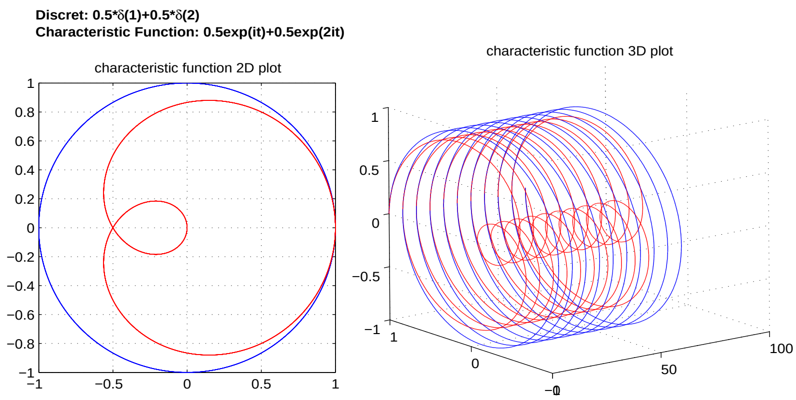

Figure 2.3: The characteristic function of discrete $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_2$ distribution

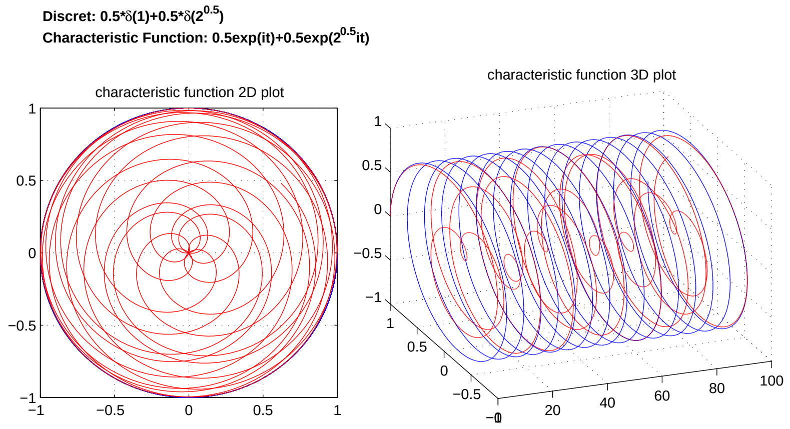

Figure 2.4: The characteristic function of discrete $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{\sqrt{2}}$ distribution

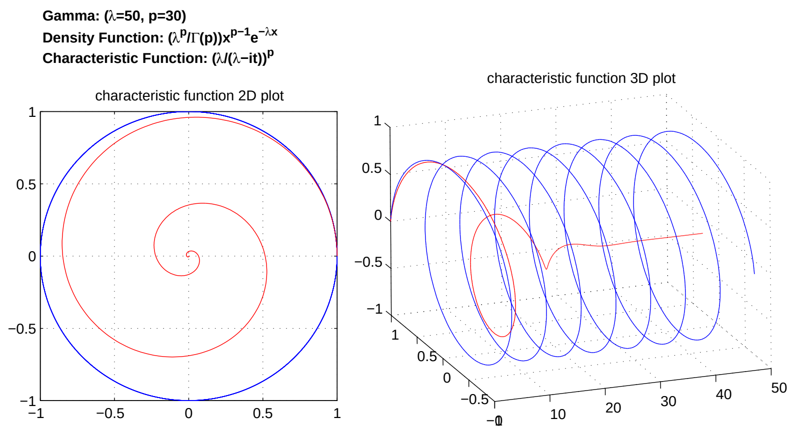

Figure 2.5: The characteristic function of Gamma $\Gamma(50, 30)$ distribution

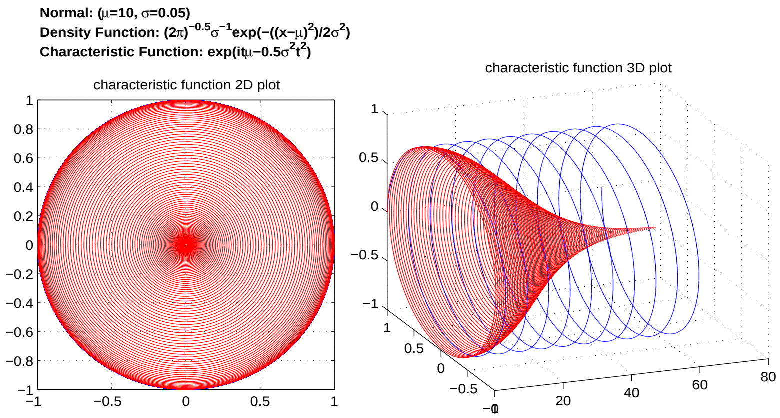

Figure 2.6: The characteristic function of Gaussian $N(10; 0.05)$ distribution

The next theorem describes the relation between the characteristic function and the distribution function of a random variable. It states also that under some regularity conditions we can recognize from the characteristic function if the distribution has a density function and even calculate this density. The formula in statement (c) of Theorem 2.1.2 is called the Fourier Inversion Formula.

Theorem 2.1.2. *Let φ be a characteristic function of the random variable X with distribution $\mathbf{P}_X(A) = \mathbf{P}\{X \in A\}$ and distribution function F. Then:*

a) for every $a < b$

$$G = \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\, dt = \mathbf{P}_X\left((a,b)\right) + \frac{1}{2}\mathbf{P}_X\left(\{a\}\right) + \frac{1}{2}\mathbf{P}_X\left(\{b\}\right);$$

b) if F is continuous at the points a and b, $a < b$, then

$$G = F(b) - F(a);$$

c) if $\int_{-\infty}^{\infty} |\varphi(t)|dt < \infty$, then F has a density function $f(\cdot)$, where

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t) dt.$$

Proof. Let us consider the following integral

$$G(R) = \frac{1}{2\pi} \int_{-R}^{R} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\, dt = \frac{1}{2\pi} \int_{-R}^{R} \frac{e^{-ita} - e^{-itb}}{it} \int_{-\infty}^{\infty} e^{itx} dF(x) dt.$$

The integrand in the last formula is bounded because

$$\left|\frac{e^{-ita}-e^{-itb}}{it}\,e^{itx}\right| = \left|\frac{e^{-ita}-e^{-itb}}{it}\right| = \left|\int_a^b e^{itx}\,dx\right| \le \int_a^b dx = b-a.$$

Using the Fubini Theorem we obtain

$$G(R) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-R}^{R}\frac{e^{it(x-a)}-e^{it(x-b)}}{it}\,dt\,dF(x).$$

Writing $e^{-itc} = \cos(tc) + i\sin(tc)$, we notice that the integral of the function $\cos(tc)/it$ over the symmetric set $[-R, R]$ vanishes, thus

$$\begin{aligned} G(R) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{2\int_0^R \frac{\sin t(x-a)}{t}\,dt - 2\int_0^R \frac{\sin t(x-b)}{t}\,dt\right\}dF(x) \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\int_0^{R(x-a)}\frac{\sin y}{y}\,dy - \int_0^{R(x-b)}\frac{\sin y}{y}\,dy\right\}dF(x) \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{R(x-b)}^{R(x-a)}\frac{\sin y}{y}\,dy\,dF(x). \end{aligned}$$

We will show now that the function $\int_0^z \frac{\sin y}{y}dy$ is bounded. Since the integrand is an odd function it is enough to consider $z > 0$. If $0 < z < \frac{\pi}{2}$, then

$$\left|\int_0^z \frac{\sin y}{y}dy\right| \le \int_0^z \left|\frac{\sin y}{y}\right| dy \le \int_0^z dy = z < \frac{\pi}{2}.$$

For $z > \frac{\pi}{2}$ we obtain integrating by parts,

$$\begin{aligned} \left|\int_0^z \frac{\sin y}{y}dy\right| &\le \left|\int_0^{\pi/2}\frac{\sin y}{y}dy\right| + \left|\frac{\cos y}{y}\right|_{\pi/2}^{z} + \int_{\pi/2}^{z}\frac{\cos y}{y^2}dy\right| \\ &\le \frac{\pi}{2} + \frac{1}{z} + \frac{2}{\pi} + \int_{\pi/2}^{\infty}\frac{1}{y^2}dy \le \frac{\pi}{2} + \frac{6}{\pi} = const. \end{aligned}$$

By Lebesgue's Theorem on bounded convergence we have:

$$\begin{aligned} &\lim_{R\to\infty} G(R) = \lim_{R\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{R(x-b)}^{R(x-a)}\frac{\sin y}{y}dydF(x) \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\lim_{R\to\infty}\int_{R(x-b)}^{R(x-a)}\frac{\sin y}{y}dy\right\}dF(x). \end{aligned}$$

Now we have to consider the following cases:
– if $a < x < b$, then

$$\lim_{R\to\infty}\frac{1}{\pi}\int_{R(x-b)}^{R(x-a)}\frac{\sin y}{y}dy = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin y}{y}dy = 1;$$

– if $x > b$, then $\lim_{R\to\infty} R(x-b) = \infty$. Since the function $\frac{\sin y}{y}$ is integrable then

$$\lim_{R\to\infty} \frac{1}{\pi} \int_{R(x-b)}^{R(x-a)} \frac{\sin y}{y} dy = 0;$$

– if $x < a$, similarly as before $\lim_{R\to\infty} R(x-a) = -\infty$ and

$$\lim_{R\to\infty} \frac{1}{\pi} \int_{R(x-b)}^{R(x-a)} \frac{\sin y}{y} dy = 0;$$

– for $x = a$ we have:

$$\lim_{R\to\infty} \frac{1}{\pi} \int_{-R(b-a)}^{0} \frac{\sin y}{y} dy = \frac{1}{\pi} \int_{-\infty}^{0} \frac{\sin y}{y} dy = \frac{1}{2};$$

– if $x = b$, then

$$\lim_{R\to\infty} \frac{1}{\pi} \int_{0}^{R(b-a)} \frac{\sin y}{y} dy = \frac{1}{\pi} \int_{0}^{\infty} \frac{\sin y}{y} dy = \frac{1}{2}.$$

We see that

$$\begin{aligned} \lim_{R\to\infty} G(R) &= \int_{-\infty}^{\infty} \left\{ \mathbf{1}_{(a,b)}(x) + \frac{1}{2}\mathbf{1}_{\{a\}}(x) + \frac{1}{2}\mathbf{1}_{\{b\}}(x) \right\} dF(x) \\ &= \mathbf{P}((a,b)) + \frac{1}{2}\mathbf{P}(\{a\}) + \frac{1}{2}\mathbf{P}(\{b\}). \end{aligned}$$

Property (b) is a simple consequence of (a), because at the continuity points of the distribution function F we have $\mathbf{P}(\{a\}) = \mathbf{P}(\{b\}) = 0$. In order to prove (c) assume that $\int_{-\infty}^{\infty} |\varphi(t)| dt < \infty$. Then the following function

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t) dt$$

is well defined, continuous and integrable on every interval $[a,b]$. Consider the integral

$$\begin{aligned} \int_a^b f(x)dx &= \int_a^b \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t) dt\, dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(t) \int_a^b e^{-itx} dx\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(t) \frac{e^{-ita} - e^{-itb}}{it} dt \\ &= \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \varphi(t) \frac{e^{-ita} - e^{-itb}}{it} dt = \mathbf{P}((a,b)) + \frac{1}{2}\mathbf{P}(\{a\}) + \frac{1}{2}\mathbf{P}(\{b\}). \end{aligned}$$

Since f is continuous, its integral is also a continuous function of the variables a and b, so

$$\forall\, a < b \qquad \int_a^b f(x)dx = F(b) - F(a),$$

which, by definition means that f is the density function corresponding to the distribution function F.

□

As a simple consequence of Theorem 2.1.2 we have the following.

Theorem 2.1.3. *The characteristic function uniquely determines the distribution of the corresponding random variable.*

There exists a strong relation between the existence of k-th derivative of the characteristic function at zero and the existence of k-th moment of the corresponding random variable. However we shall be careful in using the following theorem since it gives the implication only in one direction, not equivalence.

Theorem 2.1.4. *If there exists $n \geq 1$ such that $\mathbf{E}|X|^n < \infty$, then, for every $r \leq n$, the derivatives $\varphi^{(r)}(t)$ exist and*

$$\varphi^{(r)}(t) = \int_{-\infty}^{\infty} (ix)^r e^{itx} dF(x);$$

$$\mathbf{E}X^r = \frac{\varphi^{(r)}(0)}{i^r}.$$

Proof. If $\mathbf{E}|X|^n < \infty$, then of course $\mathbf{E}|X|^r < \infty$ for every $0 \leq r \leq n$. Consider

$$\frac{\varphi(t+h) - \varphi(t)}{h} = \mathbf{E}\left[e^{itX}\frac{e^{ihX} - 1}{h}\right].$$

Since $|e^{ihX} - 1| \leq |hX|$ and $\mathbf{E}|X| < \infty$, then there exists the limit of this difference quotient for $h \to 0$ and

$$\varphi'(t) = \lim_{h\to 0} \mathbf{E}\left[e^{itX}\frac{e^{ihX} - 1}{h}\right] = \mathbf{E}\left[e^{itX}\lim_{h\to 0}\frac{e^{ihX} - 1}{h}\right] = \mathbf{E}\left[iXe^{itX}\right].$$

By mathematical induction we obtain

$$\varphi^{(r)}(t) = \int_{-\infty}^{\infty} (ix)^r e^{itx} dF(x).$$

Now it is enough to apply this formula for $t = 0$.

□

Remark 2.1.2. The opposite implication in Theorem 2.1.4 does not hold. It can happen that the characteristic function has k-th derivative at zero but the corresponding random variable does not have finite k-th moment. It can be shown however that if the derivative $\varphi^{(2k)}(0)$ exists, then $\mathbf{E}X^{2k} < \infty$. We will prove here only the following

Theorem 2.1.5. *Let φ be the characteristic function of the random variable X. If φ is twice differentiable at zero then $\mathbf{E}X^2 < \infty$.*

Proof. Since $\varphi(0) = 1$, then the function φ cannot be convex around zero. It is not possible then that $\varphi''(0) > 0$. Applying de l'Hospital's formula we obtain:

$$\begin{aligned}
\varphi''(0) &= \lim_{h\to 0} \frac{1}{2}\left[\frac{\varphi'(2h)-\varphi'(0)}{2h} + \frac{\varphi'(0)-\varphi(-2h)}{2h}\right] \\
&= \lim_{h\to 0} \frac{\varphi'(2h)-\varphi'(-2h)}{4h} = \lim_{h\to 0} \frac{\varphi(2h)-2\varphi(0)+\varphi(-2h)}{4h^2} \\
&= \lim_{h\to 0} \int_{-\infty}^{\infty} \left(\frac{e^{ihx}-e^{-ihx}}{2h}\right)^2 dF(x) = -\lim_{h\to 0} \int_{-\infty}^{\infty} \left(\frac{\sin(hx)}{hx}\right)^2 x^2 dF(x).
\end{aligned}$$

By Fatou's Lemma we finally have

$$\begin{aligned}
-\varphi''(0) &= \lim_{h\to 0} \int_{-\infty}^{\infty} \left(\frac{\sin hx}{hx}\right)^2 x^2 dF(x) \\
&\geq \int_{-\infty}^{\infty} \lim_{h\to 0} \left(\frac{\sin hx}{hx}\right)^2 x^2\, dF(x) = \int_{-\infty}^{\infty} x^2 dF(x),
\end{aligned}$$

which was to be shown.

□

The next theorem, known in the literature under the name of Pólya's Lemma, will be needed in the next chapter.

Theorem 2.1.6 (Pólya's Lemma). *Let $\varphi(t)$ be a real-valued and continuous function on $\mathbb{R}$ which satisfies the following conditions:*

(i) $\varphi(0) = 1$;

(ii) $\varphi(-t) = \varphi(t)$;

(iii) $\varphi(t)$ *is convex for* $t > 0$;

(iv) $\lim_{t\to\infty} \varphi(t) = 0$.

Then $\varphi(t)$ is the characteristic function of an absolutely continuous distribution $F(x)$.

Proof. Since $\varphi(t)$ is a convex and symmetric function it has everywhere a right-hand derivative which we denote by $\varphi'(t)$. The function $\varphi'(t)$ is non-decreasing for $t > 0$. Moreover $\varphi'(t) \leq 0$ for $t > 0$ and $\lim_{t\to\infty} \varphi'(t) = 0$.
It is easily seen that the function $p(x)$ given by

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \varphi(t) dt = \frac{1}{\pi} \int_{0}^{\infty} \cos(tx)\, \varphi(t) dt$$

is well defined for all $x \neq 0$. It can be easily checked that

$$\varphi(t) = \int_{-\infty}^{\infty} e^{-itx} p(x) dx,$$

so in particular $\int_{-\infty}^{\infty} p(x)dx = 1$ since $\varphi(0) = 1$. This implies that the proof is completed as soon as we show that $p(x) \geq 0$.
Integrating by parts in the formula defining $p(x)$ and writing $g(t) = -\varphi'(t)$ we get

$$p(x) = \frac{1}{\pi|x|} \int_0^\infty g(t) \sin(t|x|)dt,$$

where $g(t)$ is a non-increasing, non-negative function with $\lim_{t\to\infty} = 0$. Then

$$p(x) = \frac{1}{\pi} \sum_{j=0}^{\infty} \int_{\pi j}^{\pi(j+1)} g(t) \sin(t|x|)dt = \frac{1}{\pi|x|} \int_0^{\pi/|x|} \left[\sum_{j=0}^{\infty} (-1)^j g\left(t + \frac{\pi j}{|x|}\right) \right] \sin(t|x|)dt.$$

The series

$$\sum_{j=0}^{\infty} (-1)^j g\left(t + \frac{\pi j}{|x|}\right)$$

is an alternating series whose terms are non-increasing in absolute value; since the first term of the series is non-negative one sees that the integrand is non-negative. Thus $p(x) \geq 0$.

□

2.2 Characteristic functions for random vectors

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Definition 2.2.1. *Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a random vector with distribution μ on $\mathbb{R}^d$. The characteristic function of $\mathbf{X}$ is the function $\varphi_{\mathbf{X}}$ on $\mathbb{R}^d$ taking values in the complex plane defined by*

$$\varphi_{\mathbf{X}}(\xi) = \mathbf{E} \exp\{i < \xi, \mathbf{X} >\} = \mathbf{E} \exp\left\{ i \sum_{k=1}^{d} \xi_k X_k \right\}.$$

Notice that the characteristic function of the random vector $\mathbf{X}$ at the point $t\xi$, $t \in \mathbb{R}$, $\xi \in \mathbb{R}^d$, is equal to the characteristic function of the random variable $< \mathbf{X}, \xi >= \sum_{k=1}^{d} \xi_k X_k$ at the point t, i.e.

$$\varphi_{\mathbf{X}}(\xi t) = \varphi_{<\mathbf{X},\xi>}(t).$$

This means that the main properties of multidimensional characteristic functions can be derived from the properties of one dimensional characteristic functions. We give here only one example: if $\mathbf{X}$ has a d-dimensional Gaussian distribution with the mean vector $\mathbf{m} = (m_1, \ldots, m_d)$ and the covariance matrix $\Sigma = (\sigma_{i,j})_{ij=1}^{d}$ then the $< X, \xi >$ is a Gaussian random variable with mean

$$m = \mathbf{E} < \mathbf{X}, \xi >= \mathbf{E} \sum_{k=1}^{d} X_k \xi_k = \sum_{k=1}^{d} m_k \xi_k =< \mathbf{m}, \xi >$$

and variance

$$\mathrm{Var}\big(< \mathbf{X}, \xi >\big) = \mathrm{Cov}\left(\sum_{k=1}^{d} X_k \xi_k, \sum_{j=1}^{d} X_j \xi_j\right) = \sum_{k,j=1}^{d} \xi_k \sigma_{ij} \xi_j = \xi \Sigma \xi^T.$$

Now it is enough to use the formula for the characteristic function of the Gaussian random variable $N(m, \sigma)$ (see Table 2.1.2) to obtain that

$$\varphi_{\mathbf{X}}(\xi) = \exp\left\{ i < \mathbf{m}, \xi > -\frac{1}{2}\xi\Sigma\xi^T \right\}.$$

2.3 Infinitely divisible distributions and one method for their computer simulation

In modelling real events we know that some of variables shall be considered as sums of very many very small independent factors. It turns out that the characteristic functions of such variables have very special forms, which we describe here. Let us start from the following definition.

Definition 2.3.1. *The random variable X (its distribution F or its characteristic function φ) is infinitely divisible if for every $n \in \mathbb{N}$, X has the same distribution as the sum of n independent identically distributed random variables $X_1^n, \ldots, X_n^n$ (with common distribution F_n and characteristic function φ_n). In other words for each $n \in \mathbb{N}$*

$$F = F_n^{*n}, \qquad \varphi(t) = \big(\varphi_n(t)\big)^n,$$

where F_n^{*n} denotes the n-th convolution power of distribution F_n.

In particular it means that φ is the characteristic function of an infinitely divisible random variable if and only if $(\varphi(t))^{1/n}$ is a characteristic function for each $n \in \mathbb{N}$. Now it is easy to see that

- if X has Gaussian $N(m, \sigma)$ distribution, then $\varphi(t) = \exp\{itm - \sigma^2 t^2/2\}$, thus $(\varphi(t))^{1/n} = \exp\{it/nm - \sigma^2 t^2/2n\}$ which is the characteristic function of the Gaussian $N(m/n, \sigma/\sqrt{n})$. Consequently Gaussian distribution is infinitely divisible.

- If X has $\Gamma(p, \lambda)$ distribution then

$$\varphi(t) = \left(\frac{\lambda}{\lambda - it}\right)^p.$$

  Consequently $(\varphi(t))^{1/n}$ is the characteristic function of $\Gamma(p/n, \lambda)$ distribution, which means that the Gamma distributed random variable is infinitelly divisible.

Of course the definition of an infinitely divisible random variable can be easily extended to infinitely divisible random vector, so we present here the famous Lévi-Khintchine formula in the verion for infinitely divisible random vectors. More details one can find in chapter XVII in Feller [17].

Theorem 2.3.1. *A function $\varphi: \mathbb{R}^n \to \mathbb{C}$ is the characteristic function of an infinitely divisible random vector $\mathbf{X}$ if and only if there exist $\mu \in \mathbb{R}^n$, a non-negative definite $n \times n$-matrix $\Re$ and a measure m on $\mathbb{R}^n$ with $\int (1 \wedge \|\mathbf{x}\|^2) m(dx) < \infty$ such that*

$$\begin{aligned} \ln \varphi(\xi) &= i < \mu, \xi > -\frac{1}{2} < \xi, \Re \xi > \\ &+ \int_{\mathbb{R}^n} \left(1 - e^{i<\xi, \mathbf{x}>} - i < \xi, \mathbf{x} > \mathbf{1}_{\{\|\mathbf{x}\|<1\}}\right) m(d\mathbf{x}). \end{aligned}$$

The measure m is called the Lévy measure of the infinitely divisible vector $\mathbf{X}$.

Notice that for $m = 0$ we obtain here the Gaussian characteristic function, but since $\Re$ is not strictly positive definite, the corresponding Gaussian random vector can be degenerated to a subspace of $\mathbb{R}^n$. Notice also that in Theorem 2.3.1 the Lévy measure m on $\mathbb{R}^n$ does not have to be finite, however m restricted to the complement of any neighborhood of zero is finite. If it happens that $m(\mathbb{R}^n) < \infty$ then the function $< \xi, \mathbf{x} >$ is integrable on the set $\{\|\mathbf{x}\| < 1\}$ thus the Lévy-Khintchine formula can be written in the form

$$\ln \varphi(\xi) = i < \mu_1, \xi > -\frac{1}{2} < \xi, \Re \xi > + \int_{\mathbb{R}^n} \left(1 - e^{i<\xi, \mathbf{x}>}\right) m(d\mathbf{x}).$$

Computer simulation for infinitely divisible random vector with finite Lévy measure

Assume that the random vector $\mathbf{X}$ without Gaussian component has finite Lévy measure m. This means that

$$\mu = 0, \qquad \Re = 0, \qquad m\big(\mathbb{R}^n\big) = \lambda < \infty.$$

We define a probability distribution on $\mathbb{R}^n$ by

$$\mathbf{P}_m(A) = \frac{1}{\lambda} m(A), \qquad \text{for } A \in \mathcal{B}(\mathbb{R}^n).$$

Now let the random variable $N = N_{\mathbf{X}}$ have the Poisson distribution with parameter λ, and let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be independent, identically distributed with distribution $\mathbf{P}_m$. If N is independent of $\mathbf{X}_1, \mathbf{X}_2, \ldots$ and $X_0 \equiv 0$, then

$$\mathbf{X} \stackrel{d}{=} \sum_{k=0}^{N} \mathbf{X}_k.$$

To see this, it is enough to calculate the characteristic function of $\sum_{k=0}^{N} \mathbf{X}_k$:

$$\begin{aligned}
\mathbf{E}\exp\left\{i < \xi, \sum_{k=0}^{N} \mathbf{X}_k >\right\} &= \mathbf{E}\exp\left\{i \sum_{k=0}^{N} < \xi, \mathbf{X}_k >\right\} \\
&= \sum_{n=0}^{\infty} \mathbf{E}\Big(\exp\left\{i < \xi, \sum_{k=0}^{n} \mathbf{X}_k >\right\} \Big| N = n\Big)\mathbf{P}(N = n) \\
&= \sum_{n=0}^{\infty} \Big(\mathbf{E}\exp\{i < \xi, \mathbf{X}_1 >\}\Big)^n \frac{\lambda^n}{n!} e^{-\lambda} \\
&= \exp\{-\lambda + \lambda \mathbf{E}\exp\{i < \xi, \mathbf{X}_1 >\}\} \\
&= \exp\left\{-\lambda \int_{\mathbb{R}^n} \big(1 - \exp\{i < \xi, \mathbf{X}_1 >\}\big)\mathbf{P}_m(d\mathbf{x})\right\} \\
&= \exp\left\{-\int_{\mathbb{R}^n} \big(1 - \exp\{i < \xi, \mathbf{X}_1 >\}\big)m(d\mathbf{x})\right\}.
\end{aligned}$$

Now we are ready for computer simulation of the random vector $\mathbf{X}$. The procedure is the following

1. Sample the value n of the random variable N by sampling $(n+1)$ independent variables $T_1, \ldots, T_{n+1}$ with the exponential distribution $\Gamma(1, 1)$. The number n shall be chosen in such a way that

$$T_1 + \ldots + T_n \leq \lambda, \quad T_1 + \ldots + T_{n+1} > \lambda.$$

   It is easy to calculate that

$$\mathbf{P}\{T_1 + \ldots + T_n \leq \lambda, \ T_1 + \ldots + T_{n+1} > \lambda\} = \frac{\lambda^n}{n!} e^{-\lambda}.$$

2. Sample n independent, identically distributed random vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n$ with distribution $\mathbf{P}_m$.

3. Calculate the following

$$\widetilde{\mathbf{X}} = \sum_{k=0}^{n} \mathbf{X}_k.$$

Remark. In all these calculations we assumed that $\sum_{k=0}^{0} X_k \equiv 0$. This means in particular that if the infinitely divisible random vector $\mathbf{X}$ has finite Lévy measure m then it has distribution containing at least one atom:

$$\mathbf{P}\{\mathbf{X} = 0\} \geq \mathbf{P}\{N_{\mathbf{X}} = 0\} = \exp\big\{-m\big(\mathbb{R}^n\big)\big\} = e^{-\lambda}.$$

On the other hand, if we denote the distribution of $\mathbf{X}$ by ν then we see that the probability measure

$$\nu_1 = \frac{1}{1 - e^{-\lambda}} \left(\nu - e^{-\lambda}\delta_0\right)$$

can be absolutely continuous with respect to the Lebesgue measure. This means that, after removing atom at zero, the infinitely divisible vector $\mathbf{X}$ with finite Lévy measure may have a density function.

2.4 Positive definite norm dependent matrices

Definition 2.4.1. *An $n \times n$ matrix $\Sigma = (\sigma_{ij})_{ij=1}^{n}$, $\sigma_{ij} \in \mathbb{R}$, is positive definite if for every $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$ we have $\mathbf{c}\Sigma\mathbf{c}^T \geq 0$.*

The following theorem describes the connection between positive definite real matrices and positive definite real functions. The definition of positive definite function was given in the statement 5) of Theorem 2.1.1. We shall notice here that if the function is real then it is enough to consider real constants c_j, $j = 1, \ldots, n$.

Theorem 2.4.1. *Let f be a function on $\mathbb{R}^d$ taking values in $\mathbb{R}$. The function f is positive definite if and only if for every $n \in \mathbb{N}$ and every choice of $x_1, \ldots, x_n \in \mathbb{R}^d$ the matrix*

$$\Sigma = \Big(f(x_i - x_j)\Big)_{i,j=1}^{n} = \begin{bmatrix} f(0) & f(x_1 - x_2) & \ldots & f(x_1 - x_n) \\ f(x_2 - x_1) & f(0) & \ldots & f(x_2 - x_n) \\ \ldots & \ldots & \ldots & \ldots \\ f(x_n - x_1) & f(x_n - x_2) & \ldots & f(0) \end{bmatrix}$$

is positive definite.

Proof. It is enough to notice that positive definiteness of the matrix Σ means that for every vector $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ we have $c\Sigma c^T \geq 0$. But

$$c\Sigma c^T = \sum_{ij=1}^{n} c_i c_j f(x_i - x_j),$$

which is equivalent with positive definiteness of the real function f. □

The Theorem 2.3.1 can be of great help for the construction of positive definite matrices (for example a covariance matrix) or if you want to check whether a given matrix of the form defined in the Theorem 2.3.1 is positive definite. Consider for example the following function defined on $\mathbb{R}^d$:

$$f_{p,q}(\mathbf{x}) = \exp\left\{ -\Big(\sum_{k=1}^{d} |x_k|^p\Big)^{q/p} \right\} = \exp\Big\{-\|\mathbf{x}\|_p^q\Big\},$$

Theorem 2.4.2. *The function $f_{p,q}(\mathbf{x})$ is positive definite on $\mathbb{R}^2$ if and only if the following condition holds*

$$0 < q \leq p \leq 2 \qquad \text{or} \qquad p \in (2, \infty] \text{ and } q \leq 1.$$

Proof. If $0 < q \leq p \leq 2$, we will see in the next chapter that the function $f_{p,q}(\mathbf{x})$ is the characteristic function of q-stable, p-substable random vector, and

therefore positive definite. This holds for every dimension d, not only for $d = 2$. This fact was known already in 1937 to P. Lévy [43]. He proved also that the parameter q has to belong to the interval $(0, 2]$ and this fact also does not depend on the dimension of the space.
We continue with the case $p > 2$. In Theorem 2.1 of [15] Dor proved that the function $f_{p,q}$ is not positive definite if $1 < q < 2 < p$. In the case when $p > 1$ and $q = 1$ positive definiteness of the function $f_{p,q}(\mathbf{x})$ on $\mathbb{R}^2$ follows from the well known theorem stating that every two-dimensional Banach space (e.g. $(\mathbb{R}^2, \|\cdot\|_p)$) embeds isometrically into some L_1-space. This theorem has been proven by several authors in different ways in different areas of mathematics; see e.g.: Ferguson 1962 [18], Herz 1963 [25], Lindenstrauss 1964 [44], Assouad 1979-1980 [3–5] or Misiewicz and Ryll-Nardzewski 1989 [53]. In this last paper one can find the explicit formula for the corresponding measure ν on $[0, 2\pi)$: Notice first that two dimensional Banach space is nothing but the plane $\mathbb{R}^2$ equipped with a norm $c\colon \mathbb{R}^2 \to [0, \infty)$. If this norm $c(x, y)$ is a smooth function on $\mathbb{R}^2$ then

$$c(x, y) = \frac{1}{4} \int_0^{2\pi} \left| x \cos\theta + y \sin\theta \right| \left(q''(\theta - \frac{\pi}{2}) + q(\theta - \frac{\pi}{2}) \right) \, d\theta,$$

where $q(\theta) = c(\cos\theta, \sin\theta)$. Now it is enough to know that if $f_{p,q}$ is positive definite then for every $0 < s < q$ the function $f_{p,s}$ is also positive definite, which will be clear for us in the next chapter.

□

Theorem 2.4.3. *The function $f_{p,q}(\mathbf{x})$ is positive definite on $\mathbb{R}^d$, $d \geq 3$, if and only if $0 < q \leq p \leq 2$.*
Moreover if for some $p \in (2, \infty]$ a function $f(\|\mathbf{x}\|_p)$ is a positive definite function on $\mathbb{R}^d$, $d \geq 3$, then $f \equiv 1$.

Proof. History of the proof The proof of necessity of the condition $0 < q \leq p \leq 2$ has a long history going back to the first investigations of symmetric stable random vectors [43], and the first Schoenberg problem [66]. In 1963 Herz [25] proved that if $1 < q < 2$ and $f_{p,q}$ is positive definite on $\mathbb{R}^d$ then $q < p < q(1-q)^{-1}$. In 1973 Witsenhausen [69] proved that if $p > 2.7$, $d \geq 3$ then $f_{p,1}$ is not a positive definite function. In 1976 Dor [15] (see also [11]) proved that if $p, q \in [1, \infty)$ and $f_{p,q}$ is positive definite on $\mathbb{R}^d$ then $1 \leq q \leq p \leq 2$. In 1991 Koldobsky [36] proved that if $p > 2$ and $d \geq 3$ then $f_{p,q}(\mathbf{x})$ is not positive definite for any $q \leq 2$. Note that the result of Koldobsky solves finally, after 53 years, the first Schoenberg question. And in 1995, Grząślewicz and Misiewicz [21] noticed that the previous considerations do not include all cases when $p < 1$ or $q < 1$. They proved in full generality that if $0 < p < q \leq 2$ then the function $f_{p,q}(\mathbf{x})$ is not positive definite.
In 1989 Misiewicz (see [54]) proved that if the function $f(\|\mathbf{x}\|_\infty)$ is positive definite on $\mathbb{R}^d$, $d \geq 3$ then $f \equiv 1$. It is known also that if the function $f(\|\mathbf{x}\|_p)$ is positive definite on $\mathbb{R}^d$, $d \geq 3$, $p > 2$, then $f \equiv 1$. This result was proven

independently by two authors: Lisitsky in 1991 [45] and Zastawny in 1991 [73]. □

Exercises.

1. Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables and $S_n = \sum_{k=1}^{n} X_k$. Show that
   a) if X_k's are Gaussian then S_n is Gaussian;
   b) if X_k's have Gamma distribution then S_n also has Gamma distribution;
   c) if X_k's have Bernoulli distribution, then S_n has Bernoulli distribution.

2. Show that none of the following functions can be a characteristic function of some probability distribution: a) $\varphi(t) = \cos t^2$, b) $\varphi(t) = \exp\{-t^{15}\}$.

3. Using the Fourier inversion formula find numerically the density function for a probability distribution with the following characteristic function:
   a) $\varphi(t) = \exp\{-|t|^{3/2}\}$,
   b) $\varphi(t) = \exp\{-|t|^{\sqrt{2}}\}$.
   Explain why a density exists in these cases. In both cases find the distribution of $Y = 2X + 3X'$, where X, X' are independent with the same distribution and the characteristic function $\varphi(t)$.

3

Stable random variables

In 1850 Augustino Cauchy noticed that the function f_α satisfying the equation

$$\int_{-\infty}^{\infty} e^{itx} f_\alpha(x)dx = \exp\{-|t|^\alpha\}, \qquad \alpha > 0$$

has a very special convolution property, namely:

$$\big(af_\alpha(a\cdot)\big) * \big(bf_\alpha(b\cdot)\big) = cf_\alpha(c\cdot),$$

for every $a, b > 0$ and properly chosen $c = c(a, b)$. He showed also that $f_\alpha \geq 0$ for $\alpha = 1$ and $\alpha = 2$
In 1923 George Pólya proved his famous result (see Theorem 2.3.4). From Pólya's result we obtain that $f_\alpha \geq 0$ for $\alpha \in (0, 1]$ as a density function of the corresponding probability distribution.
In 1924 Paul Lévy proved that $f_\alpha \geq 0$ for every $\alpha \in (0, 2]$. He introduced the name *stable distribution* for distribution with density f_α (with possible rescaling). Actually his definition was as follows: the random variable X is stable if $aX + bX' \stackrel{d}{=} cX + d$, for any $a, b > 0$, X' is an independent copy of X, $\stackrel{d}{=}$ denotes equality of distributions, and $c = c(a, b)$, $d = d(a, b)$ are properly chosen constants.
The easiest way to prove that for $\alpha > 2$ the function f_α cannot be a density function for any probability distribution is using a result of Durret obtained in 1991:
if $\varphi(t)$ is a characteristic function of a symmetric random variable X and

$$\lim_{t \to 0^+} \frac{\varphi(t) - 1}{t^2} = -\frac{\sigma^2}{2} > -\infty,$$

then $\mathbf{E}X = 0$ and $\mathbf{E}X^2 = \sigma^2$.
If we assume that f_α is a density of the random variable X for some $\alpha > 2$ then we would have

$$\lim_{t \to 0^+} \frac{exp\{-|t|^\alpha\} - 1}{t^2} = 0,$$

thus, by the Durret's result, $\mathbf{E}X = 0$, $\mathrm{Var}X = 0$, so $X = 0$ with probability 1 and $\mathbf{E}e^{itX} \equiv 1 \neq \exp\{-|t|^\alpha\}$ which is a contradiction.
Now the stable random variables and vectors play a crucial role in probability theory. Their investigations started in 1920's and in 1930's by Paul Lévy and Aleksandr Yakovlevich Khinchine. The literature is very rich on this topic; see, e.g. P. Lévy [43], Gnedenko, Kolmogorov [19], Zolotariev [74], Ibragimov, Linnik [27]. Not long ago appeared two books completely devoted to stable stochastic processes: one of Samorodnitsky and Taqqu [63], the second of Janicki and Weron [28]. Also Ledoux and Talagrand published a book "Probability in Banach Spaces" (see [41]), where stable random variables, vectors and processes are shown to play an important role in structure theorems of Banach spaces.

3.1 Equivalent definitions of stable random variables

In this section we will present five equivalent definitions of a stable random variables. We will also give the detailed proof of their equivalence. The last presented definition, definition E, is called the representation of the characteristic function for stable random variable.

Definition 3.1.1 (A). *A random variable X is stable (or it has a stable distribution) if*

$$\forall\, a,b>0\ \exists\, c=c(a,b)\geq 0\ \exists\, d\in\mathbb{R} \qquad aX_1+bX_2 \stackrel{d}{=} cX+d, \tag{I}$$

where X_1, X_2 are independent copies of X, and $\stackrel{d}{=}$ denotes equality in distribution.

If for the random variable X the condition (I) holds with $d = 0$ then X is called *strictly stable*. If the distribution of X is symmetric then the condition (I) holds for every $a, b \in \mathbb{R}$ (also the negative constants are possible) and X is called *symmetric stable.*

Remark 3.1.1. Notice that if φ is the characteristic function of a stable random variable then $|\varphi|^2$ is the characteristic function of symmetric random variable $X - X'$ which is stable with the same function $c(a, b)$. Indeed, for φ the condition (I) can be written as

$$\forall\, a,b>0\ \exists\, c=c(a,b)\geq 0\ \exists\, d\in\mathbb{R}\ \forall\, t\in\mathbb{R} \qquad \varphi(at)\varphi(bt)=\varphi\left(c(a,b)t\right)e^{idt}.$$

This, together with the property $\overline{\varphi(t)} = \varphi(-t)$, implies that

$$\begin{aligned} |\varphi(at)|^2|\varphi(bt)|^2 &= \varphi(at)\varphi(bt)\varphi(-at)\varphi(-bt) \\ &= \varphi\left(c(a,b)t\right)e^{idt}\varphi\left(-c(a,b)t\right)e^{-idt} = \left|\varphi\left(c(a,b)t\right)\right|^2. \end{aligned}$$

The function c, which by definition is defined only on the set $[0,\infty)^2$ can then be extended to the whole $\mathbb{R}^2$ by putting $c(a, b) = c(|a|, |b|)$. We will use this simple remark in order to simplify proofs which concern only the properties of the function c. It is easy to see that

a) $c(a,b) = c(b,a)$ for all $a, b \in \mathbb{R}$, and

b) $c(at, bt) = |t|c(a,b)$ for all $a, b, t \in \mathbb{R}$.

Notice also that the random variable concentrated at zero is symmetric stable. From now on we will assume that this trivial case does not hold, i.e. we will assume that

c) $c(a,b) = 0$ if and only if $a = b = 0$.

Lemma 3.1.1. *Let X be a random variable with characteristic function $\varphi \not\equiv 1$. If there exist positive constants c_2 and $d_2 \in \mathbb{R}$ such that*

$$X + X' \stackrel{d}{=} c_2 X + d_2,$$

where X' is an independent copy of X, then $\varphi(t) \neq 0$ for every $t \in \mathbb{R}$.

Proof. Without loss of generality we can assume that X is symmetric, which implies also that $d_2 = 0$. From the assumptions we have that

$$\varphi(t)^2 = \varphi(c_2 t) \qquad \forall\, t \in \mathbb{R}.$$

Since $\varphi \not\equiv 1$ then $c_2 \neq 1$. Assume that there exists $t_0 > 0$ such that $\varphi(t_0) = 0$, and let $\tau_0 = \inf\{t > 0 : \varphi(t) = 0\}$. Of course $\tau_0 > 0$ because $\varphi(0) = 1$. If $c_2 < 1$ then we would have

$$0 = \varphi(\tau_0)^2 = \varphi(c_2\tau_0),$$

which is impossible since $0 < c_2\tau_0 < \tau_0$. If $c_2 > 1$ then

$$0 = \varphi(t_0) = \varphi(c_2^{-1}t_0)^2,$$

which is also impossible since $0 < c_2^{-1}\tau_0) < \tau_0$.

□

Definition 3.1.2 (B). *A random variable X is stable iff*

$$\forall\, n \in \mathbb{N}\ \exists\, c_n > 0\ \exists\, d_n \in \mathbb{R} \qquad X_1 + \ldots + X_n \stackrel{d}{=} c_n X + d_n, \qquad \text{(II)}$$

where $X_1, \ldots, X_n$ are independent copies of X.

Lemma 3.1.2. *If random variable X satisfies the equation* (II)*, then there exists $\alpha > 0$ such that $c_n = n^{1/\alpha}$.*

Proof. Without loss of generality we can assume that $d_n = 0$. Notice that

$$\begin{aligned} c_{nm} X_1 &\stackrel{d}{=} (X_1 + \ldots + X_n) + \ldots + (X_{(n-1)m+1} + \ldots + X_{nm}) \\ &\stackrel{d}{=} c_n (X_1 + \ldots + X_m) \stackrel{d}{=} c_n c_m X_1. \end{aligned}$$

Thus we obtain that $c_{nm} = c_n c_m$, and consequently $c_{n^k} = c_n^k$.
Now we show that the sequence (c_n) is monotonically increasing. Since the random variables $X_i, i \in \mathbb{N}$ are symmetric then for every $x > 0$ we have

$$\begin{aligned} \mathbf{P}\{c_{n+m} X_1 > c_m x\} &= \mathbf{P}\{c_n X_1 + c_m X_2 > c_m x\} \\ &\geq \mathbf{P}\{c_n X_1 \geq 0\} \mathbf{P}\{c_m X_2 > c_m x\} = \frac{1}{2} \mathbf{P}\{X_2 > x\}. \end{aligned}$$

For a fixed $x > 0$ the right hand side of this inequality is a positive constant thus the fraction $\frac{c_m}{c_{n+m}}$ has to be bounded for $n \to \infty$. Let $m = k^s$ and $n+m = (k+1)^s$ for some fixed $k \in \mathbb{N}$ and $s \to \infty$. Since

$$\frac{c_m}{c_{n+m}} = \left(\frac{c_k}{c_{k+1}} \right)^s$$

is bounded when $s \to \infty$ we conclude that $c_k \leq c_{k+1}$.
Now let $j, k \in \mathbb{N}$. For every $n \in \mathbb{N}$ there exists exactly one $\lambda \in \mathbb{N}$ such that

$$j^\lambda \leq k^n < j^{\lambda+1}.$$

From the previous considerations it follows that

$$c_j^\lambda \leq c_k^n \leq c_j^{\lambda+1}.$$

In particular it means that $c_j \geq 1$. In the case when $c_j = 1$ for some $j > 1$ we would have $\varphi(t)^j = \varphi(t)$ for the characteristic function φ of the variable X. Since we assume that X is symmetric this would imply that $\varphi \equiv 1$, $\mathbf{P}\{X = 0\} = 1$ and the Lemma 3.1.2 holds for every $\alpha > 0$. Assume then that $c_j > 1$ for every $j \in \mathbb{N}$. Then

$$\frac{\lambda}{\lambda+1} \frac{\ln j}{\ln c_j} \leq \frac{\ln k}{\ln c_k} \leq \frac{\lambda+1}{\lambda} \frac{\ln j}{\ln c_j}.$$

Since λ can be arbitrary large, then $\frac{\ln k}{\ln c_k}$ is independent of k, i.e. $c_k = k^{1/\alpha}$ for some α. Of course $\alpha > 0$ since $c_1 = 1$ and the sequence $c_n, n \in \mathbb{N}$ is increasing. □

Theorem 3.1.1. *Definitions A and B are equivalent. Moreover, if X is stable, there exists $\alpha > 0$ such that*

$$\forall\, a, b > 0\ \exists\, d \in \mathbb{R} \qquad a X_1 + b X_2 \stackrel{d}{=} \left(a^\alpha + b^\alpha\right)^{1/\alpha} X + d.$$

where X_1, X_2 are independent copies of X. The constant α is called the index of stability or the characteristic exponent. A symmetric stable random variable X with index α is called symmetric α-stable (notation $S\alpha S$).

Proof. Assume that the functional equation (II) holds. From Lemma 3.1.2 it follows that

$$\forall\ k,n,m \qquad \left(\frac{n}{k}\right)^{1/\alpha} X_1 + \left(\frac{m}{k}\right)^{1/\alpha} X_2 \stackrel{d}{=} \left(\frac{n}{k}+\frac{m}{k}\right)^{1/\alpha} X_1 + \frac{d_{n+m}-d_n-d_m}{k^{1/\alpha}}.$$

This means that

$$\forall\ r,s \in Q_+\ \exists d(r,s) \in \mathbb{R} \qquad r^{1/\alpha}X_1 + s^{1/\alpha}X_2 \stackrel{d}{=} (r+s)^{1/\alpha}X_1 + d(r,s).$$

For φ the characteristic function of X this implies that

$$\forall\ r,s \in Q_+\ \exists\, d(r,s)\ \forall\, t \in \mathbb{R} \qquad \varphi\left(r^{1/\alpha}t\right)\varphi\left(s^{1/\alpha}t\right) = \varphi\left((r+s)^{1/\alpha}t\right)e^{id(r,s)t}.$$

Continuity of the function φ implies that for $r_n \to a$, $s_n \to b$, $r_n, s_n \in Q_+$ we have $\varphi(r_n^{1/\alpha}t) \to \varphi(a^{1/\alpha}t)$, $\varphi(s_n^{1/\alpha}t) \to \varphi(b^{1/\alpha}t)$, $\varphi((r_n+s_n)^{1/\alpha}t) \to \varphi((a+b)^{1/\alpha}t)$ for every fixed $t \in \mathbb{R}$. Consequently there exists also the limit $d(r_n, s_n)$, which we denote by $d = d(a,b)$, which ends the proof of implication (II) $\Rightarrow$ (I) together with the specification of the constant $c = c(a,b)$ in definition (I). Implication (I) $\Rightarrow$ (II) is a consequence of a simple application of mathematical induction.

□

In 1983 Zolotariev proved that a stable random variable can be equivalently defined as the one for which the conditions of definition B hold only for $n = 2$ and $n = 3$. In fact the particular choice of these two natural numbers can be replaced by any two natural numbers k, ℓ under the assumption that they are relatively prime i.e. they do not have any joint factor except 1.

Definition 3.1.3 (C). *A symmetric random variable X is stable if there exist $c_2, c_3 > 0$ and $d_2, d_3 \in \mathbb{R}$ such that*

$$X_1 + X_2 \stackrel{d}{=} c_2 X + d_2, \quad \text{and} \quad X_1 + X_2 + X_3 \stackrel{d}{=} c_3 X + d_3, \qquad \text{(III)}$$

where X_1, X_2, X_3 are independent copies of X.

Lemma 3.1.3. *Definitions B and C are equivalent.*

Proof. We need to show that definition C implies that for every $n \in \mathbb{N}$ there exist $c_n > 0$ and $d_n \in \mathbb{R}$ such that

$$\varphi^n(t) = \varphi(c_n t)e^{id_n t}.$$

It follows from Lemma 3.1.1 that the function $\Psi(t) = \ln\varphi(t)$ is well defined. For the function Ψ, Definition C implies that

$$2\Psi(t) = \Psi(c_2 t) + id_2 t, \qquad 3\Psi(t) = \Psi(c_3 t) + id_3 t,$$

and by mathematical induction we can obtain that there exists a family of real numbers d_{jk}, $j, k \in \mathbb{N}$ such that

$$2^j 3^k \Psi(t) = \Psi(c_2^j c_3^k t) + i d_{jk} t. \qquad (*)$$

Since the set $\{j + k \log_2 3 : j, k = 0, \pm 1, \pm 2, \ldots\}$ is dense in $\mathbb{R}$ then the set $\{2^j 3^k : j, k = 0, \pm 1, \pm 2, \ldots\}$ is dense in $(0, \infty)$, so for every $n \in \mathbb{N}$ there exists a sequence $r_m = 2^{j_m} 3^{k_m}$ such that $r_m \to n$. Let $c(m) = c_2^{j_m} c_3^{k_m}$. Comparing the real parts of equation $(*)$ we obtain

$$r_m \Re e \Psi(t) = \Re e \Psi(c(m)t).$$

If the sequence $c(m)$ is unbounded then there exists a subsequence $c(m') \to \infty$ and then

$$\Re e \Psi(t) = r_{m'} \Re e \Psi(t / c(m')) \to 0$$

for every $t \in \mathbb{R}$ in contradiction with the assumption that X is not trivial. This means that the sequence $c(m)$ is bounded and it contains a subsequence $c(m')$ converging to some $c_n > 0$. Coming back to equation $(*)$ we obtain

$$d_{j_{m'} k_{m'}} = \frac{i}{t} \left(\Psi(c(m')t) - r_{m'} \Psi(t) \right) \to \frac{i}{t} \left(\Psi(c_n t) - n \Psi(t) \right).$$

Since the left hand side is a numerical sequence then its limit also cannot depend on t, thus there exists d_n such that $n\Psi(t) = \Psi(c_n t) + i t d_n$, which ends the proof. □

The next definition is very important. It states that stable distributions are the only possible distributions which can be obtained as weak limits of sequences of rescaled and shifted sums of independent identically distributed random variables. The Central Limit Theorem is the most known theorem of such type. It states that under the assumption of finite variance of components we can find numbers (a_n) and (b_n) such that

$$\frac{X_1 + \ldots + X_n - a_n}{b_n} \Rightarrow X_{N(0,1)}.$$

Since definition D is one of the possible equivalent definitions of stable random variables we know that these variables can be considered as an effect of accumulation of very many very small factors.

Definition 3.1.4 (D). *A random variable X is stable if it has a domain of attraction, i.e. if there exists a sequence of independent identically distributed random variables $Y_1, Y_2, \ldots$ such that for a suitable constants $a_n \in \mathbb{R}$, $b_n > 0$*

$$\frac{Y_1 + \ldots Y_n - a_n}{b_n} \stackrel{d}{\Longrightarrow} X, \quad \text{for} \quad n \to \infty,$$

where $\stackrel{d}{\Rightarrow}$ denotes convergence in distribution.

Theorem 3.1.2. *Definitions B and D are equivalent.*

Proof. Notice first that definition B implies definition D. It is enough to take (Y_i) sequence of independent identically distributed random variables with the same distribution as X and define $a_n = d_n$, $b_n = c_n$, where c_n, d_n are the constants appearing in definition B. Then we have that for every $n \in \mathbb{N}$

$$\frac{Y_1 + \ldots Y_n - a_n}{b_n} \stackrel{d}{=} X.$$

To see the opposite implication choose $k \in \mathbb{N}$. Then we have

$$\begin{aligned} \frac{Y_1 + \ldots + Y_{nk} - a_{nk}}{b_{nk}} &= \frac{Y_1 + \ldots + Y_n - a_n}{b_n}\frac{b_n}{b_{nk}} \\ + \quad \ldots + &\frac{Y_{(k-1)n+1} + \ldots + Y_{nk} - a_n}{b_n}\frac{b_n}{b_{nk}} + \frac{ka_n - a_{nk}}{b_{nk}}. \end{aligned}$$

Denoting by φ_n the characteristic function of $(Y_1 + \ldots + Y_n - a_n)/b_n$ we obtain

$$\varphi_{nk}(t) = \left[\varphi_n\left(\frac{b_n}{b_{nk}}t\right)\right]^k \exp\left\{it\frac{ka_n - a_{nk}}{b_{nk}}\right\}. \qquad (*)$$

The sequence $\frac{b_n}{b_{nk}}$, as a sequence of positive numbers has an accumulation point $c_k^{-1} \in [0, \infty]$. It is easy to see that in both cases $c_k^{-1} = 0$ and $c_k^{-1} = \infty$ we would have $|\varphi(t)| \equiv 1$ in contradiction to our assumptions, thus $0 < c_k^{-1} < \infty$. Let (n') be subsequence of natural numbers such that $b_{n'}/b_{n'k} \to c_k^{-1}$. Then we have

$$\varphi_{n'k}(t) \to \varphi_X(t), \qquad \varphi_{n'}\left(\frac{b_{n'}}{b_{n'k}}t\right) \to \varphi_X(c_k^{-1}t),$$

which, in view of equality $(*)$, implies that there exists also the limit

$$\lim_{n'\to\infty} \frac{ka_{n'} - a_{n'k}}{b_{n'k}} \stackrel{def}{=} -\frac{d_k}{c_k}.$$

For $n' \to \infty$ in equality $(*)$ we obtain then

$$\varphi_X(t) = \left[\varphi_X(c_k^{-1}t)\right]^k \exp\left\{-it\frac{d_k}{c_k}\right\},$$

which means that $X_1 + \ldots + X_k \stackrel{d}{=} c_k X_1 + d_k$, which was to be shown. □

Definition 3.1.5 (E). *A random variable X with the characteristic function $\varphi(t)$ is stable if there are parameters $\alpha \in (0, 2]$, $\sigma > 0$, $\beta \in [-1, 1]$ and $\mu \in \mathbb{R}$ such that*

$$\varphi(t) = \begin{cases} \exp\left\{-\sigma^\alpha |t|^\alpha \left(1 - i\beta \operatorname{sgn}(t) \tan\frac{\pi\alpha}{2}\right) + i\mu t\right\} & \text{if } \alpha \neq 1; \\ \exp\left\{-\sigma |t| \left(1 + i\beta\frac{\pi}{2}\operatorname{sgn}(t)\ln|t|\right) + i\mu t\right\} & \text{if } \alpha = 1. \end{cases}$$

The parameters $\alpha \in (0,2]$, $\sigma > 0$, $\beta \in [-1,1]$ and $\mu \in \mathbb{R}$ uniquely determine the characteristic function of stable random variable and consequently its distribution. We will use the notation $\mathcal{S}_\alpha(\sigma,\beta,\mu)$ for such distribution. The definition E is called the Lévy spectral representation for a stable distribution.

Lemma 3.1.4. *If X is stable with an exponent $\alpha \neq 1$ then there exists $m \in \mathbb{R}$ such that $(X+m)$ is strictly stable.*

Proof. We follow here the proof based on Definition (B) presented by Feller [17] Th. VI.1.2. Let S_k be the sum of k independent random variables each distributed as X. Accordingly

$$S_{mn} \stackrel{d}{=} c_n S_m + m d_n \stackrel{d}{=} c_n c_m X + c_n d_m + m d_n.$$

Since m and n play the same role this means that we have identically

$$(c_n - n) d_m = (c_m - m) d_n.$$

Since $\alpha \neq 1$ then $c_n \neq n$ and this equation implies that $d_n/(c_n - n)$ does not depend on n, thus there exists m such that $d_n = m(c_n - n)$. It is easy to check now that the random variable $(X+m)$ is strictly stable.

□

Lemma 3.1.5. *Assume that X is a strictly α-stable random variable with characteristic function φ. If $\alpha \neq 1$ then there exist a positive constant $\sigma > 0$ and $\gamma \in (-1,1)$ such that*

$$\varphi(t) = \exp\left\{-\sigma^\alpha |t|^\alpha \left(1 - i sgn(t) \tan(\frac{\pi\gamma}{2})\right)\right\}.$$

Proof. It follows from Theorem 3.1.1 that there exists a positive constant α such that

$$\forall\, a,b \geq 0\ \forall\, t \in \mathbb{R} \qquad \varphi(a^{1/\alpha} t)\varphi(b^{1/\alpha} t) = \varphi\left((a+b)^{1/\alpha}\, t\right).$$

Moreover $\varphi(t) \neq 0$ for every $t \in \mathbb{R}$ by Lemma 3.1.1. Putting $t = 1$ and $\Phi(s) = \varphi(s^{1/\alpha})$ for $s > 0$ we obtain

$$\forall\, a,b \geq 0 \qquad \Phi(a)\Phi(b) = \Phi(a+b).$$

This is a classical functional equation of the Cauchy type, which can be easily solved in the following way: First by mathematical induction we show that $\Phi(ns) = \Phi(s)^n$. Substituting $s \to \frac{s}{n}$ we obtain also that $\Phi(\frac{s}{n}) = \Phi(s)^{1/n}$. Consequently

$$\forall\, k,n \in \mathbb{N} \qquad \Phi\left(\frac{k}{n}\right) = \Phi(1)^{k/n}.$$

Continuity of the function Φ implies that $\Phi(s) = \Phi(1)^s$ for every $s \geq 0$.

By the same arguments for $t=-1$ we obtain that for every $s>0$ $\Phi(-s)=\Phi(-1)^s$. Since $\Phi(-1)=\overline{\Phi(1)}$ and the real part of $\Phi(1)$ has to be less than 1, we can represent $\Phi(1)$ in the following form

$$\Phi(1)=\exp\left\{-Ae^{i\pi\gamma/2}\right\},$$

for some $\gamma\in(-1,1)$. Finally we obtain

$$\varphi(t)=\Phi(|t|^\alpha \mathrm{sgn}(t))=\exp\left\{-A|t|^\alpha\left(\cos\frac{\pi\gamma}{2}+i\mathrm{sgn}(t)\sin\frac{\pi\gamma}{2}\right)\right\}.$$

Now it is enough to define $\sigma^\alpha=A\cos\frac{\pi\gamma}{2}$.

□

Lemma 3.1.6. *The number α is the index of stability for some symmetric stable random variable iff $\alpha\in(0,2]$.*

Proof. Let X be an α-stable random variable with the characteristic function φ. According to Remark 3.1.1 it is enough to consider only the case when X has a symmetric distribution and then $\varphi(t)=\exp\{-A|t|^\alpha\}$.

• If $\alpha>2$ then the function $\varphi(t)=\exp\{-A|t|^\alpha\}$ has second derivative and $\varphi'(0)=0$, $\varphi''(0)=0$. Thus, if $\varphi(t)$ were a characteristic function of some random variable X then $\mathbf{E}X=0$ and $\mathrm{Var}(X)=0$. Consequently $\mathbf{P}\{X=0\}=1$ and $\varphi_X(t)\equiv 1\neq\varphi(t)$.

• If $\alpha\in(0,1]$ then it is easy to check that the function $\varphi(t)=\exp\{-A|t|^\alpha\}$ is symmetric, convex and decreasing on $(0,\infty)$ thus Polya's Lemma implies that φ is a characteristic function of some symmetric random variable X.

• For $\alpha=2$ the function $\varphi(t)=\exp\{-At^2\}$ is the characteristic function of a symmetric Gaussian distribution $N(0,\sqrt{2A})$.

• For $\alpha\in(1,2)$ we define a sequence of probability measures

$$\mu_n\stackrel{def}{=}\exp\{m_n\}=e^{-m_n(\mathbb{R})}\sum_{k=0}^{\infty}\frac{1}{k!}m_n^{*k},$$

where ν^{*k} denotes the k'th convolution power of the measure ν, and

$$m_n(dx)=Ac\frac{1}{|x|^{\alpha+1}}\,\mathbf{1}_{[n^{-1},\infty)}(|x|)\,dx,$$

with positive constants A and c are. Calculating the characteristic function of μ_n we obtain

$$\begin{aligned}\widehat{\mu_n}(t) &= \exp\left\{-Ac\int_{|x|>n^{-1}}\left(1-\cos(tx)\right)\frac{dx}{|x|^{\alpha+1}}\right\}\\ &= \exp\left\{-Ac|t|^\alpha\int_{|x|>n^{-1}|t|}\frac{1-\cos y}{|y|^{\alpha+1}}dy\right\}.\end{aligned}$$

Let $h(y) = (1-\cos y)|y|^{-1-\alpha}$. Since $h(y) \leq |y|^{1-\alpha}$ for $|y| < 1$ and $h(y) \leq 2|y|^{-\alpha-1}$ for $|y| > 1$ then $h(y)$ is integrable and we can choose c such that

$$c^{-1} = \int_{\mathbb{R}} \frac{1-\cos y}{|y|^{\alpha+1}} dy.$$

Now we have

$$\widehat{\mu_n}(t) \longrightarrow \exp\{-A|t|^\alpha\} \quad \text{as} \quad n \to \infty.$$

Since $\varphi(t) = \exp\{-A|t|^\alpha\}$ is continuous and it is a pointwise limit of characteristic functions we conclude that φ is also a characteristic function.

□

Lemma 3.1.7. *If the random variable X is 1-stable with the characteristic function φ then there exist $\sigma > 0$, $\mu, \gamma \in \mathbb{R}$ such that*

$$\varphi(t) = \exp\{-\sigma|t|(1 + i\gamma sgn(t)\ln|t|) + i\mu t\}.$$

Proof. We will show that the definition A for $\alpha = 1$ implies the result. From Lemma 3.1.1 we know that the function $\Phi(t) = -\ln\varphi(t)$ is well defined, thus for every $a, b > 0$ we have

$$\Phi(at) + \Phi(bt) = \Phi((a+b)t) + id(a,b)t.$$

Consequently

$$id(a,b) = \frac{1}{t}\left(\Phi(at) + \Phi(bt) - \Phi((a+b)t)\right),$$

so substituting $t = 1$ we have also

$$id(a,b) = \Phi(a) + \Phi(b) - \Phi(a+b).$$

The last two equations imply that

$$\left(\frac{\Phi(at)}{t} - \Phi(a)\right) + \left(\frac{\Phi(bt)}{t} - \Phi(b)\right) = \left(\frac{\Phi((a+b)t)}{t} - \Phi(a+b)\right).$$

Let $\Psi(a,t) = \Phi(at)/t - \Phi(a)$ for $a > 0$, $t \in \mathbb{R}$. Treating this function as a function of a we see that for every $a, b > 0$

$$\Psi(a,t) + \Psi(b,t) = \Psi((a+b),t),$$

which is again the classical Cauchy functional equation (see proof of Lemma 3.1.5), so there exists $h(t)$ such that

$$\Psi(a,t) = h(t)a,$$

and consequently

$$\Phi(at) = \Phi(a)t + h(t)at. \tag{$*$}$$

Let $t > 0$. Since $\Phi(a) = \Phi(1 \cdot a) = \Phi(1)a + h(a)a$ we obtain

$$\Phi(at) = \Phi(1)at + h(a)at + h(t)at.$$

In further considerations we will use the notation $\Phi(1) = \sigma + i\mu$. Substitute now $a = e^v$, $t = e^u$, $G(u) = \Phi(e^u)e^{-u} - \Phi(1)$, $H(u) = h(e^u)$. Now we have

$$G(u + v) = H(u) + H(v).$$

Since $G(0) = 0$ then $0 = 2H(0)$ and $G(u) = H(u)$. This means that $G(u + v) = G(u) + G(v)$ and there exists a constant which we denote by $k\gamma\sigma$, where k is some complex number, $|k| = 1$, $\gamma \in \mathbb{R}$, such that $G(u) = H(u) = k\gamma\sigma u$. Coming back to the function Φ we obtain $\Phi(e^u) = \Phi(1)e^u + k\gamma\sigma e^u u$, so consequently for every $t > 0$

$$\Phi(t) = (\sigma + i\mu)t + k\gamma\sigma t \ln t.$$

Calculating $\Phi(-at)$ for $a, t > 0$ from the equation $(*)$ we obtain $\Phi(-at) = \Phi(-1)at - ath(a) - ath(b)$. From the previous considerations $h(1) = 0$, Thus for every $t > 0$ we have

$$\Phi(-t) = \Phi(-1)t - tk\gamma\sigma \ln t.$$

Now it is enough to notice that $\Phi(-1) = \overline{\Phi(1)} = \sigma - i\mu$ and $\Phi(-t) = \overline{\Phi(t)}$ thus we can assume that $k = i$.

□

Theorem 3.1.3.

$$(A) \Leftrightarrow (B) \Leftrightarrow (C) \Leftrightarrow (D) \Leftrightarrow (E).$$

Proof. The equivalence of definitions A, B, C and D has been already proven. It is easy to see that for the random variable X with the characteristic function φ defined in definition E conditions of definition A are satisfied, thus the characteristic function of the form given in definition E is stable. For $\alpha \neq 1$ the representation E follows from Lemma 3.1.5., and for $\alpha = 1$ it follows from l Lemma 3.1.7.

□

3.2 Properties of stable random variables

For $\alpha = 2$, the random variable X with distribution $\mathcal{S}_2(\sigma, 0, 0)$ (see the notation introduced by Definition E) has simply $N(0, \sqrt{2}\sigma)$ distribution, so $\mathbf{E}|X|^p < \infty$ for every $p > 0$. If $0 < \alpha < 2$ then it is not even true that the random variable X with the distribution $\mathcal{S}_\alpha(\sigma, 0, 0)$ has α-moment. However, it can be shown that in this case

$$\lim_{t \to \infty} t^\alpha \mathbf{P}\{|X| > t\} = c_\alpha^\alpha \cdot \sigma^\alpha,$$

where $c_\alpha > 0$ depends only on α. Therefore, X has moments of order r for every $0 < r < \alpha$ and

$$\mathbf{E}|X|^r = \sigma^r \mathbf{E}|X_{s,\alpha}|^r \equiv \sigma^r c_{\alpha,r}^r, \tag{3.1.2}$$

and $\mathbf{E}|X|^r = \infty$ for all $r \geq \alpha$, where $X_{s,\alpha}$ is the standard symmetric α-stable random variable with distribution $\mathcal{S}_\alpha(1,0,0)$. We call a distribution with infinite second moment a *heavy tail distribution*, thus all α-stable distributions have heavy tails.

If $\alpha \geq 1$ then the support of any α-stable, not necessarily symmetric, random variable is equal to the whole $\mathbb{R}$, but if $\alpha \in (0,1)$, then it is possible to construct a strictly positive α-stable random variable. We will need and use a special kind of such α-stable random variables; namely variables Θ_α, $\alpha \in (0,1)$ such that $\mathbf{P}\{\Theta_\alpha \geq 0\} = 1$ and their Laplace transform is of the form:

$$\mathbf{E}\exp\Big\{-\xi\Theta_\alpha\Big\} = \exp\left\{-\xi^\alpha\right\}, \qquad \xi > 0. \tag{3.1.3}$$

It easily follows from the equality of the corresponding Laplace transforms that Θ_α is an α-stable random variable. We will use the notation γ_α^+ for the distribution of Θ_α. Only in one case the density of γ_α^+ is given in an explicit form, namely

$$\gamma_{1/2}^+(dx) = \frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{x^3}}e^{-1/(4x)}dx, \qquad x > 0;$$

for details see Feller [17], examples II.4(f) and XIII.3(b). If $\alpha \in (0,1)$ and $\alpha \neq 1/2$, then the density of Θ_α can be obtained by the inverse Fourier transform of its characteristic function. Namely, we have:

$$\gamma_\alpha^+(dx) = \frac{1}{x\pi}\int_0^\infty \exp\Big\{-t - t^\alpha x^{-\alpha}\cos(\frac{\pi}{2}\alpha)\Big\}\sin\Big\{t^\alpha x^{-\alpha}\sin(\frac{\pi}{2}\alpha)\Big\}\,dt \times dx,$$

and the proof of this formula can be found in [27], Theorem 2.3.1(3).

The probability densities of α-stable random variables exist and are continuous but, with a few exceptions, they are not known in closed form (see [74]). The exceptions are:

(a) Gaussian distribution $S_2(\sigma,0,\mu) = N(\mu,2\sigma^2)$;

(b) Cauchy distribution $S_1(\sigma,0,\mu)$ with density

$$\frac{\sigma}{\pi((x-\mu)^2+\sigma^2)}.$$

and the cumulative distribution function

$$\mathbf{P}\{X < x\} = \frac{1}{2} + \frac{1}{\pi}\mathrm{arctg}\left(\frac{x-\mu}{\sigma}\right);$$

(c) The Lévy distribution $S_{1/2}(\sigma,1,\mu)$ with the density

$$\left(\frac{\sigma}{2\pi}\right)^{1/2}\frac{1}{(x-\mu)^{3/2}}\exp\left\{-\frac{\sigma}{4(x-\mu)}\right\}$$

is concentrated on (μ, ∞). If $X \sim S_{1/2}(\sigma, 1, 0)$ then

$$\mathbf{P}\{X < x\} = 2\left(1 - \Phi\left(\sqrt{\sigma/x}\right)\right),$$

where Φ is the cumulative distribution function for $N(0,1)$ distribution.

The density function for the symmetric $S_\alpha(\sigma, 0, 0)$ distribution can be easily expressed by the inverse Fourier transform

$$f_{\alpha,\sigma}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itx}e^{-\sigma^\alpha t^\alpha}\,dt = \frac{1}{\pi}\int_0^\infty \cos(tx)e^{-\sigma^\alpha t^\alpha}\,dt.$$

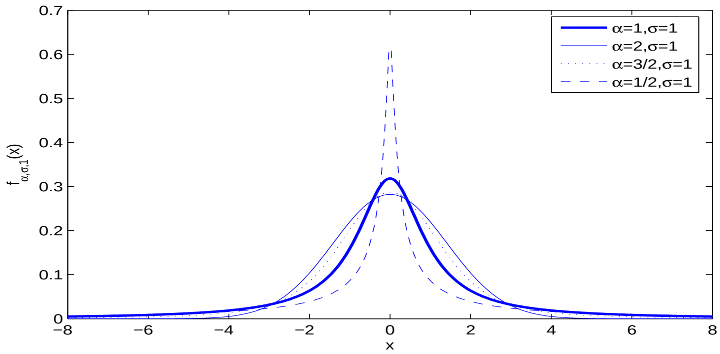


Figure 3.1: Examples of symmetric stable density functions

Directly implementing the general formula for the characteristic function of stable random variables into the Fourier inverse formula in order to get the density function may cause some numerical troubles. The cumulation of very many very small approximations in computer calculations could give us an imaginary part in the resulting density function. Thus, it is better to use the following expressions for density of $S_\alpha(\sigma, \beta, 0)$ distribution:

$$f_{\alpha,\sigma,\beta}(x) = \frac{1}{\pi}\int_0^\infty \exp\{-\sigma^\alpha|t|^\alpha\}\cos\left(\beta\sigma^\alpha|t|^\alpha(\mathrm{sgn}\ t)\mathrm{tg}(\pi\alpha/2) - tx\right)dt,$$

for $\alpha \neq 1$, and, for $\alpha = 1$:

$$f_{1,\sigma,\beta}(x) = \frac{1}{\pi}\int_0^\infty \exp\{-\sigma|t|\}\cos\left(tx - \beta\sigma\frac{2}{\pi}t\ln|t|\right)dt.$$

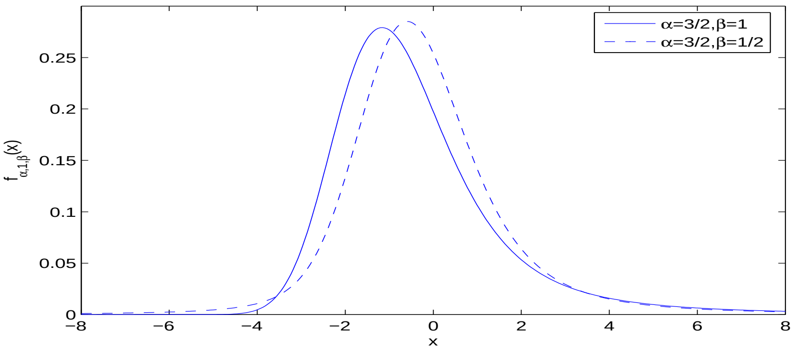


Figure 3.2: $\frac{3}{2}$-stable density function with $\mu = 0$ and different β's

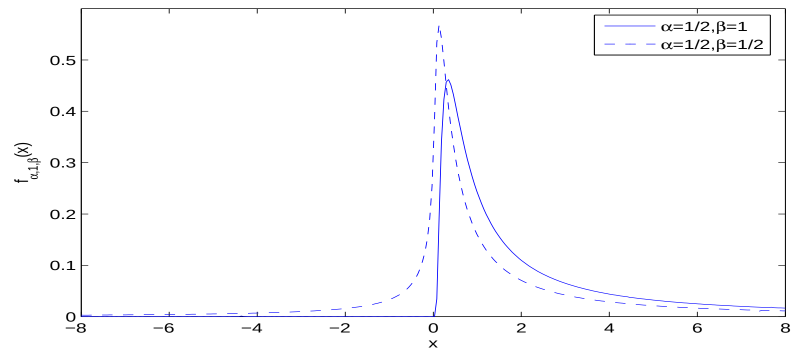


Figure 3.3: $\frac{1}{2}$-stable density function with $\mu = 0$ and different β's

3.3 Simulating stable random variables.

Consider the Lévy distribution $S_{1/2}(\sigma, 1, \mu)$. It is easy to check that this is the density function for the random variable

$$X = \sigma Z^{-2} + \mu,$$

where Z has the $N(0, 1)$ distribution. This representation gives a simple method for sampling $S_{1/2}(\sigma, 1, \mu)$ distribution, namely by sampling a symmetric Gaussian random variable. For $\alpha \neq 1/2$ we have much more trouble.

Proposition 3.3.1 (Generating completely skewed stable variables.). *Let $\alpha \in (0,1)$. If θ with uniform distribution on $(0,\pi)$ and W exponential with mean 1 are independent then*

$$X = \frac{\sin \alpha\theta}{\left(\cos \frac{\pi\alpha}{2}\right)^{1/\alpha} (\sin\theta)^{1/\alpha}} \left(\frac{\sin((1-\alpha)\theta)}{W} \right)^{(1-\alpha)/\alpha}$$

has the $S_\alpha(1,1,0)$ distribution.

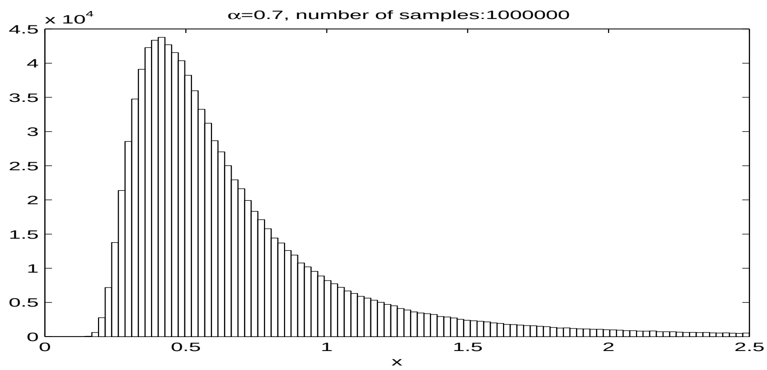


Figure 3.4: Histogram of completely skewed 0.7 stable variable

The following two propositions give the method for generating symmetric stable random variables by sampling two much simpler distributions. We present here the description given in Samorodnitsky, Taqqu (1994) [63].

Proposition 3.3.2. *For $0 < \alpha \le 2$, $\alpha \ne 1$ and $0 < \theta < \frac{\pi}{2}$ we define*

$$U_\alpha(\theta) = \left(\frac{\sin\alpha\theta}{\cos\theta} \right)^{\alpha/(1-\alpha)} \frac{\cos((1-\alpha)\theta)}{\cos\theta}$$

and let $\mathcal{L}(X) = S_\alpha(1,0,0)$. Then for $x \ge 0$,

$$\frac{1}{\pi}\int_0^{\pi/2} exp\left\{ -x^{\alpha/(\alpha-1)} U_\alpha(\theta) \right\} d\theta = \begin{cases} \mathbf{P}\{0 < X \le x\} & \textit{if } 0 < \alpha < 1; \\ \mathbf{P}\{X \ge x\} & \textit{if } 1 < \alpha \le 2. \end{cases}$$

Proof. The result follows from Zolotarev (1986), Remark 1, page 78. This gives us the integral representation for the cumulative distribution function for symmetric α-stable variables with $\alpha \ne 1$. □

Proposition 3.3.3 (Generating symmetric stable variables.). *Let θ with uniform distribution on $(-\pi/2, \pi/2)$ and W exponential with mean 1 be independent. Then*

$$X = \frac{\sin \alpha\theta}{(\cos\theta)^{1/\alpha}} \left(\frac{\cos((1-\alpha)\theta)}{W} \right)^{(1-\alpha)/\alpha}$$

has $S_\alpha(1,0,0)$ distribution.

Proof. For $\theta \in (0, \pi/2)$ the right-hand side of this formula can be expressed as

$$\left(\frac{a(\theta)}{W} \right)^{(1-\alpha)/\alpha},$$

where

$$a(\theta) = \left(\frac{\sin\alpha\theta}{\cos\theta} \right)^{\alpha/(1-\alpha)} \cos((1-\alpha)\theta).$$

(a) Case $\alpha \in (0,1)$.

$$\begin{aligned} \mathbf{P}\{0 \le X \le x\} &= \mathbf{P}\{0 \le X \le x, \theta > 0\} \\ &= \mathbf{P}\left\{0 \le (a(\theta)/W)^{(1-\alpha)/\alpha} < x, \theta > 0\right\} \\ &= \mathbf{P}\left\{W \ge x^{-\alpha/(1-\alpha)} a(\theta), \theta > 0\right\} \\ &= \mathbf{E} \exp\left\{-x^{-\alpha/(1-\alpha)} a(\theta)\right\} \mathbf{1}_{\{\theta>0\}} \\ &= \frac{1}{\pi} \int_0^{\pi/2} \exp\left\{-x^{-\alpha/(1-\alpha)} a(\theta)\right\} d\theta. \end{aligned}$$

Now the previous proposition gives the result.

(b) Case $\alpha \in (1,2]$. We start with $\mathbf{P}\{0 \le X \le x\} = \mathbf{P}\{0 \le X \le x, \theta > 0\}$ and proceed as above, making use of the inequality $(1-\alpha) < 0$.

(c) Case $\alpha = 1$. The formula reduces to $X = \tan\theta$, and it is easy to see that X has a symmetric Cauchy distribution.

□

The next proposition generalizes the previous construction to strictly stable random variables for $\alpha \notin \{1, 2\}$. We follow here the description given in Janicki (1996) [29].

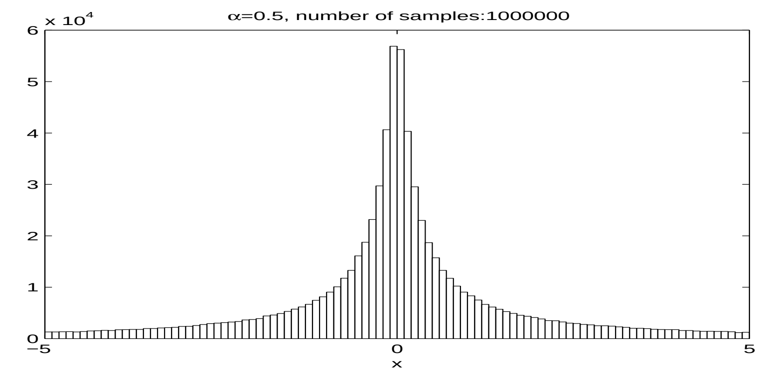


Figure 3.5: Histogram of symmetric 0.5 stable variable

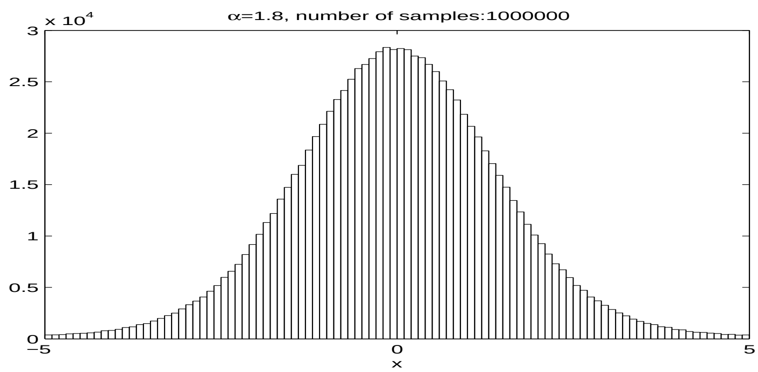


Figure 3.6: Histogram of symmetric 01.8 stable variable

Proposition 3.3.4 (Generating strictly stable random variables). *Let θ with uniform distribution on the interval $(-\pi/2, \pi/2)$ and the exponential random variable W with mean 1 be independent. For $\alpha \in (0,1) \cup (1,2)$ and $\beta \in [-1,1]$ we define*

$$C_{\alpha,\beta} = \frac{\arctan(\beta \tan(\pi\alpha/2))}{\alpha},$$

and

$$D_{\alpha,\beta} = \big(\cos(\arctan(\beta \tan(\pi\alpha/2)))\big)^{-1/\alpha}.$$

Then the random variable

$$X = D_{\alpha,\beta} \cdot \frac{\sin(\alpha(\theta + C_{\alpha,\beta}))}{(\cos\theta)^{1/\alpha}} \left[\frac{\cos(\theta - \alpha(\theta + C_{\alpha,\beta}))}{W}\right]^{(1-\alpha)/\alpha}$$

has a strictly stable distribution $S_\alpha(1, \beta, 0)$.

This representation applied to completely skewed stable random variables with distribution $S_\alpha(1,1,0)$ for $\alpha < 1$ gives

$$C_{\alpha,1} = \frac{\pi}{2}, \qquad D_{\alpha,1} = \left(\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1/\alpha},$$

thus $S_\alpha(1,1,0)$ distribution can be sampled using the formula

$$X = \left(\cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1/\alpha} \cdot \frac{\sin\left(\alpha(\theta+\frac{\pi}{2})\right)}{(\cos\theta)^{1/\alpha}} \left[\frac{\cos\left(\theta-\alpha(\theta+\frac{\pi}{2})\right)}{W}\right]^{(1-\alpha)/\alpha}$$

Proposition 3.3.5 (Full family of stable random variables). *Let θ with uniform distribution on the interval $(-\pi/2,\pi/2)$ and the exponential random variable W with mean 1 be independent.*
For $\alpha \neq 1$ and $\beta \in [-1,1]$ we define

$$C_{\alpha,\beta} = \frac{\arctan(\beta\tan(\pi\alpha/2))}{\alpha},$$

and

$$D_{\alpha,\beta,\sigma} = \sigma\big(\cos(\arctan(\beta\tan(\pi\alpha/2)))\big)^{-1/\alpha},$$

and

$$B_{\alpha,\beta,\sigma,\mu} = \mu - \beta\sigma^\alpha \tan\frac{\pi\alpha}{2}.$$

Then the random variable

$$X = D_{\alpha,\beta,\sigma} \cdot \frac{\sin\left(\alpha(\theta+C_{\alpha,\beta})\right)}{(\cos\theta)^{1/\alpha}} \left[\frac{\cos\left(\theta-\alpha(\theta+C_{\alpha,\beta})\right)}{W}\right]^{(1-\alpha)/\alpha} + B_{\alpha,\beta,\sigma,\mu}$$

has stable distribution $S_\alpha(\sigma,\beta,\mu)$.
• *For $\alpha = 1$ we have that*

$$X = \sigma\frac{2}{\pi}\left[\left(\frac{\pi}{2}+\beta\theta\right)\tan\theta - \beta\ln\left(\frac{\pi W\cos\theta}{\pi+2\beta\theta}\right)\right] + B_{\beta,\sigma,\mu},$$

where

$$B_{\beta,\sigma,\mu} = \mu + \frac{2}{\pi}\beta\sigma\ln\sigma$$

has $S_1(\sigma,\beta,\mu)$ distribution.

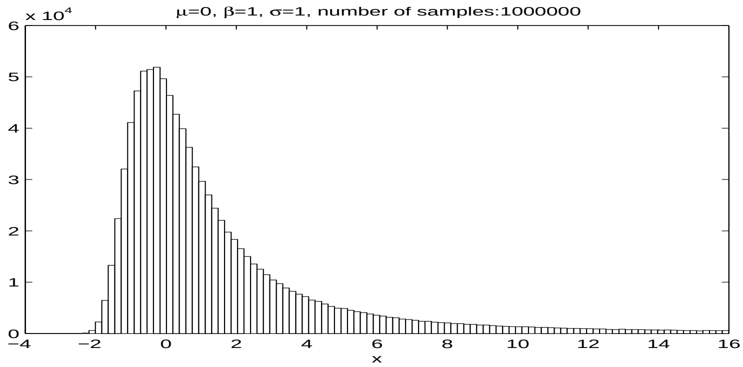


Figure 3.7: histogram for Cauchy distribution with $\beta = \sigma = 1$, $\mu = 0$

4

Stable random vectors

4.1 Basic Properties of symmetric stable vectors

Definition 4.1.1. *A random vector* $\mathbf{X} = (X_1, \ldots, X_n)$ *is said to be stable in* $\mathbb{R}^n$ *if for every choice of* a, $b \in \mathbb{R}$ *there exists a positive constant* c *and vector* $d \in \mathbb{R}^n$ *such that*

$$a\mathbf{X}_1 + b\mathbf{X}_2 \stackrel{d}{=} c\mathbf{X} + d,$$

where $\mathbf{X}_1$, $\mathbf{X}_2$ *are independent copies of* $\mathbf{X}$.

Theorem 4.1.1. *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *be a symmetric stable random vector in* $\mathbb{R}^n$. *Then there exists* $\alpha \in (0, 2]$ *such that all linear combinations of the components of* $\mathbf{X}$ *are symmetric* α*-stable random variables.*

Proof. The definition of a symmetric stable random vector $\mathbf{X}$ is equivalent to the following condition for its characteristic function $\varphi_{\mathbf{X}}$: for every a, $b \in \mathbb{R}$ there exists a positive constant c such that for every $\xi \in \mathbb{R}^n$

$$\varphi_{a\mathbf{X}}(\xi)\varphi_{b\mathbf{X}}(\xi) = \varphi_{c\mathbf{X}}(\xi).$$

Put $\xi = \big(\xi_1, 0, \ldots, 0\big) \in \mathbb{R}^n$. Then the characteristic function φ_{X_1} of the first component X_1 of the random vector $\mathbf{X}$ has the property $\varphi_{aX_1}(\xi_1)\varphi_{bX_1}(\xi_1) = \varphi_{cX_1}(\xi_1)$, which means that X_1 is a stable random variable with some index of stability $\alpha \in (0, 2]$. Evidently, X_1 is symmetric as a component of a symmetric random vector $\mathbf{X}$, hence $c^\alpha = |a|^\alpha + |b|^\alpha$. Consider now a random variable $Y = \langle \xi, \mathbf{X} \rangle = \sum_1^n \xi_k X_k$. Calculating the characteristic function φ_Y of Y we get:

$$\varphi_{aY}(t)\varphi_{bY}(t) = \varphi_{a\mathbf{X}}(t\xi)\varphi_{b\mathbf{X}}(t\xi) = \varphi_{c\mathbf{X}}(t\xi) = \varphi_{cY}(t),$$

which means that the random variable Y is stable. As Y is a linear combination of the components of a symmetric random vector $\mathbf{X}$, it is also symmetric. If the index of stability for Y is β, then $c^\beta = |a|^\beta + |b|^\beta$. Comparing with the stability of X_1 we have

$$\left(|a|^\beta + |b|^\beta\right)^{1/\beta} = \left(|a|^\alpha + |b|^\alpha\right)^{1/\alpha}$$

for every a, $b \in \mathbb{R}$; this however is possible only if $\alpha = \beta$. Now, Y is a symmetric α-stable random variable and thus there exists a positive constant $c(\xi)$ such that

$$\varphi_Y(t) = \exp\left\{-c(\xi)^\alpha |t|^\alpha\right\}, \qquad t \in \mathbb{R}.$$

□

Corolary 4.1.1. *If every linear combination of the components of the random vector* $\mathbf{X} = (X_1, \ldots, X_n)$ *in* $\mathbb{R}^n$ *is symmetric* α*-stable then* $\mathbf{X}$ *is symmetric* α*-stable.*

The next two theorems are known in the literature as the *Lévy spectral representation for symmetric stable random vectors in* $\mathbb{R}^n$ (see [43] §63). In the language of geometry of Banach space theory they can be expressed as follows:

Let $\mathbb{E}$ *be a finite dimensional linear space equipped with a quasi-norm, i.e. a continuous function* $c : \mathbb{E} \mapsto [0, \infty)$ *such that* $c(x) = 0 \iff x = 0$ *and* $c(tx) = |t| c(x)$ *for every* $t \in \mathbb{R}, x \in \mathbb{E}$. *Then the function* $\exp\{-c(x)^\alpha\}$ *is positive definite on* $\mathbb{E}$ *if and only if 1)* $\alpha \in (0, 2]$, *and 2)* $(\mathbb{E}, c)$ *embeds isometrically into some* L_α*-space.*
Lévy was using finite measures ν on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ (so his "embedding" was taken into the space $L_\alpha(S^{n-1}, \nu)$ with the correspondence $x \leftrightarrow\ < x, y >$) in order to obtain uniqueness of the representation. For the purpose of this book however it is better to omit this restriction.

Theorem 4.1.2. *Let* $\alpha \in (0, 2]$. *For every finite symmetric measure* ν *on* $\mathbb{R}^n$ *such that* $\int \ldots \int \left|< \xi, x >\right|^\alpha \nu(dx) < \infty$ *for every* $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, *the following formula*

$$\varphi(\xi) = \exp\left\{ - \int_{\mathbb{R}^n} \ldots \int \left|< \xi, x >\right|^\alpha \nu(dx) \right\}, \qquad \xi \in \mathbb{R}^n$$

defines a characteristic function of some symmetric α*-stable random vector* $\mathbf{X} = (X_1, \ldots, X_n)$ *on* $\mathbb{R}^n$.

Proof. For $\alpha = 2$ we see that $-\ln(\varphi(\xi)$ can be treated as an Euclidean norm on $\mathbb{R}^n$, thus it defines an inner product in $\mathbb{R}^n$ which can always be written in the form $2\ \xi \Sigma \xi^T$ for the $n \times n$-matrix Σ with

$$2\sigma_{i,j} = \frac{1}{4}\left[-\ln(\varphi(e_i + e_j)) - \ln(\varphi(e_i - e_j))\right],$$

where $e_i = (0, ..., 1, 0, ..., 0)$ (1 on the i'th position), $i = 1, ..., n$. The last equality follows from the parallelogram equation for $2\sigma_{i,j}$ being the inner product of e_i and e_j. Now it is easy to see that $\varphi(\xi)$ is the characteristic function of symmetric Gaussian random vector with the correlation matrix Σ.
Assume that $\alpha < 2$. Let ν be a positive finite symmetric measure on $\mathbb{R}^n$. If the characteristic function of the random vector $\mathbf{X} = (X_1, \dots, X_n)$ is given by the function φ, then evidently $\mathbf{X}$ is symmetric α-stable. So we need only to show that the function φ is indeed a characteristic function of some random vector. To show this let us define a family of probability measures:

$$\mathbf{E}\mathrm{xp}(m_\varepsilon) = \exp\left(-m_\varepsilon(\mathbb{R}^n)\right) \sum_{k=0}^{\infty} \frac{m_\varepsilon^{*k}}{k!},$$

where

$$m_\varepsilon(A) \equiv a^{-1} \int_\varepsilon^\infty \nu(A/s) s^{-\alpha-1}\, ds,$$

for every Borel set $A \subset \mathbb{R}^n$, and where the constant a is defined by

$$a = \int_0^\infty \big(1 - \cos s\big) s^{-\alpha-1}\, ds.$$

It is easy to see now that φ is the characteristic function of the probability measure which is the weak limit of probability measures $\mathbf{E}\mathrm{xp}(m_\varepsilon)$ as $\varepsilon \searrow 0$, because

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \underset{R^n}{\int \dots \int} e^{i<\xi,x>}\, \mathbf{E}\mathrm{xp}(m_\varepsilon)(dx) \\
&= \lim_{\varepsilon \searrow 0} \exp\Big\{ - \underset{R^n}{\int \dots \int} \big(1 - \cos <\xi, x>\big)\, m_\varepsilon(dx) \Big\} \\
&= \lim_{\varepsilon \searrow 0} \exp\Big\{ -a^{-1} \underset{\mathbb{R}^n}{\int \dots \int} |<\xi, x>|^\alpha\, \nu(dx) \int_{\varepsilon|<\xi,x>|}^\infty \big(1 - \cos s\big) s^{-\alpha-1} ds \Big\} \\
&= \exp\Big\{ - \underset{\mathbb{R}^n}{\int \dots \int} |<\xi, x>|^\alpha\, \nu(dx) \Big\}.
\end{aligned}$$

□

Theorem 4.1.3. *If a random vector $\mathbf{X} = (X_1, \dots, X_n)$ on $\mathbb{R}^n$ is symmetric α-stable then there exists a positive finite measure ν on $\mathbb{R}^n$ such that*

$$\mathbf{E}e^{i<\xi,\mathbf{X}>} = \exp\Big\{ - \underset{\mathbb{R}^n}{\int \dots \int} |<\xi, x>|^\alpha\, \nu(dx) \Big\}, \qquad \textit{for every } \xi \in \mathbb{R}^n.$$

The measure ν is called the (canonical) spectral measure for the $S\alpha S$ random vector $\mathbf{X}$.

Proof. We will follow here the proof given by Ledoux and Talagrand in [41]. Recall that if the random variable Y has a symmetric α-stable distribution $\mathcal{S}_\alpha(c,0,0)$ and $r<\alpha$ then $\mathbf{E}|Y|^r = c^r_{\alpha,r}c^r$ (see formula (3.1.2)). It follows that for every $\xi=(\xi_1,\ldots,\xi_n)\in\mathbb{R}^n$,

$$\mathbf{E}\exp\Big\{i\sum_{k=1}^n \xi_k X_k\Big\} + \exp\Big\{-c(\xi)^\alpha\Big\} = \exp\Big\{-c^{-\alpha}_{\alpha,r}\Big(\mathbf{E}\Big|\sum_{k=1}^n \xi_k X_k\Big|^r\Big)^{\alpha/r}\Big\},$$

where $c(\xi)$ is the scale parameter for the random variable $\sum_{k=1}^n \xi_k X_k$. For every $r<\alpha$ define then a positive finite measure m_r on the unit sphere $S=S^{n-1}_\infty$ for the sup-norm $\|\cdot\|_\infty$ on $\mathbb{R}^n$ by setting, for every bounded measurable function φ on S,

$$\int_S\!\!\ldots\!\int \varphi(y)\, m_r(dy) = c^{-r}_{\alpha,r}\int_{\mathbb{R}^n}\!\!\ldots\!\int \varphi\Big(x/\|x\|_\infty\Big)\|x\|^r_\infty\, \mathbf{P_X}(dx),$$

where $\mathbf{P_X}$ is the distribution of $\mathbf{X}=(X_1,\ldots,X_n)$. Hence for any $\xi=(\xi_1,\ldots,\xi_n)\in\mathbb{R}^n$

$$\mathbf{E}\exp\Big\{i\sum_{k=1}^n \xi_k X_k\Big\} = \exp\Big\{-\Big(\int_S\!\!\ldots\!\int\Big|\sum_{k=1}^n \xi_k x_k\Big|^r m_r(dx)\Big)^{\alpha/r}\Big\}.$$

Now the total mass $|m_r|$ of the measure m_r is easily seen to be majorized by

$$|m_r| \;\leq\; \Big[\inf_{x\in S}\sum_{k=1}^n |x_k|^r\Big]^{-1}\sum_{k=1}^n c(e_k)^r,$$

where e_k, $1\leq k\leq n$ are the unit vectors of $\mathbb{R}^n$. Therefore $\sup_{r<\alpha}|m_r|<\infty$. Let m be a cluster point (in the $*$-weak sense) of $\{m_r : r<\alpha\}$; m is a positive finite measure which is clearly the spectral measure of $\mathbf{X}$.

□

Remark 4.1.1. It is easy to see that for every finite positive symmetric measure ν on $\mathbb{R}^n$ such that $\int\ldots\int|<\xi,x>|^\alpha\,\nu(dx)<\infty$ for every $\xi=(\xi_1,\ldots,\xi_n)\in\mathbb{R}^n$ we can construct a finite positive symmetric measure ν_1 on $S^{n-1}=\{x\in\mathbb{R}^n : \sum_1^n x_k^2=1\}$ such that for every $\xi\in\mathbb{R}^n$

$$\int_{\mathbb{R}^n}\!\!\ldots\!\int\Big|<\xi,x>\Big|^\alpha\,\nu(dx) = \int_{S^{n-1}}\!\!\ldots\!\int\Big|<\xi,x>\Big|^\alpha\,\nu_1(dx).$$

It is enough to use spherical variables and integrate out the radial part. If the characteristic function of a symmetric α-stable random vector $\mathbf{X}$ is of the form:

$$\exp\Big\{-\int_{S^{n-1}}\!\!\ldots\!\int\Big|<\xi,x>\Big|^\alpha\,\nu_1(dx)\Big\},$$

then the symmetric measure ν_1 is called the *canonical spectral measure* of $\mathbf{X}$. For $0 < \alpha < 2$ the canonical spectral measure of a symmetric α-stable random vector is uniquely determined. If the characteristic function of a symmetric α-stable random vector $\mathbf{X}$ is of the form:

$$\exp\left\{-\int\limits_{\mathbb{R}^n}\!\!\cdots\!\!\int |< \xi, x >|^\alpha \, \nu_1(dx)\right\},$$

then the symmetric measure ν_1 is called the *spectral measure* of $\mathbf{X}$.

Example 4.1.1. A random vector $(X_1, \ldots, X_n)$ is symmetric Gaussian if there exists a symmetric positive definite $n \times n$-matrix $\mathcal{R}$ such that the characteristic function

$$\mathbf{E}\exp\left\{i\sum^n \xi_k X_k\right\} = \exp\left\{-\frac{1}{2}\langle \xi, \mathcal{R}\xi\rangle\right\}.$$

This means that for every $\xi \in \mathbb{R}^n$ the random variable $\sum^n \xi_k X_k$ has the same distribution as $\left(\langle \xi, \mathcal{R}\xi\rangle\right)^{1/2} X_0$, where the random variable X_0 has distribution $N(0,1)$. It is easy to see that a symmetric Gaussian random vector is symmetric 2-stable, thus for every symmetric positive definite $n \times n$-matrix there exists a finite positive measure ν on S^{n-1} such that

$$\frac{1}{2}\langle \xi, \mathcal{R}\xi\rangle = \int\limits_{S^{n-1}}\!\!\cdots\!\!\int \, | < \xi, x > |^2 \, \nu(dx), \qquad \xi \in \mathbb{R}^n.$$

However, in the case of symmetric Gaussian random vectors the spectral measure ν is not uniquely determined; we have e.g.:

$$\begin{aligned}\sum_{k=1}^n \xi_k^2 &= \int\limits_{S^{n-1}}\!\!\cdots\!\!\int \, | < \xi, x > |^2 \cdot \tfrac{1}{2}\sum_{k=1}^n \left(\delta_{e_k} + \delta_{-e_k}\right)(dx) \\ &= \int\limits_{S^{n-1}}\!\!\cdots\!\!\int \, | < \xi, x > |^2 \, c\lambda(dx),\end{aligned}$$

where $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$, λ is the uniform distribution on the unit sphere S^{n-1} and c is a suitable constant.

Example 4.1.2. If the spectral measure of a symmetric α-stable random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is of the form

$$\nu(dx) = \tfrac{1}{2}\sum_{k=1}^n a_k\left(\delta_{e_k} + \delta_{-e_k}\right)(dx)$$

for some positive constants $a_1, \ldots, a_n$, then the characteristic function of $\mathbf{X}$ can be written as:

$$\varphi_{\mathbf{X}}(\xi) = \exp\left\{-\sum_{k=1}^n a_k|\xi_k|^\alpha\right\}.$$

It is easy to see that in this case $\mathbf{X}$ has independent components. The opposite implication also holds, i.e. if a symmetric α-stable random vector $\mathbf{X}$ has independent components, then its spectral measure is of the form $\nu(dx) = \frac{1}{2}\sum_{k=1}^{n} a_k \big(\delta_{e_k} + \delta_{-e_k}\big)(dx)$, for some positive constants $a_1, \ldots, a_n$.

4.2 Covariation

The *covariance* function is an extremely powerful tool in studying properties of Gaussian random vectors. It is also useful in studying other random vectors for which the second moment exists. However it is not defined for vectors without second moment. The *covariation* is designed to replace covariance for symmetric α-stable random vectors. In this section we describe only the basic properties of the covariation function; one can find more details in the book of G. Samorodnitsky and M. Taqqu [63].
We consider here only $\alpha \in (1, 2]$. Let us define the *signed power* $a^{<p>}$

$$a^{<p>} = |a|^p \mathrm{sign} a = \begin{cases} a^p & \text{if } a \geq 0; \\ -|a|^p & \text{if } a < 0. \end{cases}$$

Notice that with this notation the derivative of the function $f(x) = |x|^r$ can be simply written as $f'(x) = r x^{<r-1>}$ for every $x \neq 0$.

Definition 4.2.1. *Let (X_1, X_2) be symmetric α-stable random vector with spectral measure ν. The covariation of X_1 on X_2 (it is not symmetric!) is the real number*

$$[X_1, X_2]_\alpha = \int_{S^1} s_1 s_2^{<\alpha-1>} \nu(ds_1, ds_2)$$

If $\alpha = 2$ then we obtain $[X_1, X_2]_2 = \frac{1}{2}\mathrm{Cov}(X_1, X_2)$.
Notice that if (X_1, X_2) is symmetric α stable random vector then for every $a, b \in \mathbb{R}$ the variable $aX_1 + bX_2$ is also symmetric α stable with distribution $S_\alpha(\sigma_{aX_1+bX_2}, 0, 0)$. Of course $\sigma_{aX_1+bX_2}$ is the scale parameter here. It is easy to see that the covariation $[X_1, X_2]_\alpha$ can be also expressed in the following formula

$$[X_1, X_2]_\alpha = \frac{1}{\alpha}\frac{\partial}{\partial a}\sigma^\alpha_{aX_1+bX_2}\Big|_{a=0,b=1} = \frac{1}{\alpha}\frac{\partial}{\partial a}\int_{S^1} |as_1 + bs_2|^\alpha \nu(ds_1, ds_2)\Big|_{a=0,b=1}.$$

Example 4.2.1. Let (X_1, X_2, X_3) be a symmetric α-stable random vector with the spectral measure $\nu = \frac{1}{2}\sum_i(\delta_{e_i} + \delta_{-e_i})$, where $e_i, i = 1, 2, 3$, are the unit versors in $\mathbb{R}^3$. Let $X = \sum p_i X_i$ and $Y = \sum q_i X_i$. In order to calculate $[X, Y]_\alpha$ we shall find first

$$\begin{aligned} \mathbf{E}\exp\{i(aX + bY)\} &= \mathbf{E}\exp\left\{ i\sum_{i=1}^{3}(ap_i + bq_i)X_i \right\} \\ &= \exp\left\{ -\sum_{i=1}^{3}|ap_i + bq_i|^\alpha \right\} = \exp\left\{-\sigma^\alpha_{aX+bY}\right\}. \end{aligned}$$

Now we obtain

$$[X_1, X_2]_\alpha = \frac{1}{\alpha}\frac{\partial}{\partial a}\sum_{i=1}^{3}|ap_i + bq_i|^\alpha\Big|_{a=0,b=1} = \sum_{i=1}^{3} p_i \cdot q_i^{<\alpha-1>}.$$

Lemma 4.2.1. *Let (X, Y, Z) be symmetric α-stable random vector, $\alpha \in (1, 2]$. Then we have*

1. $[X + Y, Z]_\alpha = [X, Z]_\alpha + [Y, Z]_\alpha$*;*
2. $[aX, bZ]_\alpha = ab^{<\alpha-1>}[X, Z]_\alpha$*,*
3. if X and Y are independent, then $[X, Y]_\alpha = 0$; the opposite implication does not hold.
4. covariation is not linear in the second argument, but if Y and Z are independent, then $[X, Y + Z]_\alpha = [X, Y]_\alpha + [X, Z]_\alpha$;
5. $\big|[X, Y]_\alpha\big| \leq \sigma_X \sigma_Y^{\alpha-1}$*.*

Proof. The properties 1 and 2 easily follow from the definition. If we consider the symmetric α-stable random vector (X, Y) with the spectral measure ω_2 uniform on the unit sphere $S^1 \subset \mathbb{R}^2$ then we have

$$\begin{aligned}
&\mathbf{E}\exp\{i(aX + bY)\} = \exp\left\{-\int_0^{2\pi}|a\cos\theta + b\sin\theta|^\alpha d\theta\right\} \\
&= \exp\left\{-(a^2+b^2)^{\alpha/2}\int_0^{2\pi}|cos(\varphi-\theta)|^\alpha d\theta\right\} \\
&= \exp\left\{-(a^2+b^2)^{\alpha/2}\int_0^{2\pi}|cos(\theta)|^\alpha d\theta\right\} = \exp\left\{-C(a^2+b^2)^{\alpha/2}\right\},
\end{aligned}$$

where $\cos\varphi = a/\sqrt{a^2+b^2}$, $\sin\varphi = b/\sqrt{a^2+b^2}$, and C is a suitable constant. We see now that this function cannot be written as a product of function dependent only on a and a function dependent only on b, thus X and Y are not independent. We On the other hand

$$[X, Y]_\alpha = \frac{\partial}{\partial a}C\left(a^2+b^2\right)^{\alpha/2}\Big|_{a=0,b=1} = 0,$$

which proves that $[X, Y]_\alpha = 0$ does not imply independence. However, if X and Y are independent then we have $\sigma^\alpha_{aX+bY} = |a|^\alpha\sigma^\alpha_X + |b|^\alpha\sigma^\alpha_Y$, and consequently $[X, Y]_\alpha = 0$.
It is easy to see that the linearity does not hold for the second argument of the covariation function. The proof that it does hold if Y and Z are independent is more complicated and it can be found in [63], Section 2.8. To prove property 5, we use the Hölder inequality:

$$\big|[X, Y]_\alpha\big| = \Big|\int_{S^1} u_1 u_2^{<\alpha-1>}\nu(d\mathbf{u})\Big|$$

$$
\begin{aligned}
&\le \left(\int_{S^1} |u_1|^\alpha \nu(d\mathbf{u})\right)^{1/\alpha} \left(\int_{S^1} |u_2|^{(\alpha-1)(1-1/\alpha)^{-1}} \nu(d\mathbf{u})\right)^{1-1/\alpha} \\
&= \sigma_X \sigma_Y^{\alpha(1-1/\alpha)} = \sigma_X \sigma_Y^{\alpha-1}.
\end{aligned}
$$

□

Definition 4.2.2. *Let (X, Y) be a symmetric α-stable random vector. The covariation ratio $\rho_\alpha(X, Y)$ of X to Y is defined by*

$$
\rho_\alpha(X, Y) = \frac{[X, Y]_\alpha}{\sigma_X \sigma_Y^{\alpha-1}} = \frac{1}{\sigma_X} \frac{\partial}{\partial a} \sigma_{aX+bY} \Big|_{a=0, b=1}.
$$

According to the property 5 in Lemma 4.2.1 we have that

$$
-1 \le \rho_\alpha(X, Y) \le 1.
$$

The idea of covariation ratio turns out to be especially useful in the situation when the parameter α is unknown and in the case of pseudo-isotropic distributions.

The scale constant σ_X for symmetric α-stable random variable X plays a very similar role in the theory of stable variables as the variance σ in the theory of Gaussian variables. Moreover, if we denote by $\mathcal{S}_\alpha$ the space of jointly stable random variables, $\alpha > 1$, then the function

$$
\mathcal{S}_\alpha \ni X \Rightarrow \|X\|_\alpha \stackrel{def}{=} \sigma_X = ([X, X]_\alpha)^{1/\alpha}
$$

defines a norm on $\mathcal{S}_\alpha$. To see this note that

1. if $\sigma_X = 0$ for the random variable $X \sim S(\sigma_X, 0, 0)$ then $\mathbf{P}\{X = 0\} = 1$;
2. $\sigma_{aX} = |a| \sigma_X$;
3. if $X, Y \in \mathcal{S}_\alpha$ then they are jointly symmetric α-stable, thus there exists a spectral measure ν for the random vector (X, Y) such that

$$
\begin{aligned}
\|X + Y\|_\alpha &= \sigma_{X+Y} = \left(\int_{S^1} |u_1 + u_2|^\alpha \nu(\mathbf{u})\right)^{1/\alpha} \\
&\le \left(\int_{S^1} |u_1|^\alpha \nu(\mathbf{u})\right)^{1/\alpha} + \left(\int_{S^1} |u_2|^\alpha \nu(\mathbf{u})\right)^{1/\alpha} \\
&= \sigma_X + \sigma_Y = \|X\|_\alpha + \|Y\|_\alpha.
\end{aligned}
$$

4.3 Stable can be sub-stable

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a symmetric α-stable random vector on $\mathbb{R}^n$ with spectral measure ν and let Θ_β with distribution γ_β^+, where $\beta \in (0, 1)$, be independent of $\mathbf{X}$. Consider the random vector $\mathbf{Y} = \mathbf{X}\Theta_\beta^{1/\alpha}$. The characteristic function of $\mathbf{Y}$ is of the form

$$\begin{aligned}\mathbf{E}e^{i<\xi t,\mathbf{Y}>} &= \mathbf{E}\exp\Big\{-\Theta_\beta|t|^\alpha \int_{S^{n-1}}\!\!\dots\!\int |<\xi,x>|^\alpha\,\nu(dx)\Big\} \\ &= \exp\Big\{-|t|^{\alpha\beta}\Big(\int_{S^{n-1}}\!\!\dots\!\int |<\xi,x>|^\alpha\,\nu(dx)\Big)^\beta\Big\},\end{aligned}$$

for every $\xi \in \mathbb{R}^n$ and $t \in \mathbb{R}$, which means that all linear combinations of the components of $\mathbf{Y}$ are symmetric $(\alpha\beta)$-stable random variables. From Corollary 4.1.1 we have that the random vector $\mathbf{Y}$ is also symmetric $(\alpha\beta)$-stable, so from Theorem 4.1.4. we get that there exists a finite positive measure ν_1 on S^{n-1} such that

$$\mathbf{E}e^{i<\xi t,\mathbf{Y}>} = \exp\Big\{-|t|^{\alpha\beta}\int_{S^{n-1}}\!\!\dots\!\int |<\xi,x>|^{\alpha\beta}\,\nu_1(dx)\Big\}.$$

Finally we get that for every $\alpha \in (0,2]$, $\kappa < \alpha$ and every finite positive measure ν on S^{n-1} there exists a finite positive measure ν_1 on S^{n-1} such that

$$\Big(\int_{S^{n-1}}\!\!\dots\!\int |<\xi,x>|^\alpha\,\nu(dx)\Big)^{1/\alpha} = \Big(\int_{S^{n-1}}\!\!\dots\!\int |<\xi,x>|^\kappa\,\nu_1(dx)\Big)^{1/\kappa}$$

To see this it is enough to put $\beta = \kappa/\alpha$ in the previous considerations.
Notice that the last equality implies that for every $\xi \in \mathbb{R}^n$ the scale parameter $\sigma_{<\xi,\mathbf{X}>}$ for $S\alpha S$ random vector $\mathbf{X}$ is equal to the scale parameter $\sigma_{<\xi,\mathbf{Y}>}$ for $S\kappa S$ random vector $\mathbf{Y} = \mathbf{X}\Theta_\beta^{1/\alpha}$. This means also that

$$\rho_\alpha(X_i, X_j) = \rho_\kappa(Y_i, Y_j), \qquad i,j \in \{1,\dots,n\}.$$

The measure ν is called the spectral measure for $S\alpha S$ random vector $\mathbf{X}$. If ν is concentrated on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ then it is called the canonical spectral measure for $\mathbf{X}$. The canonical spectral measure for given $S\alpha S$ vector $\mathbf{X}$ is uniquely determined.

Definition 4.3.1. *An $S\alpha S$ random vector is called β-substable, $\alpha < \beta \le 2$, if there exists a symmetric β-stable random vector $\mathbf{Y}$ such that*

$$\mathbf{X} \stackrel{d}{=} \mathbf{Y}\Theta^{1/\beta},$$

where $\Theta \ge 0$ is a α/β-stable random variable with the Laplace transform $\exp\{-t^{\alpha/\beta}\}$, $\mathbf{Y}$ and Θ are independent.

Definition 4.3.2. *An $S\alpha S$ random vector $\mathbf{X}$ is maximal if for every $\beta \ge \alpha$ and every $S\beta S$ random vector $\mathbf{Y}$, and every Θ independent of $\mathbf{Y}$ the equality $\mathbf{X} \stackrel{d}{=} \mathbf{Y}\Theta$ implies that $\alpha = \beta$ and $\Theta = constant$.*

Almost all $S\alpha S$ random vectors and stochastic processes studied in the literature are maximal; and even more, almost all of them have pure atomic spectral measures. The following, surprisingly simple, theorem characterizes maximal symmetric α-stable random vectors on $\mathbb{R}^n$:

Theorem 4.3.1. *Assume that a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is symmetric α-stable and β-substable for some $\beta \in (\alpha, 2]$. Then the canonical spectral measure ν for the vector $\mathbf{X}$ has continuous density function $f(u)$ with respect to the Lebesgue measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, and $f(u) > 0$ for every $u \in S^{n-1}$.*

Proof. From the assumptions we have that there exists a symmetric β-stable random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ such that $\mathbf{X} \stackrel{d}{=} \mathbf{Y}\Theta^{1/\beta}$, where $\Theta > 0$ independent of $\mathbf{Y}$ is α/β-stable with a Laplace transform $\exp\{-t^{\alpha/\beta}\}$. Assume that

$$\mathbf{E}\exp\{it < \xi, \mathbf{Y} >\} = \exp\{-c(\xi)^\beta |t|^\beta\}.$$

This means that for every ξ we have $< \xi, \mathbf{Y} > \stackrel{d}{=} c(\xi) Y_0$, where $\mathbf{E}e^{itY_0} = e^{-|t|^\beta}$. In particular

$$\mathbf{E}| < \xi, \mathbf{Y} > |^\alpha = c(\xi)^\alpha \mathbf{E}|Y_0|^\alpha.$$

Since $\alpha < \beta$ we have that $c^{-1} = \mathbf{E}|Y_0|^\alpha < \infty$ and $c(\xi)^\alpha = c\mathbf{E}| < \xi, \mathbf{Y} > |^\alpha$. Calculating now the characteristic function for the vector $\mathbf{X}$ we obtain:

$$\begin{aligned} \mathbf{E}\exp\{i < \xi, \mathbf{X} >\} &= \mathbf{E}\exp\{i < \xi, \mathbf{Y}\Theta^{1/\beta} >\} \\ &= \mathbf{E}\exp\{-c(\xi)^\beta \Theta\} = \exp\{-c(\xi)^\alpha\} \\ &= \exp\{-\mathrm{c}\mathbf{E}| < \xi, \mathbf{Y} > |^\alpha\} = \exp\{-\int \ldots \int_{\mathbf{R}^n} | < \xi, \mathbf{x} > |^\alpha \mathrm{c} f_\beta(\mathbf{x}) d\mathbf{x}\}, \end{aligned}$$

where $f_\beta(\mathbf{x})$ denotes the density function of the $S\beta S$ random vector $\mathbf{Y}$. This means that the function $c f_\beta(\mathbf{x})$ is the density of a spectral measure for the random vector $\mathbf{X}$.
To get the canonical spectral measure ν_0 for the $S\alpha S$ random vector $\mathbf{X}$ from this spectral measure it is enough to make the spherical substitution $\mathbf{x} = r\mathbf{u}$ and integrate out the radial part. So for every Borel set $A \subset S^{n-1}$ we obtain

$$\nu_0(A) = \int \ldots \int_A \int_0^\infty \mathrm{c}\, f_\beta(r\mathbf{u}) r^{n-1+\alpha} dr\, w(d\mathbf{u})$$

here w is the Lebesgue measure on S^{n-1}

$$= \int \ldots \int_A \underbrace{\int r^{n-1+\alpha} \mathrm{c}\, f_\beta(r\mathbf{u}) dr}_{g(\mathbf{u})}\, w(d\mathbf{u}).$$

Since f_β is continuous on $\mathbb{R}^n$ and $f_\beta > 0$ everywhere, it follows that $g(\mathbf{u})$ is a continuous function and $g(\mathbf{u}) > 0$ everywhere. The uniqueness of the canonical spectral measure implies that the function $g(\mathbf{u})$ is the density of the measure ν_0, which ends the proof.

□

Corolary 4.3.1. *Every random vector with a pure atomic spectral measure is maximal. In fact for maximality of the $S\alpha S$ random vector it is enough that its spectral measure μ is zero on a subset of S^{n-1} of positive Lebesgue measure.*

Corolary 4.3.2. *If an $S\alpha S$ random vector $\mathbf{X}$ is not maximal, i.e. if $\mathbf{X}$ is β-substable for some $\beta > \alpha$, then there exists a symmetric Gaussian random vector $\mathbf{Z}$ and a maximal $S\alpha S$ random vector $\mathbf{Y}$ such that*

$$\mathbf{X} \stackrel{d}{=} \mathbf{Z}\Theta^{1/2} + \mathbf{Y},$$

where $\Theta \geq 0$ has the Laplace transform $\exp\{-t^{\alpha/2}\}$, $\mathbf{Z}$, $\mathbf{Y}$ and Θ are independent.

Proof. Since every continuous function attains its extremes on every compact set we have that

$$A = \inf\left\{g(\mathbf{u}) : \mathbf{u} \in S^{n-1}\right\} > 0,$$

where $g(\mathbf{u})$ is the density of the canonical spectral measure for $\mathbf{X}$ obtained in Theorem 4.3.1. Now it is easy to see that $\mathbf{X} \stackrel{d}{=} \mathbf{Z}\Theta^{1/2} + \mathbf{Y}$, for the Gaussian random vector $\mathbf{Z}$ with the characteristic function $\exp\{-A^{1/\alpha}\sum_{k=1}^{n}\xi_k^2\}$, and the $S\beta S$ random vector $\mathbf{Y}$ with the spectral measure given by the density function $f(\mathbf{u}) = g(\mathbf{u}) - A$.

□

Remark 4.3.1. The representation obtained in the Corollary 4.3.2 is not unique. In fact, for every $S\alpha S$ β-substable random vector $\mathbf{X}$ and every symmetric Gaussian random vector $\mathbf{Z}$ taking values in the same space $\mathbb{R}^n$ there exist a constant $c > 0$ and a maximal $S\alpha S$ random vector $\mathbf{Y}$ such that

$$\mathbf{X} \stackrel{d}{=} c\mathbf{Z}\Theta^{1/2} + \mathbf{Y},$$

where Θ as in Corollary 4.3.2, $\mathbf{Y}$, $\mathbf{Z}$ and Θ are independent.

Proof. Let us recall that the canonical spectral measure ν for Gaussian random vector ($\alpha = 2$) is not unique. In fact ν can always be taken here from the class of pure atomic measures on S^{n-1}, but such representation is not useful for our construction. We will use the measure ν_A constructed as follows:
Let $\nu = \omega_n$ be the uniform distribution on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, and let $U = (U_1, \ldots, U_n)$ be the random vector with the distribution ν. Then we have

$$\exp\left\{-\int_{S^{n-1}}\!\!\ldots\!\int |<\xi, \mathbf{u}>|^2 c_n \nu(d\mathbf{u})\right\} = \exp\left\{-\frac{1}{2} <\xi, \xi>\right\},$$

where $c_n^{-1} = 2\mathbf{E}U_1^2$. Now let Σ be the covariance matrix for the random vector $\mathbf{Z}$ and let $\Sigma = AA^T$. If by ν_1 we denote the distribution of the random vector AU, then

$$\exp\left\{-\int_{\mathbb{R}^n}\cdots\int <\xi,\mathbf{x}>^2 c_n\nu_1(d\mathbf{x})\right\} = \exp\left\{-\int_{S^{n-1}}\cdots\int <\xi, A\mathbf{u}>^2 c_n\nu(d\mathbf{u})\right\}$$

$$= \exp\left\{-\int_{S^{n-1}}\cdots\int |<A^T\xi,\mathbf{u}>|^2 c_n\nu(d\mathbf{u})\right\}$$

$$= \exp\left\{-\frac{1}{2} <A^T\xi, A^T\xi>\right\} = \exp\left\{-\frac{1}{2}<\xi,\Sigma\xi>\right\},$$

which is the characteristic function for the Gaussian vector $\mathbf{Z}$. It is easy to see now that for a suitable constant $a > 0$

$$\exp\left\{-\int_{\mathbb{R}^n}\cdots\int |<\xi,\mathbf{x}>|^\alpha c_n\nu_1(d\mathbf{x})\right\} = \exp\left\{-a\left(<\xi,\Sigma\xi>\right)^{\alpha/2}\right\},$$

which is a characteristic function of the sub-Gaussian vector $\mathbf{Z}\Theta^{1/2}$. We define now the measure ν_A as the projection (in the sense described in the proof of Theorem 4.3.1) of the measure ν_1 on the sphere S^{n-1} and we obtain

$$\int_{\mathbb{R}^n}\cdots\int |<\xi,\mathbf{x}>|^\alpha c_n\nu_1(d\mathbf{x}) = \int_{S^{n-1}}\cdots\int |<\xi,\mathbf{u}>|^\alpha \nu_A(d\mathbf{u}).$$

Since ν_1 is absolutely continuous with respect to the Lebesgue measure, it follows that ν_A has the same property and $\nu_A(d\mathbf{u}) = f_A(\mathbf{u})\omega(d\mathbf{u})$ for some positive function f_A. If $g(\mathbf{u})$ is the density of the spectral measure for $\mathbf{X}$ then there exists $c_0 > 0$ such that

$$c_0 = \sup\left\{c > 0 : g(\mathbf{u}) - cf_A(\mathbf{u}) \geq 0\right\}.$$

Now it is enough to define the maximal $S\alpha S$ random vector $\mathbf{X}$ by its canonical spectral measure absolutely continuous with respect to the Lebesgue measure with the density $h(\mathbf{u}) = g(\mathbf{u}) - c_0 f_A(\mathbf{u})$ and put $c = c_0^{1/\alpha}$.

□

The next four figures ilustrate how different can be characteristic functions for α-stable random vectors even if we fix α and assume that vector is symmetric. We consider here different types of two-dimensional, symmetric Cauchy random vector.

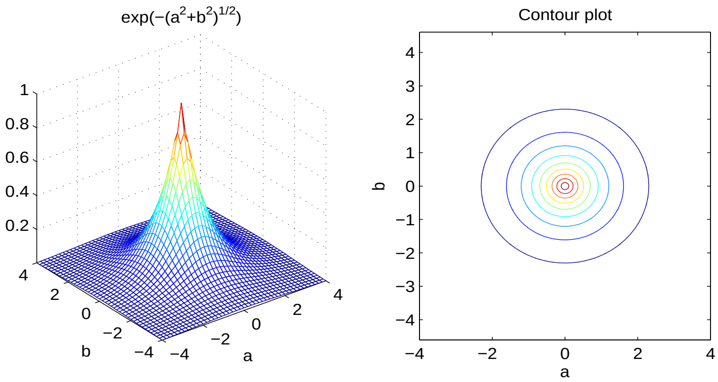


Figure 4.1: The characteristic function of the sub-Gaussian (rotationally invariant two-dimensional Cauchy distribution

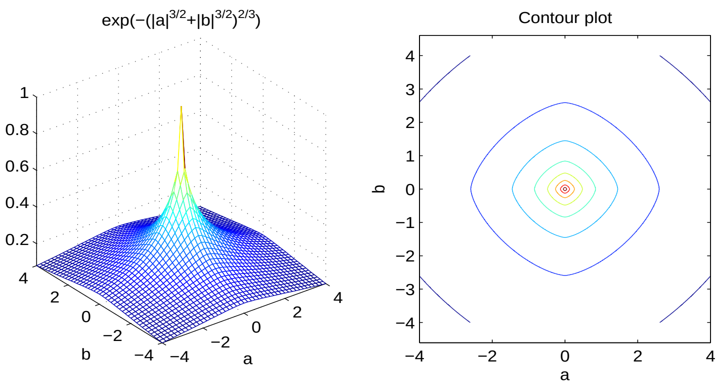


Figure 4.2: The characteristic function of a two-dimensional Cauchy 3/2-substable distribution

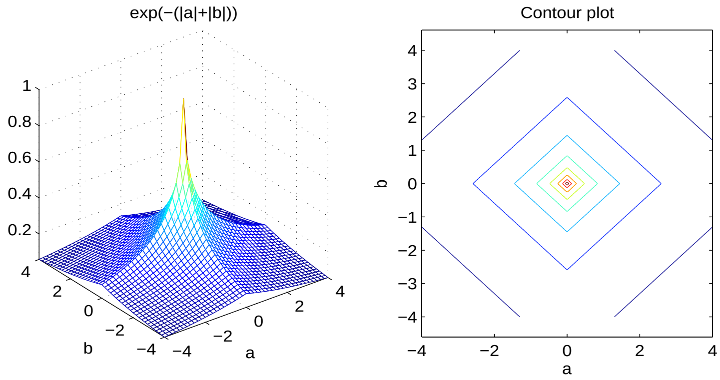


Figure 4.3: The characteristic function of a two-dimensional Cauchy distribution with independent marginals (it is maximal)

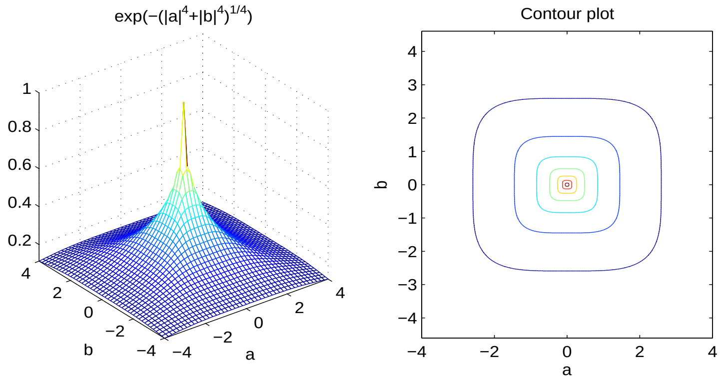


Figure 4.4: The characteristic function of a two-dimensional maximal Cauchy distribution

Figure 4.4 is especially surprising. The level curves are smooth and strictly convex, they resemble the level curves presented in Figure 4.2 corresponding to the 3/2-substable Cauchy distribution. And they are the level curves of a *maximal* 1-stable distribution. To see this we shall go back to the last statement in the history of the proof of Theorem 2.3.2. In the case of Cauchy distribution with the characteristic function

$$\exp\left\{ - \left(|a|^{\alpha} + |b|^{\alpha} \right)^{1/\alpha} \right\},$$

with $\alpha > 2$ we have $c(a,b) = \left(|a|^{\alpha} + |b|^{\alpha} \right)^{1/\alpha}$, and $q(\varphi) = \left(|\cos\varphi|^{\alpha} + |\sin\varphi|^{\alpha} \right)^{1/\alpha}$. To calculate the density of the corresponding spectral measure notice first that $[|x|^p]' = p x^{<p-1>}$ and $[x^{<p>}]' = p|x|^{p-1}$ in the sense defined in Section 4.2. Now we have

$$q'(\varphi) = q^{1-\alpha}(\varphi) \left[(\sin\varphi)^{<\alpha-1>} \cos\varphi - (\cos\varphi)^{<\alpha-1>} \sin\varphi \right],$$

and

$$
\begin{aligned}
q''(\varphi) &= (1-\alpha)q^{1-\alpha}(\varphi)\left[(\sin\varphi)^{<\alpha-1>}\cos\varphi - (\cos\varphi)^{<\alpha-1>}\sin\varphi\right]^2 \\
&+ (\alpha-1)q^{1-\alpha}(\varphi)\left[|\sin\varphi|^{\alpha-2}\cos^2\varphi - |\cos\varphi|^{\alpha-2}\sin^2\varphi\right] - q(\varphi) \\
&= (\alpha-1)q^{1-2\alpha}(\varphi)|\sin\varphi\cos\varphi|^{\alpha-2} - q(\varphi).
\end{aligned}
$$

This means that the density of the spectral measure has the form

$$
\frac{1}{4}\Big(q''\Big(\varphi+\frac{\pi}{2}\Big)+q\Big(\varphi+\frac{\pi}{2}\Big)\Big) = (\alpha-1)q^{1-2\alpha}(\varphi)|\sin\varphi\cos\varphi|^{\alpha-2}.
$$

Now it is easy to see that this density attains value zero at points $k\frac{\pi}{4}$ for $k = 1, 3, 5, 7$, thus the corresponding Cauchy distribution is maximal.

4.4 Copulae as a spectral measure for $S\alpha S$ random vector

We are not obliged to use the canonical spectral measure in the canonical representation for the characteristic function of a symmetric α stable random vector. Using other distributions on $\mathbb{R}^n$ we loose uniqueness of the representation, but this has no influence on the geometrical properties of the corresponding $S\alpha S$ random vector or its characteristic function.

Remark 4.4.1. Assume that $n = 2$ and assume that the canonical measure ν in the spectral representation of the $S\alpha S$ random vector (X, Y) is absolutely continuous with the density function $f(x, y)$. Then we can write:

$$
\begin{aligned}
-\ln\varphi_{(X,Y)}(\xi) &= \int\!\!\dots\!\!\int_{\mathbb{R}^2} |<\xi, x>|^\alpha \nu(dx) \\
&= \int_0^{2\pi} |\xi_1\cos t + \xi_2\sin t|^\alpha \int_0^\infty r^{\alpha+1} f(r\cos t, r\sin t)\,dr\,dt.
\end{aligned}
$$

This means that the canonical spectral measure ν_0 for this random vector has density given by:

$$
g(\mathbf{u}) = \int_0^\infty f(r\mathbf{u})r^{\alpha+1}dr, \qquad \mathbf{u}\in S^1\subset\mathbb{R}^2.
$$

In general, by the term copula we understand two dimensional (or n-dimensional) distribution with given marginals, which usualy are assumed to be uniform. The inversion method restricts the problem of constructing such distributions into constructing distributions on $[0,1]^2$ (or $[-1,1]^2$) having uniform marginals on the interval $[0,1]$ (or $[-1,1]$ respectively). Many types of copulas are well known in the literature. Recently there appeared a book written by Nelsen [61] which is entirely devoted to to the theory of copulae and two dimensional distribution on $[0,1]^2$. In this section, based on the paper [7] written by J.

Bojarski and J. Misiewicz, we will use copulae from a very wide class constructed by J. Bojarski in [6]. The construction as follows:

Construction:

Let Z be a random variable with a density function $f(z)$, concentrated on the interval $[-2, 2]$ such that $f(z) = f(-z)$. We define a two-dimensional density function $g(x, y)$ concentrated on $[-1, 1]^2$ by the formula

$$g(x,y) = \begin{cases} f(x-y) + f(x+y-2) & \text{for} \quad x+y \geq 0, \\ f(x-y) + f(x+y+2) & \text{for} \quad x+y \leq 0, \end{cases} \tag{4.4.1}$$

The marginals of the density $g(x, y)$ are uniform on the interval $[-1, 1]$, thus it defines a two-dimensional copulae.

Let us recall that $x^{<p>} = |x|^p \text{sign}(x)$. This notation is very useful in describing properties and moments of random variables with infinite variance. In our considerations we will use the following formulas:

$$\begin{aligned} \int (ax+b)^{<\alpha>}\, dx &= \frac{1}{a(\alpha+1)} |ax+b|^{\alpha+1} + C, \\ \int |ax+b|^{\alpha}\, dx &= \frac{1}{a(\alpha+1)} (ax+b)^{<\alpha+1>} + C, \end{aligned} \tag{4.4.2}$$

Theorem 4.4.1. *Assume that Z is a random variable with a density function $f(z)$ concentrated on $[-2, 2]$. If the spectral measure ν of an $S\alpha S$ random vector (X, Y) has the density $g(x, y)$ given by formula (4.4.1) then the characteristic function of (X, Y) at the point (a, b) is given by $\exp\{-c(a, b)^\alpha\}$ where*

$$c(a,b)^\alpha = \frac{2(1+\alpha)^{-1}}{(a^2-b^2)} \mathbf{E}\left[b\,(b - a(1-|Z|))^{<\alpha+1>} + a\,(b(1-|Z|) - a)^{<\alpha+1>} \right].$$

The covariation of X on Y is given by:

$$\begin{aligned} [X, Y]_\alpha &\stackrel{def}{=} \int\int x^{<\alpha-1>} y \nu(dx, dy) \\ &= \frac{2}{\alpha(\alpha+1)} \mathbf{E}\left[(\alpha+1)(1-|Z|) - (1-|Z|)^{<\alpha+1>} \right]. \end{aligned}$$

Proof. The proof is only a matter of a laborious calculation, which can be simplified somewhat by the integral formulas (4.4.2) given at the beginning of this section. The formula for $[X, Y]_\alpha$ holds for every $\alpha \in (0, 2]$ as long as the right hand side makes sense. □

In the following examples we want to illustrate for different α's the relation between the distribution $f(x)$, distribution of the spectral measure $g(x, y)$, the shape of the level curves of the characteristic function of the corresponding $S\alpha S$ vector. For each example we give also

$$h(\alpha) = \rho_\alpha(X, Y) = \frac{[X, Y]_\alpha}{[X, X]_\alpha}$$

describing the dependence between α and covariation ratio. In the definition of the function $h(\alpha)$ we shall explain something more. Since $g(x, y)$ is a copula density function, it has identical marginals, and

$$\sigma_X \sigma_Y^{\alpha-1} = \sigma_X^\alpha = [X, X]_\alpha,$$

which explains why we have the denominator different than given in Definition 4.2.2.

Example 1. (Figures 4.5 and 4.6) The function f is constant on the subset of $[-2, 2]$. Notice that the shape of the level curves of the characteristic function suggests positive dependence, while in fact we have here $h(\alpha) \equiv 0$.

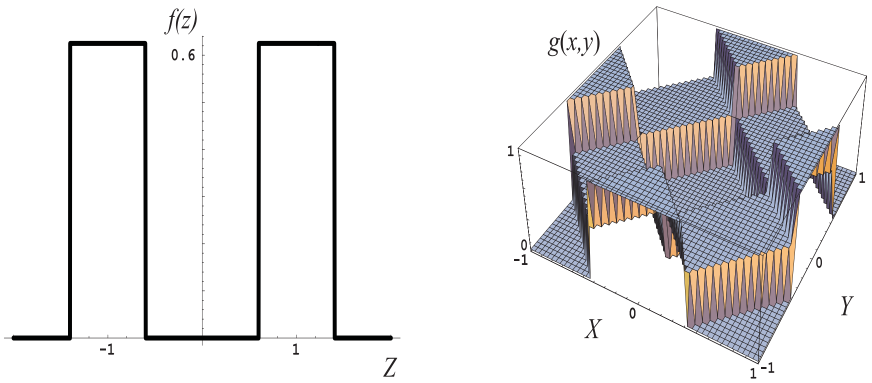


Figure 4.5: The density function $f(x)$ and the corresponding density $g(x, y)$ defining the spectral measure on $\mathbb{R}^2$

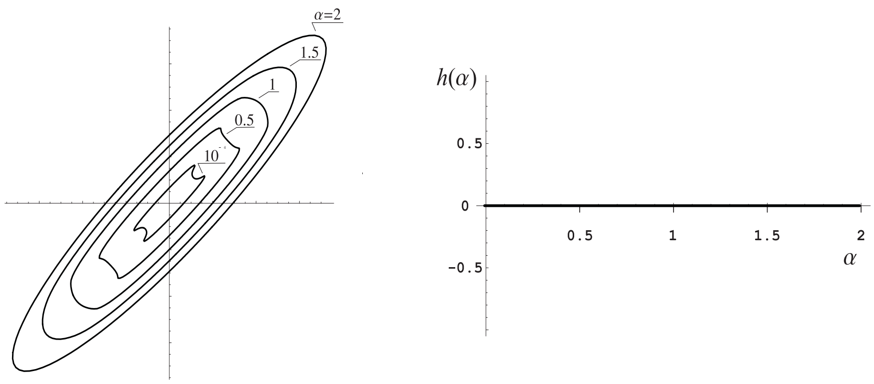


Figure 4.6: Level curves for the corresponding $S\alpha S$ random vector for different α's and the plot of the covariation ratio $h(\alpha)$

Example 2. The whole description is contained in Figures 4.7 and 4.8. The function f consists of four parabolic functions.

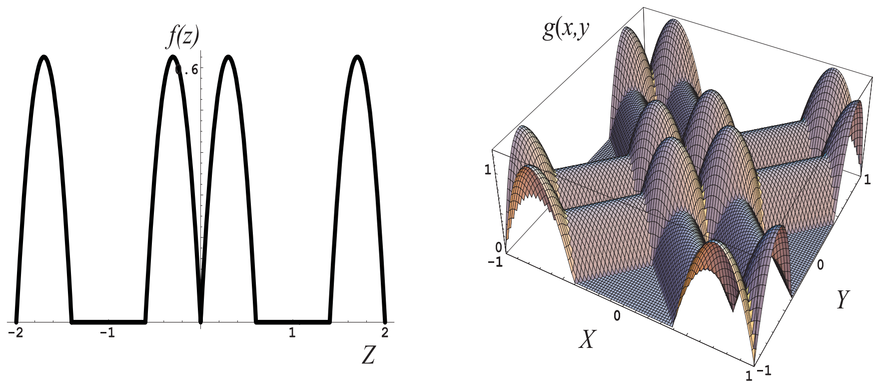


Figure 4.7: Another density function $f(x)$ and the corresponding density $g(x, y)$ defining the spectral measure on $\mathbb{R}^2$

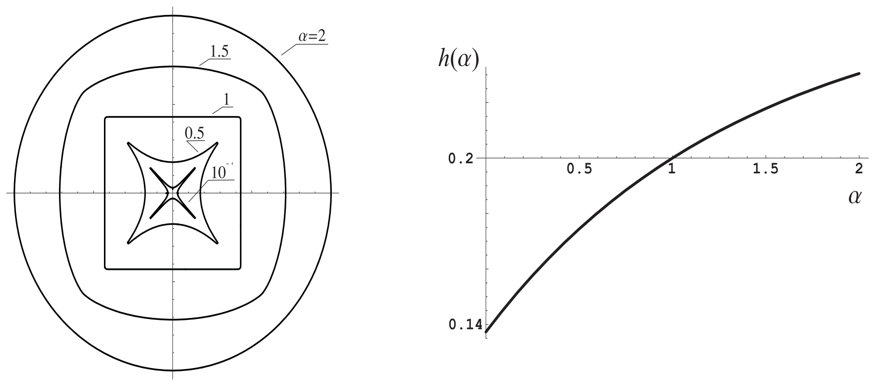


Figure 4.8: Level curves for the corresponding $S\alpha S$ random vector for different α's and the plot of the covariation ratio $h(\alpha)$

4.5 A series representation for $S\alpha S$ random vector

In the Section 7.4 of [?] R.B. Ash gave as a problem a simple method to obtain symmetric α-stable random variable as a weak limit of much simpler distributions. He started from independent, identically distributed random variables $X_1, \ldots, X_n$ with distribution uniform on the interval $[-n, n]$. Next he showed that the following random variable

$$Y_n = k \sum_{k=1}^{n} \frac{\operatorname{sign}(X_k)}{|X_k|^{1/\alpha}}$$

converges in distribution to a random variable with symmetric α-stable distribution. The construction of Ash can be extended to a series representation of symmetric α-stable random vector with given spectral measure. We start from the spectral representation of symmetric α-stable random vector $\mathbf{X}$:

$$\mathbf{E} \exp\{i < \xi, \mathbf{X} >\} = \exp\left\{ - \int \underset{\mathbb{R}^2}{\cdots} \int | < \xi, \mathbf{x} > |^{\alpha} \nu(d\mathbf{x}) \right\}.$$

We do not have to assume here that ν is the canonical spectral measure, bu we assume that $\int < \xi, \mathbf{x} >^2 \nu(dx) < \infty$ for every $\xi \in \mathbb{R}^n$.
Let $\lambda = \nu(\mathbb{R}^n)$. Then $\lambda^{-1}\nu(\cdot)$ is a probability measure on $\mathbb{R}^n$. Let X_k, $k \in \mathbb{N}$ be a sequence of independent random variables such that X_k has uniform distribution on the interval $[-k, k]$, and let $\mathbf{Y}_k$ be a sequence of independent, identically distributed random vectors with the distribution $\lambda^{-1}\nu(\cdot)$ such that (X_k) and $(\mathbf{Y}_k)$ are independent. Then the series

$$\mathbf{Z} = A \sum_{k=1}^{\infty} \mathbf{Y}_k \frac{\operatorname{sign}(X_k)}{|X_k|^{1/\alpha}}$$

converges a.s. to a random variable with the same distribution as $\mathbf{X}$. Here A is a positive constant such that

$$A^{\alpha} \int_0^{\infty} \left(1 - \cos y^{1/\alpha}\right) y^{-2} dy = \lambda.$$

First, calculate the following

$$\begin{aligned}
\mathbf{E} \exp\left\{ i < \xi, A\mathbf{Y}_k \frac{\operatorname{sign}(X_k)}{|X_k|^{1/\alpha}} \right\} &= \frac{1}{k} \int_0^k \mathbf{E} \cos\left(A < \xi, \mathbf{Y}_k > x^{-1/\alpha}\right) dx \\
&= 1 - \frac{1}{k} \mathbf{E} \int_0^k \left(1 - \cos\left(A < \xi, \mathbf{Y}_k > x^{-1/\alpha}\right)\right) dx \\
&= 1 - \frac{1}{k} A^{\alpha} \mathbf{E} | < \xi, \mathbf{Y}_k > |^{\alpha} \int_0^{\infty} \left(1 - \cos y^{1/\alpha}\right) y^{-2} dy + \frac{1}{k} h_k(\xi),
\end{aligned}$$

where

$$\begin{aligned}
h_k(\xi) &= A^{\alpha} \mathbf{E} | < \xi, \mathbf{Y}_k > |^{\alpha} \int_0^{A^{\alpha} | < \xi, \mathbf{Y}_k > |^{\alpha}/k} \left(1 - \cos y^{1/\alpha}\right) y^{-2} dy \\
&\leq A^{\alpha} \mathbf{E} | < \xi, \mathbf{Y}_k > |^{\alpha} \int_0^{A^{\alpha} | < \xi, \mathbf{Y}_k > |^{\alpha}/k} y^{2/\alpha - 2} dy \\
&= \frac{\alpha}{2 - \alpha} A^2 \mathbf{E} | < \xi, \mathbf{Y}_k > |^2 k^{1 - 2/\alpha}.
\end{aligned}$$

This means that $h_k(\xi) \to 0$ when $k \to \infty$. Now we see that

$$\begin{aligned}
&\mathbf{E}\exp\left\{i < \xi, A\sum_{k=n}^{\infty} \mathbf{Y}_k \frac{\operatorname{sign}(X_k)}{|X_k|^{1/\alpha}} >\right\} \\
&= \prod_{k=1}^{n}\left(1 - \frac{1}{k}\lambda\mathbf{E}| < \xi, \mathbf{Y}_k > |^{\alpha} + \frac{1}{k}h_k(\xi)\right) \\
&\overset{n\to\infty}{\longrightarrow} \mathbf{E}\exp\{i < \xi, \mathbf{X} >\}.
\end{aligned}$$

This shows that the series Z converges in distribution to a random variable with the same distribution as $\mathbf{X}$. The proof that Z converges almost surely requires some further justification, but this also can be done.

5

Weakly stable random vectors

The first four sections of this chapter are based on the paper of J. Misiewicz, K. Oleszkiewicz and K. Urbanik [57] published in 2005. The fifth section describing the idea of generalized convolution and generalized infinite divisibility comes from the paper J. Misiewicz [59] published in 2006. Since ideas and constructions in this chapter are not well known we give the detailed proofs of all basic properties of weakly stable distributions and generalized convolution.
In Section 5.6 we recall the characterization of the elliptically contoured distributions and show that their extreme points are weakly stable. We give also the method of simulation not only elliptically contoured random vectors, but also elliptically for contoured Lévy stochastic processes.
Section 5.7 is completely devoted to the very interesting class of weakly stable distributions defined by Cambanis, Keener and Simons [12] in 1983. We present here the results obtained by G. Mazurkiewicz in 2005 [48], thus we give the explicit formulas for k-dimensional marginal distributions and their interesting graphs. In spite of very complicated formulas for the densities, simulation of such random vectors is surprisingly simple.

5.1 Definition of weakly stable distributions

We will consider here the set of all mixtures of a fixed measure μ on $\mathbb{R}^n$, i.e.:

$$\mathcal{M}(\mu) = \{\mu \circ \lambda : \lambda \in \mathcal{P}\} = \mu \circ \mathcal{P}.$$

When it is more convenient we will write $(\widehat{\mu})$ instead of $\mathcal{M}(\mu)$. The corresponding set of characteristic functions we denote by

$$\Phi(\mu) = \{\widehat{\nu} : \nu = \mu \circ \lambda, \lambda \in \mathcal{P}\} = \left\{\varphi : \varphi(\xi) = \int \widehat{\mu}(\xi t)\lambda(dt), \lambda \in \mathcal{P},\ \xi \in \mathbb{R}^n\right\}.$$

The problem which is discussed here has a very elementary formulation. Namely, what is the characterization of those probability measures μ on $\mathbb{R}^n$, for which the set $\mathcal{M}(\mu)$ is closed under convolution?

Definition 5.1.1. *The random vector* $\mathbf{X}$ *and its distribution* μ *is weakly stable if the following condition holds:*

$$\forall \nu_1, \nu_2 \in \mathcal{M}(\mu) \qquad \nu_1 * \nu_2 \in \mathcal{M}(\mu) \tag{A}$$

Equivalently if

$$\forall \Theta_1, \Theta_2 \ \exists \Theta \qquad X\Theta_1 + X'\Theta_2 \stackrel{d}{=} X\Theta,$$

where X, X', Θ_1, Θ_2, Θ *are independent,* X *and* X' *are independent copies of* X.

The main result in this area states that the condition (A) is equivalent to the following:

$$\forall a, b \in \mathbb{R} \qquad T_a(\mu) * T_b(\mu) \in \mathcal{M}(\mu). \tag{B}$$

In the language of corresponding random vectors this condition can be formulated in the following way:

$$\forall a, b \in \mathbb{R} \ \exists \Theta = \Theta(a,b), \ X \text{ and } \Theta \text{ independent and } \quad aX + bX' \stackrel{d}{=} X\Theta.$$

Example 5.1.1. The class of symmetric distributions on $\mathbb{R}$ is closed under mixing and under convolution. It is easy to see that this class can be written as $\mathcal{M}(\tau)$ for $\tau = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$. Checking property (B) in this case is especially simple. In the language of characteristic functions we have

$$\begin{aligned} \widehat{\tau}(at)\widehat{\tau}(bt) &= \cos(at)\cos(bt) = \frac{1}{2}\cos\big((a+b)t\big) + \frac{1}{2}\cos\big((a-b)t\big) \\ &= \int_{\mathbb{R}} \cos(ts)\Big(\frac{1}{2}\delta_{a+b} + \frac{1}{2}\delta_{a-b}\Big)(ds), \end{aligned}$$

which means that for the measure λ we can take $\frac{1}{2}\delta_{a+b} + \frac{1}{2}\delta_{a-b}$. But there are many other possibilities, since, for the symmetric random vector X we know that if X and Θ are independent, then $X\Theta \stackrel{d}{=} X|\Theta|$. Thus the measure λ is not uniquely determined and the condition (B) holds for every λ_{pq}, $p, q \in [0, 1/2]$, where

$$\lambda_{pq} \stackrel{def}{=} p\delta_{a+b} + \Big(\frac{1}{2} - p\Big)\delta_{-a-b} + q\delta_{a-b} + \Big(\frac{1}{2} - q\Big)\delta_{b-a}.$$

It is easy to see that the set $K(\delta_a, \delta_b) = \{\lambda_{pq} : p, q \in [0, 1/2]\}$ is closed and convex. This property turns out to be general.

In [38, 67, 70–72] Kucharczak, Urbanik and Vol'kovich considered a very similar problem. They were studying properties of *weakly stable* random variables and measures, where a random variable X with distribution μ is said to be *weakly stable* if

$$\forall a, b > 0 \ \exists \lambda \qquad T_a\mu * T_b\mu = \mu \circ \lambda. \tag{C}$$

From now on we will say that the distribution μ for which the condition (C) holds, is $\mathbb{R}_+$*−weakly stable*, and we will say that μ is weakly stable when the condition (B) is satisfied for μ. The next example shows that these two conditions are not equivalent.

Example 5.1.2. Assume that a random vector X has a symmetric α-stable distribution μ with $\alpha \in (0, 2]$. This is equivalent with the property that for every $a, b \in \mathbb{R}$ we have $aX + bX' \stackrel{d}{=} cX$, where $c^\alpha = |a|^\alpha + |b|^\alpha$, so the condition (B) holds for $\lambda = \delta_c$. This is a little different from the definition of the strictly stable distribution of the random vector X, which is

$$\forall\, a, b > 0 \;\exists\, c > 0 \quad aX + bX' \stackrel{d}{=} cX. \tag{D}$$

Thus, a strictly stable distribution is $\mathbb{R}_+-$weakly stable, but it is not weakly stable. Symmetric stable distributions are both $\mathbb{R}_+-$weakly stable and weakly stable.

5.2 Conditions (A) and (B) are equivalent

Lemma 5.2.1. *Assume that a measure μ has property (B). Then, for every choice of discrete measures $\nu_1 = \sum_i p_i \delta_{a_i}$ and $\nu_2 = \sum_i q_i \delta_{b_i}$, the measure $(\mu \circ \nu_1) * (\mu \circ \nu_2)$ belongs to $\mathcal{M}(\mu)$.*

Proof. Let λ_{ij} be such that $T_{a_i}(\mu) * T_{b_j}(\mu) = \mu \circ \lambda_{ij}$. Then we have

$$\begin{aligned}(\mu \circ \nu_1) * (\mu \circ \nu_2) &= \sum_{i,j} p_i q_j T_{a_i}(\mu) * T_{b_j}(\mu) \\ &= \sum_{i,j} p_i q_j \mu \circ \lambda_{ij} = \mu \circ \Big(\sum_{i,j} p_i q_j \lambda_{ij}\Big).\end{aligned}$$

□

Lemma 5.2.2. *Let $\mu \neq \delta_0$ be a probability measure on $\mathbb{R}^n$ and let $\mathcal{A}$ be a subset of $\mathcal{P}$. If the set $\mathcal{B} = \{\mu \circ \lambda : \lambda \in \mathcal{A}\}$ is tight, then also the family $\mathcal{A}$ is tight.*

Proof. Let X and Q_λ independent, $\lambda \in \mathcal{A}$, be independent and such that $\mu = \mathcal{L}(X)$ and $\lambda = \mathcal{L}(Q_\lambda)$. Let ε be an arbitrary positive number. Since $\mathcal{B}$ is tight there exists a compact set $L \subset \mathbb{R}^n$ such that

$$\forall\, \lambda \in \mathcal{A} \qquad \mathbf{P}(Q_\lambda X \in L) \geq 1 - \varepsilon \mathbf{P}(X \neq 0).$$

Put $L_n = [-1/n, 1/n] \cdot L = \{sx; s \in [-1/n, 1/n], x \in L\}$. Since L is bounded and the sequence L_n is decreasing we have

$$\liminf_{n\to\infty} \mathbf{P}(X \notin L_n) \geq \mathbf{P}(X \neq 0).$$

Choose n such that $\mathbf{P}(X \notin L_n) \geq \mathbf{P}(X \neq 0)/2$. Then

$$\varepsilon \mathbf{P}(X \neq 0) \geq \mathbf{P}(Q_\lambda X \notin L) \geq \mathbf{P}(|Q_\lambda| > n, X \notin L_n) =$$

$$\mathbf{P}(|Q_\lambda| > n)\mathbf{P}(X \notin L_n) \geq \mathbf{P}(|Q_\lambda| > n)\mathbf{P}(X \neq 0)/2,$$

so that $\mathbf{P}(|Q_\lambda| > n) \leq 2\varepsilon$ for all $\lambda \in \mathcal{A}$. This implies tightness of $\mathcal{A}$. □

Lemma 5.2.3. *The set $\mathcal{M}(\mu)$ is closed in the topology of weak convergence and the set of extreme points of $\mathcal{M}(\mu)$ is $\{T_a(\mu) : a \in \mathbb{R}\}$.*

Proof. If $\mu = \delta_0$ then the assertion follows immediately, so we assume that $\mu \neq \delta_0$. Assume that $\mu \circ \lambda_n \Rightarrow \nu$. Then the set $\{\mu \circ \lambda_n : n \in \mathbb{N}\}$ is tight, and, by Lemma 2 the set $\{\lambda_n : n \in \mathbb{N}\}$ is also tight. Thus it contains a subsequence λ_{n_k} converging weakly to a probability measure λ on $\mathbb{R}$. Since the function $\widehat{\mu}(t)$ is bounded and continuous, we obtain

$$\int \widehat{\mu}(ts)\lambda_{n_k}(ds) \ \to \ \int \widehat{\mu}(ts)\lambda(ds).$$

On the other hand we have

$$\int \widehat{\mu}(ts)\lambda_n(ds) \ \to \ \widehat{\nu}(t).$$

This means that $\nu = \mu \circ \lambda$ and consequently $\nu \in \mathcal{M}(\mu)$. So $\mathcal{M}(\mu)$ is weakly closed.

If $a = 0$, then $T_a(\mu) = \delta_0$ and it is easy to check that δ_0 is an extreme point in $\mathcal{M}(\mu)$. Assume that for some $a \in \mathbb{R}$, $a \neq 0$, there exist $\lambda_1, \lambda_2 \in \mathcal{P}$ and $p \in (0,1)$ such that

$$T_a(\mu) = p\mu \circ \lambda_1 + (1-p)\mu \circ \lambda_2 = \mu \circ (p\lambda_1 + (1-p)\lambda_2).$$

This means that $aX \stackrel{d}{=} X\Theta$ for some random variable Θ independent of X with distribution $p\lambda_1 + (1-p)\lambda_2$. A result of Mazurkiewicz (see [49]) implies that $\mathbf{P}\{\Theta = a\} = 1$ if the distribution of X is not symmetric, and $\mathbf{P}\{|\Theta| = |a|\} = 1$ if X is symmetric. In the first situation we would have

$$\delta_a = p\lambda_1 + (1-p)\lambda_2,$$

so $\lambda_1 = \lambda_2 = \delta_a$ since δ_a is an extreme point in $\mathcal{P}$. If X has symmetric distribution we obtain that

$$\begin{aligned} \delta_{|a|}(A) &= p\lambda_1(A) + (1-p)\lambda_2(A) + p\lambda_1(-A) + (1-p)\lambda_2(-A) \\ &:= p|\lambda_1|(A) + (1-p)|\lambda_2|(A), \end{aligned}$$

for every Borel set $A \in (0,\infty)$. Since $\delta_{|a|}$ is an extreme point in $\mathcal{P}_+$, then $\delta_{|a|} = |\lambda_1| = |\lambda_2|$. Now, it is enough to notice that for symmetric distribution μ, the equality $\mu \circ \lambda = \mu \circ |\lambda|$ holds for every probability measure λ. Consequently, we obtain

$$T_a\mu = \mu \circ |\lambda_1| = \mu \circ \lambda_1 = \mu \circ |\lambda_2| = \mu \circ \lambda_2.$$

The above reasoning works for $\mu \in \mathcal{P}$. For $\mu \in \mathcal{P}_n$ the following two situations are possible. If μ is nonsymmetric then one can choose $\xi \in \mathbb{R}^n$ such that $<\xi, X>$ is nonsymmetric and use the result of Mazurkiewicz as before. If μ is symmetric then there exists $\xi \in \mathbb{R}^n$ such that $<\xi, X> \not\equiv 0$ since $\mu \neq \delta_0$, so that $\delta_{|a|} = |\lambda_1| = |\lambda_2|$, as before. The rest of the reasoning does not need any change.
Assume now that the probability measure ν is an extreme point of $\mathcal{M}(\mu)$. Then there exists a probability measure λ such that $\nu = \mu \circ \lambda$. If $\lambda \neq \delta_a$ for any $a \in \mathbb{R}$ then we could divide $\mathbb{R}$ into two Borel sets A and $A' = \mathbb{R} \setminus A$ such that $\lambda(A) = \alpha \in (0,1)$. Then

$$\mu = \alpha\, \mu \circ (\alpha^{-1} \lambda\big|_A) + (1-\alpha)\mu \circ ((1-\alpha)^{-1}\lambda\big|_{A'}),$$

in contradiction with the assumption that ν is extremal.

□

Lemma 5.2.4. *Assume that for a probability measure $\mu \neq \delta_0$ and some $\nu_1, \nu_2 \in \mathcal{P}$ the set*

$$K_\mu(\nu_1, \nu_2) \stackrel{def}{=} \{\lambda : (\mu \circ \nu_1) * (\mu \circ \nu_2) = \mu \circ \lambda\}$$

is not empty. Then $K_\mu(\nu_1, \nu_2)$ is convex and compact in the topology of weak convergence.

Proof. Notice that

$$\{(\mu \circ \nu_1) * (\mu \circ \nu_2)\} = \{\mu \circ \lambda : \lambda \in K_\mu(\nu_1, \nu_2)\},$$

and the set $\{(\mu \circ \nu_1) * (\mu \circ \nu_2)\}$ contains only one point. Then the compactness of $K_\mu(\nu_1, \nu_2)$ follows from Lemma 5.2.2. The convexity is trivial.

□

Lemma 5.2.5. *Assume that $\mu \neq \delta_0$ is a probability measure and $K_\mu(\nu_n^1, \nu_n^2) \neq \emptyset$ for every $n \in \mathbb{N}$, where $\nu_n^1 \to \nu_1$ weakly, $\nu_n^2 \to \nu_2$ weakly, and $\nu_n^i, \nu_i \in \mathcal{P}$. Then $K_\mu(\nu_1, \nu_2) \neq \emptyset$.*

Proof. Let $\mathcal{A} = \bigcup_{n=1}^\infty K_\mu(\nu_n^1, \nu_n^2)$ and

$$\mathcal{B} = \{\mu \circ \lambda : \lambda \in \mathcal{A}\} = \left\{\left(\mu \circ \nu_n^1\right) * \left(\mu \circ \nu_n^2\right) : n \in \mathbb{N}\right\}.$$

The set $\mathcal{B}$ is tight, thus it follows from Lemma 5.2.2 that the set $\mathcal{A}$ is also tight. Choosing now $\lambda_n \in K_\mu(\nu_n^1, \nu_n^2)$ for every $n \in \mathbb{N}$, we can find a subsequence λ_{n_k} converging weakly to a probability measure λ. Since

$$(\mu \circ \nu_{n_k}^1) * (\mu \circ \nu_{n_k}^2) = \mu \circ \lambda_{n_k},$$

then also

$$(\mu \circ \nu_1) * (\mu \circ \nu_2) = \mu \circ \lambda,$$

and consequently $\lambda \in K_\mu(\nu_1, \nu_2) \neq \emptyset$.

□

Theorem 5.2.1. *For every probability distribution μ properties (A) and (B) are equivalent.*

Proof. The implication $(A) \Rightarrow (B)$ is trivial. Assume that $\mu \neq \delta_0$ and the condition (B) holds for the measure μ. This means that for every $a, b \in \mathbb{R}$ the set $K_\mu(\delta_a, \delta_b)$ is not empty. It follows from Lemma 5.2.1 now that for every choice of discrete measures ν_1, ν_2 the set $K_\mu(\nu_1, \nu_2)$ is not empty. Consider now two probability measures $\lambda_1, \lambda_2 \in \mathcal{P}$. We can find two sequences of discrete measures $\nu_{1,n}$ and $\nu_{2,n}$ converging weakly to λ_1 and λ_2. Since for every $n \in \mathbb{N}$ the set $K_\mu(\nu_{1,n}, \nu_{2,n})$ is not empty, then it follows from Lemma 5.2.5 that also the set $K_\mu(\lambda_1, \lambda_2) \neq \emptyset$, which implies (A).

□

Proposition 5.2.1. *Let $X = (X_1, \ldots, X_n)$ be a symmetric α-stable random vector, and let random variable Θ be independent of X. Then $Y = X\Theta$ is weakly stable iff $|\Theta|^\alpha$ is $\mathbb{R}_+-$weakly stable.*

Proof. Notice that

$$aX\Theta + bX'\Theta' \stackrel{d}{=} \left(|a\Theta|^\alpha + |b\Theta'|^\alpha\right)^{1/\alpha} X,$$

where X', Θ' are independent copies of X, Θ such that X, X', Θ, Θ' are independent. Assume that Y is weakly stable. Since $X\Theta \stackrel{d}{=} X|\Theta|$ we obtain that

$$\left(|a\Theta|^\alpha + |b\Theta'|^\alpha\right)^{1/\alpha} X \stackrel{d}{=} X \cdot |\Theta| \cdot Q,$$

for some random variable Q. Without loss of generality we can assume that $Q \geq 0$. The symmetric stable distribution is cancellable (see [38], Prop.1.1) thus we obtain

$$|a|^\alpha|\Theta|^\alpha + |b|^\alpha|\Theta'|^\alpha \stackrel{d}{=} |\Theta|^\alpha Q^\alpha.$$

This implies that the random variable $|\Theta|^\alpha$ is $\mathbb{R}_+-$weakly stable. The reverse implication is trivial.

□

5.3 Symmetrizations of mixing measures are uniquely determined

Assume that a measure μ on $\mathbb{R}$ is weakly stable. As we have seen before, for every choice of $\nu_1, \nu_2 \in \mathcal{P}$, the set $K_\mu(\nu_1, \nu_2)$ is a nonempty convex and weakly compact set in $\mathcal{P}$. In this section we discuss further properties of $K_\mu(\nu_1, \nu_2)$. For a weakly stable measure μ we define

$$\Phi(\mu) = \{\widehat{\nu} : \nu = \mu \circ \lambda, \lambda \in \mathcal{P}\}.$$

Let $L(\mu)$ denote the complex linear space generated by $\Phi(\mu)$. Weak stability of μ implies that for every $f, g \in L(\mu)$ we have $fg, \overline{f} \in L(\mu)$. Since $\mu \circ \delta_0 = \delta_0$ then $L(\mu)$ contains constants.
By $\overline{\mathbb{R}} = \mathbb{R} \cup \{\Delta\}$ we denote the one-point compactification of the real line, by $\overline{\mathbb{R}_+} = \mathbb{R}_+ \cup \{\infty\}$ the one-point compactification of the nonnegative half-line. We will say that $f \in C\left(\overline{\mathbb{R}}\right)$ iff f is a continuous real function on $\overline{\mathbb{R}}$, and similarly we will say that $f \in C\left(\overline{\mathbb{R}_+}\right)$ iff f is a continuous real function on $\overline{\mathbb{R}_+}$. The set $C\left(\overline{\mathbb{R}_+}\right)$ can be identified with the set of even (symmetric) functions from $C\left(\overline{\mathbb{R}}\right)$. Now, for a probability measure μ, we define

$$A(\mu) = \left\{ f \in L(\mu) : f = \overline{f}, \lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) \right\}.$$

If the measure μ is weakly stable then $A(\mu)$ is a linear algebra, i.e. it is closed with respect to real linear combinations and products.

Lemma 5.3.1. *If a probability measure μ on $\mathbb{R}$ is not symmetric, then the set $A(\mu)$ separates points of $\overline{\mathbb{R}}$.*

Proof. Let γ be a symmetric Cauchy distribution with the Fourier transform $\widehat{\gamma}(t) = e^{-|t|}$. For every $c \in \mathbb{R}$, we define

$$h_c(t) = \left(\mu \circ (\gamma * \delta_c)\right)^{\wedge}(t) \in \Phi(\mu).$$

First we show that there exists $a \in \mathbb{R}$ such that $\Im m(h_a) \neq 0$. Assume the opposite, i.e. assume that for every $c \in \mathbb{R}$ we have $\Im m(h_c) \equiv 0$. This means that

$$\Im m\big(h_c(t)\big) = \int_{-\infty}^{\infty} e^{-|tx|} \sin(ctx)\mu(dx) = 0$$

for every $c, t \in \mathbb{R}$. Substituting $u = ct$, we obtain

$$\int_{-\infty}^{\infty} e^{-|ux|/|c|} \sin(ux)\mu(dx) = 0$$

for $u \in \mathbb{R}$ and $c \neq 0$. This implies that

$$\lim_{c\to\infty} \int_{-\infty}^{\infty} e^{-|ux|/|c|} \sin(ux)\mu(dx) = \int_{-\infty}^{\infty} \sin(ux)\mu(dx) = 0,$$

which means that the characteristic function $\widehat{\mu}$ is real which contradicts our assumption.
Now let $a, t_0 \in \mathbb{R}$ be such that $\Im m h_a(t_0) \neq 0$. For every $s \neq 0$, we define

$$g_s(t) = \Im m\left(h_a\left(\frac{t \cdot t_0}{s}\right)\right).$$

It is easy to see that $g_s(t) \in A(\mu)$, and $g_s(t) = -g_s(-t)$. We can see now that for every $r \in \mathbb{R}$, $r \neq 0$ the function $g_r(t)$ separates points r and $-r$ since

$g_r(r) = h_a(t_0) \neq g_r(-r)$. To finish the proof, it is enough to notice that the function

$$h_0(t) = \int_{-\infty}^{\infty} e^{-|tx|} \mu(dx)$$

separates points $t_1, t_2 \in \mathbb{R}$ if only $|t_1| \neq |t_2|$ including the case $t_i = \Delta$. □

Lemma 5.3.2. *If a probability measure μ on $\mathbb{R}$ is symmetric and $\mu \neq \delta_0$, then $A(\mu)$ separates points of $\overline{\mathbb{R}}_+$.*

Proof. It is enough to notice that the function $h_0(t) = \int e^{-|tx|}\mu(dx)$ separates points of $\overline{\mathbb{R}}_+$. □

Theorem 5.3.1. *If a weakly stable measure $\mu \neq \delta_0$ on $\mathbb{R}$ is not symmetric, then for every $\nu_1, \nu_2 \in \mathcal{P}$ the set $K_\mu(\nu_1, \nu_2)$ contains only one measure.*

Proof. Assume that $\lambda_1, \lambda_2 \in K_\mu(\nu_1, \nu_2)$. This means that $\mu \circ \lambda_1 = \mu \circ \lambda_2$, and consequently, for every $\lambda \in \mathcal{P}$,

$$(\mu \circ \lambda) \circ \lambda_1 = (\mu \circ \lambda) \circ \lambda_2.$$

Consequently, for every $\lambda \in \mathcal{P}$,

$$\int_{-\infty}^{\infty} (\mu \circ \lambda)^{\wedge}(tx)\lambda_1(dx) = \int_{-\infty}^{\infty} (\mu \circ \lambda)^{\wedge}(tx)\lambda_2(dx).$$

This implies that for every function $f \in A(\mu)$, the following equality holds:

$$\int_{-\infty}^{\infty} f(x)\lambda_1(dx) = \int_{-\infty}^{\infty} f(x)\lambda_2(dx). \qquad (*)$$

From Lemma 5.3.1 we know that the algebra $A(\mu)$ separates points of $\overline{\mathbb{R}}$, thus by the Stone-Weierstrass Theorem (see Theorem 4E in [46]), $A(\mu)$ is dense in $C(\overline{\mathbb{R}})$ in the topology of uniform convergence. This means that the equality $(*)$ holds for every $f \in C(\overline{\mathbb{R}})$, and consequently $\lambda_1 = \lambda_2$. □

Let $\tau = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$. By the symmetrization of the measure $\lambda \in \mathcal{P}$ we understand the measure $\lambda \circ \tau$. Notice that the measure λ is symmetric if and only if $\lambda = \lambda \circ \tau$.

Theorem 5.3.2. *If a weakly stable measure $\mu \neq \delta_0$ on $\mathbb{R}$ is symmetric and $\nu_1, \nu_2 \in \mathcal{P}$, then*

$$\lambda_1, \lambda_2 \in K_\mu(\nu_1, \nu_2) \quad \Longrightarrow \quad \lambda_1 \circ \tau = \lambda_2 \circ \tau.$$

If $\lambda_1 \circ \tau = \lambda_2 \circ \tau$ and $\lambda_1 \in K_\mu(\nu_1, \nu_2)$ then $\lambda_2 \in K_\mu(\nu_1, \nu_2)$.

Proof. The second implication is trivial because for every symmetric measure μ we have $\mu \circ \lambda = \mu \circ (\lambda \circ \tau)$. To prove the first implication assume that $\lambda_1, \lambda_2 \in K_\mu(\nu_1, \nu_2)$. This implies that $\mu \circ \lambda_1 = \mu \circ \lambda_2$, and consequently $(\mu \circ \lambda) \circ \lambda_1 = (\mu \circ \lambda) \circ \lambda_2$ for every $\lambda \in \mathcal{P}$. This means that for every even function $f \in A(\mu)$ the following equality holds:

$$\int_0^\infty f(x)(\tau \circ \lambda_1)(dx) = \int_0^\infty f(x)(\tau \circ \lambda_2)(dx). \qquad (**)$$

It follows from the proof of Lemma 5.3.2 that even functions from $A(\mu)$ separate points in $\overline{\mathbb{R}}_+$. Applying the Stone-Weierstrass Theorem again we conclude that the set of even functions from $A(\mu)$ is dense in $C(\overline{\mathbb{R}}_+)$ in the topology of uniform convergence. This means that the equality $(**)$ holds for every function $f \in C(\overline{\mathbb{R}}_+)$. This implies that measures $\tau \circ \lambda_1$ and $\tau \circ \lambda_2$ coincide on $\mathbb{R}_+$, and, by their symmetry, also on $\mathbb{R}$.

□

Remark 5.3.1. Notice that it follows from the proofs of Theorem 5.3.1 and Theorem 5.3.2, that weakly stable distributions are reducible in the sense that:

- If X, Y, Z are independent real random variables and X is nonsymmetric and weakly stable then the equality $XY \stackrel{d}{=} XZ$ implies $\mathcal{L}(Y) = \mathcal{L}(Z)$.
- If X, Y, Z are independent, Y, Z are real, and X is a nonsymmetric weakly stable random vector taking values in $\mathbb{R}^n$ then the equality $XY \stackrel{d}{=} XZ$ implies $\mathcal{L}(Y) = \mathcal{L}(Z)$. To see this it is enough to apply the previous remark to the random variable $< \xi, X >$, where $\xi \in \mathbb{R}^n$ is such that $< \xi, X >$ is not symmetric.
- If X, Y, Z are independent, Y, Z take values in $\mathbb{R}^n$ and X is a nonsymmetric weakly stable real random variable then the equality $XY \stackrel{d}{=} XZ$ implies $\mathcal{L}(Y) = \mathcal{L}(Z)$. To see this note that it suffices to prove $< \xi, Y > \stackrel{d}{=} < \xi, Z >$ for all $\xi \in \mathbb{R}^n$.
- If X, Y, Z are independent, and $X \not\equiv 0$ is symmetric weakly stable then the equality $XY \stackrel{d}{=} XZ$ implies $\mathcal{L}(Y) \circ \tau = \mathcal{L}(Z) \circ \tau$.

Remark 5.3.2. Notice that if μ is weakly stable then also $\mu \circ \tau$ is weakly stable. Indeed, if $T_a\mu * T_b\mu = \nu_1 \circ \mu$ and $T_a\mu * T_{-b}\mu = \nu_2 \circ \tau$ then

$$T_a(\mu \circ \tau) * T_b(\mu \circ \tau) = \left(\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2\right) \circ (\mu \circ \tau) .$$

5.4 Some general properties of weakly stable distributions

Lemma 5.4.1. *If a measure μ on $\mathbb{R}$ is weakly stable then $\mu(\{0\}) = 0$ or 1.*

Proof. Let X be a weakly stable variable such that $\mathcal{L}(X) = \mu$, $\mathbf{P}\{X = 0\} = p < 1$, and let X' be its independent copy. We define the random variable Y with distribution $\mathcal{L}(X|X \neq 0)$ and Y' its independent copy. The random variable Y/Y' has at most countably many atoms, so there exists $a \in \mathbb{R}$, $a \neq 0$, such that $\mathbf{P}\{Y = aY'\} = 0$. Now let Θ be a random variable independent of X, such that

$$X - aX' \stackrel{d}{=} X\Theta.$$

Then we have

$$p \leq \mathbf{P}\{X\Theta = 0\} = \mathbf{P}\{X - aX' = 0\} = p^2 + (1-p)^2\mathbf{P}\{Y - aY' = 0\} = p^2.$$

This holds only if $p = 0$, which ends the proof. □

Lemma 5.4.2. *Assume that a weakly stable probability measure $\mu \neq \delta_0$ on $\mathbb{R}$ has at least one atom. Then the discrete part of μ (normalized to a probability measure) is also weakly stable.*

Proof. Let $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$, $\alpha \in (0,1)$, where $\alpha\mu_1$ is the discrete part of the measure μ, $\mu_1(\mathbb{R}) = 1$, and μ_2 is such that $\mu_2(\mathbb{R}) = 1$ and $\mu_2(\{x\}) = 0$ for every $x \in \mathbb{R}$. If μ is weakly stable, then for every $a \in \mathbb{R}$ there exists a probability measure λ such that $\mu * T_a\mu = \mu \circ \lambda$. Now we have

$$\begin{aligned} \mu * T_a\mu &= \alpha^2\mu_1 * T_a\mu_1 + \alpha(1-\alpha)\mu_1 * T_a\mu_2 \\ &+ \alpha(1-\alpha)\mu_2 * T_a\mu_1 + (1-\alpha)^2\mu_2 * T_a\mu_2. \end{aligned}$$

Clearly for $a \neq 0$ the discrete part of $\mu * T_a\mu$ is equal to $\alpha^2\mu_1 * T_a\mu_1$. On the other hand we have

$$\mu \circ \lambda = (1-\beta)\mu \circ \lambda_2 + \alpha\beta\mu_1 \circ \lambda_1 + (1-\alpha)\beta\mu_2 \circ \lambda_1,$$

where $\lambda_1(\mathbb{R}) = \lambda_2(\mathbb{R}) = 1$, λ_1 is a discrete measure, $\lambda_2(\{x\}) = 0$ for every $x \in \mathbb{R}$ and $\lambda = \beta\lambda_1 + (1-\beta)\lambda_2$.
Let $S = \{a \in \mathbb{R} : \mu * T_a\mu(\{0\}) = 0\}$. If $a \in S$, $a \neq 0$, then $\lambda(\{0\}) = 0$ and $\mu \circ \lambda_2(\{x\}) = \beta\mu_2 \circ \lambda_1(\{x\}) = 0$, for every $x \in \mathbb{R}$, so

$$\alpha^2\mu_1 * T_a\mu_1 = \alpha\beta\mu_1 \circ \lambda_1.$$

This means that $\alpha = \beta$ and $\mu_1 * T_a\mu_1 = \mu_1 \circ \lambda_1$.
If $a \notin S$ then there exists a sequence $a_n \in S \setminus \{0\}$, $n \in \mathbb{N}$ such that $\lim_n a_n = a$. Then $\mu * T_{a_n}\mu \Rightarrow \mu * T_a\mu$ and $\mu_1 * T_{a_n}\mu_1 \Rightarrow \mu_1 * T_a\mu_1$. For every $n \in \mathbb{N}$ there exists λ_n such that $\mu_1 * T_{a_n}\mu_1 = \mu_1 \circ \lambda_n$, i.e. $\lambda_n \in K_{\mu_1}(\delta_1, \delta_{a_n})$. In view of Lemma 5.2.5 there exists $\lambda \in K_{\mu_1}(\delta_1, \delta_a)$ which ends the proof. □

Theorem 5.4.1. *Assume that a random vector X taking values in $\mathbb{R}^n$ and having distribution μ is such that $\mathbf{E}\|X\| < \infty$ and $\mathbf{E}X = a \neq 0$. Then μ is weakly stable if and only if $\mu = \delta_a$.*

Proof. Assume first that $n = 1$. If $\mu = \delta_a$ for some $a \neq 0$, then all the conditions of the theorem are satisfied. Now let μ be weakly stable and $\mathbf{E}X = a \neq 0$. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with distribution μ. The Weak Law of Large Numbers implies that

$$\frac{1}{n}\sum_{k=1}^{n} X_k \longrightarrow a$$

weakly when $n \to \infty$. The measure μ is weakly stable thus for every $n \in \mathbb{N}$ there exists a measure ν_n such that

$$\mu_n = \mathcal{L}\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right) = \left(T_{1/n}\mu\right)^{*n} = \mu \circ \nu_n.$$

Since $\mu_n \Rightarrow \delta_a$, it follows from Lemma 5.2.2 that the family $\{\nu_n\}$ is tight and it contains a sequence ν_{n_k} such that $\nu_{n_k} \Rightarrow \nu$ for some probability measure ν. Now, we obtain

$$\delta_a = \lim_{n\to\infty} \mu_n = \lim_{k\to\infty} \mu \circ \nu_{n_k} = \mu \circ \nu.$$

Since $a \neq 0$ the last equality is possible only if $\mu = \delta_x$ and $\nu = \delta_y$ for some $x, y \in \mathbb{R}$, $xy = a$. Since $\mathbf{E}X = a$, we conclude that $\mu = \delta_a$.
If X, $\mathbf{E}X = a \neq 0$, is a random vector in $\mathbb{R}^n$, then for each $\xi \in \mathbb{R}^n$ it follows from the previous considerations that $\mathbf{P}\{< \xi, X > = < \xi, a >\} = 1$. Consequently $\mathbf{P}\{X = a\} = 1$.

□

Theorem 5.4.2. *Assume that for a weakly stable measure $\mu \neq \delta_0$ on $\mathbb{R}^n$ there exists $\varepsilon \in (0, 1]$ such that for every $\xi \in \mathbb{R}^n$ and every $p \in (0, \varepsilon)$*

$$\int_{\mathbb{R}^n} | < \xi, x > |^p \mu(dx) < \infty.$$

Then $\mathcal{M}(\mu)$ contains strictly α-stable measures for every $\alpha \in (0, \alpha_0)$ for some $\alpha_0 \in [\varepsilon, 2]$.

Proof. Let $p \in (0, \varepsilon)$. The set $\mathcal{M}(\mu)$ is closed under scale mixing, thus for every $n \in \mathbb{N}$ the measure $\mu \circ m_n \in \mathcal{M}(\mu)$, where

$$m_n(dx) = c(n)x^{-p-1}\mathbf{1}_{(1/n,\infty)}dx, \qquad c(n) = pn^{-p}.$$

The set $\mathcal{M}(\mu)$ is also closed under convolution and convex linear combinations, and it is weakly closed thus for every $n \in \mathbb{N}$

$$\nu_n = \exp\{c(n)^{-1}(\mu \circ m_n)\} \in \mathcal{M}(\mu),$$

where $\exp(\kappa) \stackrel{def}{=} e^{-\kappa(\mathbb{R}^n)} \sum_{k=0}^{\infty} \kappa^{*k}/k!$ for every finite measure κ on $\mathbb{R}^n$. Notice that for every $\xi \in \mathbb{R}^n$ we have

$$\begin{aligned}\widehat{\nu_n}(\xi) &= \exp\left\{-\int_{\mathbb{R}^n}\int_{1/n}^{\infty}\left(1 - e^{i\xi(sx)}\right) s^{-p-1} ds \mu(dx)\right\} \\ &= \exp\left\{-\int_{\mathbb{R}^n} |<\xi, x>|^p \int_{|<\xi,x>|/n}^{\infty}\left(1 - e^{iu\mathrm{sgn}(<\xi,x>)}\right) u^{-p-1} du\, \mu(dx)\right\}.\end{aligned}$$

Let $h(u) = \left(1 - e^{iu\mathrm{sgn}(<\xi,x>)}\right) u^{-p-1}$. The function $h(u)$ is integrable on $[0,\infty)$ since $p \in (0,1)$ and $|h(u)| = 2\left|\sin\frac{u}{2}\right| u^{-p-1}$, thus $|h(u)| \leq u^{-p}$ for $u < 1$ and $|h(u)| \leq 2u^{-p-1}$ for $u \geq 1$. This implies that the function

$$H_p(<\xi, x>) = \int_0^{\infty}\left(1 - e^{iu\mathrm{sgn}(<\xi,x>)}\right) u^{-p-1} du$$

is well defined and bounded on $\mathbb{R}^n$ thus we can write

$$\widehat{\nu_n}(\xi) \to \exp\left\{-\int_{\mathbb{R}^n} |<\xi, x>|^p H_p(<\xi, x>)\mu(dx)\right\} \stackrel{def}{=} \widehat{\gamma_p}(\xi).$$

It is easy to see now that the function γ_p is the characteristic function of a strictly $p-$stable random variable and the corresponding measure γ_p belongs to $\mathcal{M}(\mu)$ since this class is weakly closed. Now we define

$$\alpha_0 = \sup\Big\{\alpha \in (0,2] : \mathcal{M}(\mu) \text{ contains strictly } \alpha\text{-stable measure }\Big\}.$$

To end the proof it is enough to remind that for every $0 < \beta < \alpha \leq 2$ and every strictly α-stable measure γ_α we have that the measure $\gamma_\alpha \circ \lambda_{\beta/\alpha}$ is strictly β-stable, where $\lambda_{\beta/\alpha}$ is the distribution of the random variable $\Theta_{\beta/\alpha}^{1/\alpha}$, for $\Theta_{\beta/\alpha} \geq 0$ such that $\mathbf{E}\exp\{-t\Theta_{\beta/\alpha}\} = \exp\{-t^{\beta/\alpha}\}$.

□

Remark 5.4.1. Notice that if a weakly stable measure $\mu \neq \delta_0$ on $\mathbb{R}^n$ is such that $\int |<\xi, x>|^p \mu(dx) < \infty$ for every $\xi \in \mathbb{R}^n$ and $p \in (0,\varepsilon)$ for some $\varepsilon \in (0,2]$ then $\mathcal{M}(\mu)$ contains symmetric p-stable measures for every $p \in (0,\varepsilon)$.
To see this it is enough to notice that for symmetric measure μ the measure ν_n constructed in the proof of Theorem 5.4.2 is also symmetric. Consequently

$$\widehat{\nu_n}(\xi) = \exp\left\{-\int_{\mathbb{R}^n} |<\xi, x>|^p \int_{|<\xi,x>|/n}^{\infty} (1 - \cos u)\, u^{-p-1} du\, \mu(dx)\right\}.$$

Let $h(u) = (1-\cos u)u^{-p-1}$. Then $|h(u)| < u^{1-p}$ for $u < 1$, and $|h(u)| < 2u^{-p-1}$ for $u > 1$ so the function $h(u)$ is integrable on $[0, \infty)$ for every $p \in (0, 2)$. For the constants

$$H_p = \int_0^\infty (1 - \cos u)u^{-p-1} du$$

we obtain that

$$\widehat{\nu_n}(\xi) \longrightarrow \exp\left\{-H_p \int_{\mathbb{R}^n} |<\xi, x>|^p \,\mu(dx)\right\},$$

which is the characteristic function of a symmetric p-stable random vector. If the measure μ is not symmetric then we shall use in this construction the measure $\mu \circ \tau$ instead of μ. It is possible since $\mu \circ \tau$ is symmetric, belongs to $\mathcal{M}(\mu)$ and it has the same moments as the measure μ.

Remark 5.4.2. If in the situation described in Remark 5.4.1 we have $n = 1$ then $\mathcal{M}(\mu)$ contains also a symmetric ε-stable random variable. It follows from Remark 5.4.1 that

$$\exp\left\{-H_p \int_{\mathbb{R}} |tx|^p \,\mu(dx)\right\} = \exp\left\{-|t|^p H_p \int_{\mathbb{R}} |x|^p \,\mu(dx)\right\}$$

is the characteristic function of some measure from $\mathcal{M}(\mu)$. Since rescaling is admissible we have also that $\exp\{-|t|^p\}$ is the characteristic function of some measure from $\mathcal{M}(\mu)$. Now it is enough to notice that

$$\lim_{p \nearrow \varepsilon} \exp\{-|t|^p\} = \exp\{-|t|^\varepsilon\},$$

and use Lemma 5.2.3.

Remark 5.4.3. There exist measures μ such that $\mu \circ \nu = \gamma_\alpha$, where γ_α is symmetric α-stable, for some probability measure ν but μ is not weakly stable. Any measure of the form $\mu = q\delta_{-1} + (1-q)\delta_1$ for $q \in (0,1) \setminus \{\frac{1}{2}\}$ can serve as an example.

Lemma 5.4.3. *Let X be a real random variable with distribution μ. If μ is weakly stable and μ is supported on a finite set then either there exists $a \in \mathbb{R}$ such that $\mu = \delta_a$ or there exists $a \neq 0$ such that $\mu = \frac{1}{2}\delta_a + \frac{1}{2}\delta_{-a}$.*

Proof. Let X' be an independent copy of X. Assume that $\mu \neq \delta_a$ for all $a \in \mathbb{R}$. Theorem 5.4.1 implies that X must take on both negative and positive values with positive probability. Let $V = \{x \in \mathbb{R} : \mu(\{x\}) > 0\}$. By Lemma 5.4.1 we have $0 \notin V$. Let b be the greatest and let $-a$ be the least element of V. Certainly, $a, b > 0$. We will prove first that $a = b$.
Assume that $b > a$. For $\lambda \in \mathbb{R}$ let us define the set of the values taken on by the random variable $X - \lambda X'$ with positive probability: $V_\lambda = \{v - \lambda w : v, w \in V\}$. Clearly, for $\lambda \in (0,1)$ the greatest element of V_λ is equal to $b + \lambda a$, whereas $-(a + \lambda b)$ is the least element of V_λ. Moreover $a + \lambda b < b + \lambda a$ (hence $b + \lambda a$

has strictly the greatest value among all elements of V_λ). Since μ is weakly stable there exists a real random variable Y_λ independent of X and such that $Y_\lambda X \stackrel{d}{=} X - \lambda X'$. One can easily see that Y_λ is also finitely supported. We have $b + \lambda a \in V_\lambda$ so that there exist $c, d \neq 0$ such that $\mathbf{P}\{Y_\lambda = c\} > 0$, $d \in V$ and $cd = b + \lambda a$. Also for any $d' \in V$ there is $cd' \in V_\lambda$, so that $|d'| > |d|$ would imply $|cd'| > b + \lambda a$, contrary to the fact that $b + \lambda a$ has the maximal absolute value out of all elements of V_λ. Hence d must have maximal absolute value among all elements of V and therefore $d = b$ so that $c = 1 + \lambda a/b$.

We deduce that $-\frac{a}{b}(b + \lambda a) = c \cdot (-a) \in V_\lambda$ and therefore there exist $v, w \in V$ such that $-\frac{a}{b}(b + \lambda a) = v - \lambda w$, and consequently $\lambda(w - \frac{a^2}{b}) = v + a$. Let us assume that $a^2/b \notin V$. Then the last equation may be satisfied for finitely many values of the parameter λ only (because v and w can be chosen from a finite set only). It was proved for all $\lambda \in (0, 1)$, however. Hence $a^2/b \in V$. Therefore $\frac{a^2}{b^2}(b + \lambda a) = c \cdot a^2/b \in V_\lambda$ and again, there exist $v, w \in V$ such that $\frac{a^2}{b^2}(b + \lambda a) = v - \lambda w$ so that $\lambda(w + \frac{a^3}{b^2}) = v - \frac{a^2}{b}$. Like before we infer that $-\frac{a^3}{b^2} \in V$. By iterating this reasoning we prove that for every $k \in \mathbb{N}$ there is $(-1)^{k+1} a^{k+1}/b^k \in V$. Since $0 < a/b < 1$ this implies that V contains an infinite subset, contradicting our assumptions. The case $a > b$ is excluded in a similar way. Hence $a = b$.

Now, let $-\alpha$ be the greatest negative element of V and let β denote the least positive element of V. Let us consider $X - \lambda X'$ for $0 < \lambda < \min(\alpha, \beta)/a$. Certainly, the least positive element of V_λ is equal to $\beta - \lambda a$ whereas $-(\alpha - \lambda a)$ is the greatest negative element of V_λ. Let us assume without loss of generality that $\beta \leq \alpha$ so that $\beta - \lambda a$ has the least absolute value among all elements of V_λ (otherwise one can consider $-X$ instead of X). Again, we choose Y_λ and parameters $c, d \neq 0$ such that $\mathbf{P}\{Y_\lambda = c\} > 0$, $d \in V$ and $cd = \beta - \lambda a$. We obtain $d \in \{-\alpha, \beta\}$ with a similar reasoning as before - no element can be both at the same side of zero as d and closer to zero than d because multiplying by c we would get a positive element of V_λ less than $(\beta - \lambda a)$. Hence $c \in \{(\beta - \lambda a)/\beta, -(\beta - \lambda a)/\alpha\}$. However, $ca \in V_\lambda$ so that there exist $v, w \in V$ such that $ca = v - \lambda w$, which means that $\lambda(w - \frac{a^2}{\beta}) = v - a$ or $\lambda(w + \frac{a^2}{\alpha}) = v + \frac{a\beta}{\alpha}$. Since we proved this alternative for infinitely many λ's and we know that v and w can have only finitely many values we infer that $a^2/\beta \in V$ (if $d = \beta$) or $-a^2/\alpha \in V$ (if $d = -\alpha$). We have proved that $V \subset [-a, a]$, so that $\beta = a$ if $d = \beta$, or $\alpha = a$ if $d = -\alpha$. Anyway, $|d| = a$ so that $|c| = \frac{\beta}{a} - \lambda$. Since $\{-a, a\} \subset V$ we have $\{-ca, ca\} \subset V_\lambda$ and therefore also $-(\beta - \lambda a) \in V_\lambda$. We have assumed though that $\beta - \lambda a$ has the least absolute value among all elements of V_λ so in particular $-(\alpha - \lambda a) \leq -(\beta - \lambda a)$. Since $-(\alpha - \lambda a)$ is the greatest negative element of V_λ we have also $-(\alpha - \lambda a) \geq -(\beta - \lambda a)$. Hence $\alpha = \beta$.

We have proved earlier that $\alpha = a$ or $\beta = a$, so finally $\alpha = \beta = a$ and the support of μ is equal $\{-a, a\}$. Theorem 5.4.1 implies that μ is symmetric. □

Lemma 5.4.4. *Let X be a real random variable with distribution $\mu \neq \delta_0$ and let X' be an independent copy of X. Assume that μ is weakly stable, so that for any $\lambda \in \mathbb{R}$ there exists a real random variable Y_λ independent of X such that $X - \lambda X' \stackrel{d}{=} Y_\lambda X$. If X is symmetric we additionally assume that $Y_\lambda \geq 0$ a.s. Then the map*

$$\lambda \mapsto \mathcal{L}(Y_\lambda)$$

is well defined and continuous on $\mathbb{R}$.

Proof. The existence and the uniqueness of distribution of the random variable Y_λ follows from Theorem 5.3.1 and Theorem 5.3.2. We only need to prove that $\lambda_n \to \lambda$ implies that $Y_{\lambda_n} \xrightarrow{d} Y_\lambda$ when $n \to \infty$. Let us assume it is not true. Then we can find $\varepsilon > 0$ and a subsequence $\{n_k\}$ such that for any k the law of $Y_{\lambda_{n_k}}$ is ε-separated from the law of Y_λ in Lévy's metric. Since

$$Y_{\lambda_{n_k}} X \stackrel{d}{=} X - \lambda_{n_k} X' \xrightarrow{d} X - \lambda X' \stackrel{d}{=} Y_\lambda X,$$

by Lemma 5.2.2 we can choose a subsequence $\{n_{k_l}\} \subset \{n_k\}$ such that $Y_{\lambda_{n_{k_l}}} \xrightarrow{d} Z$ for some real random variable Z as $l \to \infty$. Hence $\mathcal{L}(Z) \neq \mathcal{L}(Y_\lambda)$. Moreover $Z \geq 0$ a.s. if X is symmetric because then all Y_{λ_n}'s are nonnegative a.s.
On the other hand, $Z'X \stackrel{d}{=} Y_\lambda X$, where Z' is a copy of Z independent of X since the map $\lambda \mapsto \mathcal{L}(X - \lambda X')$ is continuous. Therefore $Y_\lambda \stackrel{d}{=} Z' \stackrel{d}{=} Z$, by Theorem 5.3.1 and Theorem 5.3.2 (or by Remark 5.3.1). The obtained contradiction ends the proof. □

Remark 5.4.4. Let $\alpha \in [1, 2]$. Note that if a random variable X with a weakly stable distribution μ is such that $\mathbf{E}|X|^p < \infty$ for all $p \in (0, \alpha)$ then

$$1 + |\lambda| \geq \begin{cases} |Y_\lambda| \text{ a.s.} & \text{if} \quad \alpha = 2; \\ \|Y_\lambda\|_\alpha & \text{if} \quad \alpha < 2. \end{cases}$$

Indeed, by Theorem 5.4.2 there exists Θ independent of X such that $X\Theta$ is strictly α-stable. If $\alpha < 2$ then $\mathbf{E}|X\Theta|^\beta < \infty$ for every $\beta < \alpha$, thus $\mathbf{E}|X|^\beta < \infty$ for every $\beta < \alpha$. If $\alpha = 2$ then $X\Theta$ is Gaussian so $\mathbf{E}|X|^\beta < \infty$ for every $\beta > 0$. Now it is enough to notice that for $\beta \geq 1$ we have

$$\|Y_\lambda\|_\beta \|X\|_\beta = \|Y_\lambda X\|_\beta = \|X - \lambda X'\|_\beta \leq \|X\|_\beta + |\lambda| \|X'\|_\beta = (1 + |\lambda|)\|X\|_\beta.$$

The case $\beta = \alpha$ can be obtained by observing that $\|Y_\lambda\|_\alpha = \lim_{\beta \to \alpha^-} \|Y_\lambda\|_\beta$. If $\alpha = 2$ the inequality holds for all $\beta \geq 1$, which implies that $\|Y_\lambda\|_\infty \leq 1 + |\lambda|$.

Lemma 5.4.5. *Let X be a real random variable with distribution μ. If μ is weakly stable and μ is supported on a countable set then there exists $a \in \mathbb{R}$ such that $\mu = \delta_a$ or there exists $a \neq 0$ such that $\mu = \frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_a$.*

Proof. Assume that the support of μ is an infinite countable set. For $\lambda \in (0,1)$ we have $X - \lambda X' \stackrel{d}{=} Y_\lambda X$, where X' and Y_λ are defined as in the preceding lemma (so that if X is symmetric then $Y_\lambda \geq 0$ a.s.). By Lemma 5.4.4 $Y_\lambda \xrightarrow{d} Y_0 = 1$ as $\lambda \to 0$. Let

$$\mu = \sum_{n=1}^{\infty} p_n \delta_{x_n},$$

where x_n's are nonzero (by Lemma 5.4.1) and pairwise different, and $(p_n)_{n=1}^{\infty}$ is a non-increasing sequence of positive numbers. Let

$$M = \left\{ \frac{x_i - x_j}{x_k - x_l} : k \neq l \right\}.$$

Certainly, M is a countable set. We see that for $\lambda \notin M$ the equality $x_k - \lambda x_i = x_l - \lambda x_j$ implies $i = j$ and $k = l$. Finally, let $N \in \mathbb{N}$ be such that $\sum_{n>N} p_n \leq p_1^2/2$. Then for $\lambda \in (0,1) \setminus M$ we have

$$\begin{aligned} p_1^2 &= \mathbf{P}\{X = x_1, X' = x_1\} = \mathbf{P}\{X - \lambda X' = x_1 - \lambda x_1\} \\ &= \mathbf{P}\left\{Y_\lambda X = x_1(1-\lambda)\right\} = \sum_{n=1}^{\infty} \mathbf{P}\left\{Y_\lambda = \frac{x_1}{x_n}(1-\lambda)\right\} \cdot p_n \\ &\leq \mathbf{P}\left\{Y_\lambda = 1-\lambda\right\} \cdot p_1 + \frac{p_1^2}{2} + \sum_{n=2}^{N} \mathbf{P}\left\{\frac{Y_\lambda}{1-\lambda} = \frac{x_1}{x_n}\right\} \cdot p_n, \end{aligned}$$

and the summands for $2 \leq n \leq N$ tend to zero as $\lambda \to 0$ (since $\frac{Y_\lambda}{1-\lambda} \xrightarrow{d} 1$) so that

$$\liminf_{\lambda \to 0; \lambda \in (0,1)\setminus M} \mathbf{P}\left\{Y_\lambda = 1 - \lambda\right\} \geq p_1/2.$$

On the other hand, for $\lambda \in (0,1) \setminus M$ and $k \in \mathbb{N}$ we have

$$\begin{aligned} p_k^2 &= \mathbf{P}\left\{X = x_k, X' = x_k\right\} = \mathbf{P}\left\{X - \lambda X' = x_k(1-\lambda)\right\} \\ &= \mathbf{P}\left\{Y_\lambda X = x_k(1-\lambda)\right\} \geq \mathbf{P}\left\{Y_\lambda = 1-\lambda\right\} \cdot p_k, \end{aligned}$$

so that

$$\limsup_{\lambda \to 0; \lambda \in (0,1)\setminus M} \mathbf{P}\left\{y_\lambda = 1 - \lambda\right\} \leq p_k.$$

Hence $p_k \geq p_1/2$ for any $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} p_k = \infty$ which clearly is not possible. The obtained contradiction proves that μ has finite support and the assertion follows from Lemma 5.4.3.

□

Theorem 5.4.3. *Let μ be a weakly stable probability measure on $\mathbb{R}^n$. Then either there exists $a \in \mathbb{R}^n$ such that $\mu = \delta_a$ or there exists $a \in \mathbb{R}^n \setminus \{0\}$ such that $\mu = \frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_a$, or $\mu(\{a\}) = 0$ for all $a \in \mathbb{R}^n$.*

Proof. Assume first that $n = 1$. One can express μ as $p\mu_1 + (1-p)\mu_2$, where $p \in [0,1]$, μ_1 is a discrete probability measure and $\mu_2(\{x\}) = 0$ for any $x \in \mathbb{R}$. The case $p = 0$ is trivial, so we will assume that $p > 0$. Lemma 5.4.2 implies that μ_1 is weakly stable on $\mathbb{R}$ and therefore by Lemma 12 $\mu_1 = \delta_a$ for some $a \in \mathbb{R}$ or $\mu_1 = \frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_a$ for some $a \neq 0$.
Case 1. $\mu_1 = \delta_a$. If $a = 0$ then by Lemma 5.4.1 we have $p = 1$ and the proof is finished. If $a \neq 0$ let us note that for $\lambda \in (0,1)$ the random variable $X - \lambda X' \stackrel{d}{=} Y_\lambda X$ has exactly one atom with the mass p^2 at $(1-\lambda)a$. Hence Y_λ has an atom with the mass p at $(1-\lambda)$. Since $Y_\lambda \stackrel{d}{\to} Y_1$ as $\lambda \to 1$ we have $\mathbf{P}\{Y_1 = 0\} \geq p$, and therefore $\mathbf{P}\{X - X' = 0\} = \mathbf{P}\{Y_1 X = 0\} \geq p$. On the other hand $\mathbf{P}\{X - X' = 0\} = \mathbf{P}\{X = X'\} = p^2$ because X has only one atom, at a. Hence $p^2 \geq p$ so that $p = 1$.
Case 2. $\mu_1 = \frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_a$ for some $a \neq 0$. For $\lambda \in (0,1)$ the random variable $X - \lambda X' \stackrel{d}{=} Y_\lambda X$ has exactly four atoms, with mass $p^2/4$ each, at $(1-\lambda)a$, $-(1-\lambda)a$, $(1+\lambda)a$ and $-(1+\lambda)a$. Hence Y_λ has atoms with total mass $p/2$ at $(1-\lambda)$ and $-(1-\lambda)$ (and atoms with total mass $p/2$ at $(1+\lambda)$ and $-(1+\lambda)$). Since $Y_\lambda \stackrel{d}{\to} Y_1$ as $\lambda \to 1$ we have $\mathbf{P}\{Y_1 = 0\} \geq p/2$, and therefore $\mathbf{P}\{X - X' = 0\} = \mathbf{P}\{Y_1 X = 0\} \geq p/2$. On the other hand, $\mathbf{P}\{X - X' = 0\} = \mathbf{P}\{X = X' = a\} + \mathbf{P}\{X = X' = -a\} = p^2/2$ so that $p^2/2 \geq p/2$ and $p = 1$.
Let n be an arbitrary natural number. By the use of the above result for real random variables $< \xi, X >$, where $\xi \in \mathbb{R}^n$ we can easily end the proof. □

5.5 Generalized weak convolution

Definition 5.5.1. *Let $\mu \in \mathcal{P}_n$ be a nontrivial weakly stable measure, and let λ_1, λ_2 be probability measures on $\mathbb{R}$. If*

$$(\lambda_1 \circ \mu) * (\lambda_2 \circ \mu) = \lambda \circ \mu,$$

then the generalized convolution of the measures λ_1, λ_2 with respect to the measure μ (notation $\lambda_1 \oplus_\mu \lambda_2$) is defined as follows

$$\lambda_1 \oplus_\mu \lambda_2 = \begin{cases} \lambda & \text{if } \mu \text{ is not symmetric;} \\ |\lambda| & \text{if } \mu \text{ is symmetric.} \end{cases}$$

If Θ_1, Θ_2 are random variables with distributions λ_1, λ_2 respectively then the random variable with distribution $\lambda_1 \oplus_\mu \lambda_2$ we will denote by $\Theta_1 \oplus_\mu \Theta_2$. Thus we have

$$\Theta_1 \mathbf{X}' + \Theta_2 \mathbf{X}'' \stackrel{d}{=} \big(\Theta_1 \oplus_\mu \Theta_2\big)\mathbf{X},$$

where $\mathbf{X}, \mathbf{X}', \mathbf{X}''$ have distribution μ, $\Theta_1, \Theta_2, \mathbf{X}', \mathbf{X}''$ and $\Theta_1 \oplus_\mu \Theta_2, \mathbf{X}$ are independent. One can always choose such versions of $\Theta_1 \oplus_\mu \Theta_2$ and $\mathbf{X}$ that the above equality holds almost everywhere.
Now it is easy to see that the following lemma holds.

Lemma 5.5.1. *If the weakly stable measure $\mu \in \mathcal{P}(\mathbb{E})$ is not trivial then*

1) $\lambda_1 \oplus_\mu \lambda_2$ *is uniquely determined;*

2) $\lambda_1 \oplus_\mu \lambda_2 = \lambda_2 \oplus_\mu \lambda_1$;

3) $\lambda \oplus_\mu \delta_0 = \lambda$;

4) $\big(\lambda_1 \oplus_\mu \lambda_2\big) \oplus_\mu \lambda_3 = \lambda_1 \oplus_\mu \big(\lambda_2 \oplus_\mu \lambda_3\big)$;

5) $T_a\big(\lambda_1 \oplus_\mu \lambda_2\big) = \big(T_a\lambda_1\big) \oplus_\mu \big(T_a\lambda_2\big)$.

Example 5.5.1. It is known that the random vector $\mathbf{U}^n = (U_1, \ldots, U_n)$ with the uniform distribution ω_n on the unit sphere $S_{n-1} \subset \mathbb{R}^n$ is weakly stable. The easiest way to see this is using the characterizations of rotationally invariant vectors.
Let us recall that the random vector $\mathbf{X} \in \mathbb{R}^n$ is rotationally invariant (spherically symmetric) if $L(\mathbf{X}) \stackrel{d}{=} \mathbf{X}$ for every unitary linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$. It is known (see [14, 66] for the details) that the following conditions are equivalent

a) $\mathbf{X} \in \mathbb{R}^n$ is rotationally invariant,

b) $\mathbf{X} \stackrel{d}{=} \Theta\mathbf{U}^n$, where $\Theta = \|\mathbf{X}\|_2$ is independent of $\mathbf{U}^n$,

c) the characteristic function of $\mathbf{X}$ has the form

$$\mathbf{E}e^{i<\xi,\mathbf{X}>} = \varphi_{\mathbf{X}}(\xi) = \varphi(\|\xi\|_2)$$

for some symmetric function $\varphi : \mathbb{R} \to \mathbb{R}$.

Now let $\mathcal{L}(\Theta_1) = \lambda_1$, $\mathcal{L}(\Theta_2) = \lambda_2$ be such that $\Theta_1, \Theta_2, \mathbf{U}^{n1}, \mathbf{U}^{n2}$ are independent, $\mathbf{U}^{n1} \stackrel{d}{=} \mathbf{U}^{n2} \stackrel{d}{=} \mathbf{U}^n$. In order to prove weak stability of $\mathbf{U}^n$ we consider the characteristic function ψ of the vector $\Theta_1\mathbf{U}^{n1} + \Theta_2\mathbf{U}^{n2}$

$$\begin{aligned}\psi(\xi) &= \mathbf{E}\exp\big\{i < \xi, \Theta_1\mathbf{U}^{n1} + \Theta_2\mathbf{U}^{n2} >\big\} \\ &= \mathbf{E}\exp\big\{i < \xi, \Theta_1\mathbf{U}^{n1} >\big\}\,\mathbf{E}\exp\big\{i < \xi, \Theta_2\mathbf{U}^{n2} >\big\} = \varphi_1\big(\|\xi\|_2\big)\varphi_2\big(\|\xi\|_2\big).\end{aligned}$$

It follows from the condition (c) that $\Theta_1\mathbf{U}^{n1} + \Theta_2\mathbf{U}^{n2}$ is also rotationally invariant. Using condition (b) we obtain that $\Theta_1\mathbf{U}^{n1} + \Theta_2\mathbf{U}^{n2} \stackrel{d}{=} \Theta\mathbf{U}^n$ for some random variable Θ, which we denote by $\Theta_1 \oplus_{\omega_n} \Theta_2$. This means that $\mathbf{U}^n$ is weakly stable and

$$\begin{aligned}\Theta_1 \oplus_{\omega_n} \Theta_2 &= \big\|\Theta_1\mathbf{U}^{n1} + \Theta_2\mathbf{U}^{n2}\big\|_2 \\ &= \left(\sum_{k=1}^n \big(\Theta_1 U_k^{n1} + \Theta_2 U_k^{n2}\big)^2\right)^{1/2},\end{aligned}$$

where $\mathbf{U}^{ni} = (U_1^{ni}, \ldots, U_n^{ni})$, $i = 1, 2$. Since $\mathbf{U}^2 = (\cos\varphi, \sin\varphi)$ for the random variable φ with uniform distribution on $[0, 2\pi]$, then in the case $n = 2$ we get

$$\Theta_1 \oplus_{\omega_n} \Theta_2 = \left(\Theta_1^2 + \Theta_2^2 + 2\Theta_1\Theta_2 \cos(\alpha - \beta)\right)^{1/2},$$

where $\Theta_1, \Theta_2, \alpha, \beta$ are independent, α and β have uniform distribution on the interval $[0, 2\pi]$. It is easy to check that $\cos(\alpha - \beta)$ has the same distribution as $\cos(\alpha)$, thus we have

$$\Theta_1 \oplus_{\omega_n} \Theta_2 \stackrel{d}{=} \left(\Theta_1^2 + \Theta_2^2 + 2\Theta_1\Theta_2 \cos(\alpha)\right)^{1/2}.$$

Definition 5.5.2. *Let $\mathcal{L}(\Theta) = \lambda$, and let $\mu = \mathcal{L}(\mathbf{X})$ be a weakly stable measure on $\mathbb{E}$. We say that the measure λ (random variable Θ) is μ-weakly infinitely divisible if for every $n \in \mathbb{N}$ there exists a probability measure λ_n such that*

$$\lambda = \lambda_n \oplus_\mu \ldots \oplus_\mu \lambda_n, \qquad \text{(n-times)},$$

where (for the uniqueness) $\lambda_n \in \mathcal{P}_+$ if μ is weakly stable on $[0, \infty)$ or if μ is symmetric, and $\lambda_n \in \mathcal{P}$ if μ is weakly stable nonsymmetric.

Notice that if λ is μ-weakly infinitely divisible then the measure $\lambda \circ \mu$ is infinitely divisible in the usual sense. However, as it is shown in the following example if $\lambda \circ \mu$ is infinitely divisible then it does not have to imply μ-infinite divisibility of λ.

Example 5.5.2. Let m be a discrete measure:

$$m(\{x\}) = \begin{cases} 0,26 & \text{for} \quad x = 1, 2, 4, 5, \\ -0,04 & \text{for} \quad x = 3. \end{cases}$$

Since μ is a signed measure it cannot be the Lévy measure of any infinitely divisible distribution. However for some positive k the measure $\exp(km)$ defined by

$$\exp(km) = e^{-k} \sum_{n=0}^{\infty} \frac{(km)^{*n}}{n!}$$

is taking only nonnegative values and has total mass 1, thus it is a probability measure. It is not very difficult to see that $\exp(m)$ is a probability measure. Moreover

$$\exp(km)(\{3\}) = e^{-k} \left[-0,04k + 0,26^2 k^2 + \frac{1}{6} 0,26^3 k^3\right],$$

thus it is negative for

$$0 < k < \frac{\sqrt{9,96} - 3}{2 \cdot 0,26} \sim 0,3.$$

Consider now the measure $\gamma_\alpha \circ exp(m)$. It was shown in [?] that

$$\gamma_\alpha \circ exp(m) = \exp\left\{\gamma_\alpha \circ m\right\},$$

thus in order to show that $\gamma_\alpha \circ exp(m)$ is infinitely divisible we only need to show that the measure $\gamma_\alpha \circ m$ has nonnegative density. We know that $\gamma_\alpha = \gamma_2 \circ \tilde{\gamma_{\alpha/2}}$, where $\tilde{\gamma_{\alpha/2}}$ is the distribution of $\sqrt{\Theta}$ with Θ having $\alpha/2$-stable distribution on the positive half-line. Thus it is enough to show that for every $x \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{x^2}{2s^2}\right\} s^{-1} m(ds) \geq 0.$$

To see this it is enough to notice that

$$\frac{0,26}{4} \exp\left\{-\frac{x^2}{2\cdot 4^2}\right\} \geq \frac{0,04}{3} \exp\left\{-\frac{x^2}{2\cdot 3^2}\right\}.$$

Example 5.5.3. If γ_α is a strictly α-stable (symmetric α-stable) distribution on space $\mathbb{R}^n$ then it is weakly stable on $[0,\infty)$ (weakly stable). Simple application of the definition of stable distribution shows that

$$\Theta_1 \oplus_{\gamma_\alpha} \Theta_2 \stackrel{d}{=} \left(\Theta_1^\alpha + \Theta_2^\alpha\right)^{1/\alpha}, \quad \left(\Theta_1 \oplus_{\gamma_\alpha} \Theta_2 \stackrel{d}{=} \left(|\Theta_1|^\alpha + |\Theta_2|^\alpha\right)^{1/\alpha}\right).$$

Now we see that Θ is γ_α- weakly infinitely divisible if and only if Θ^α (respectively $|\Theta|^\alpha$) is infinitely divisible in the usual sense.

Lemma 5.5.2. *Let μ be a weakly stable distribution, $\mu \neq \delta_0$. If λ is μ-weakly infinitely divisible then there exists a family $\{\lambda^r : r \geq 0\}$ such that*
1) $\lambda^0 = \delta_0$, $\lambda^1 = \lambda$;
2) $\lambda^r \oplus_\mu \lambda^s = \lambda^{r+s}$, $r, s \geq 0$;
3) $\lambda^r \Rightarrow \delta_0$ if $r \to 0$.

Proof. If λ is μ-weakly infinitely divisible then for every $n \in \mathbb{N}$ there exists a measure λ_n such that

$$(\lambda_n \circ \mu)^{*n} = \lambda \circ \mu,$$

where ν^{*n} denotes the n'th convolution power of the measure ν. We define $\lambda^{1/n} := \lambda_n$. Weak stability of the measure μ implies that for every $k, n \in \mathbb{N}$ there exists a probability measure which we denote by $\lambda^{k/n}$ such that

$$\lambda^{k/n} \circ \mu = \left(\lambda^{1/n} \circ \mu\right)^{*k} = (\lambda \circ \mu)^{*k/n}.$$

The last expression follows from the infinite divisibility of the measure $\lambda \circ \mu$. We see here that for every $n, k, m \in \mathbb{N}$ we have

$$\lambda^{km/nm} = \lambda^{k/n},$$

since

$$(\lambda \circ \mu)^{*km/nm} = (\lambda \circ \mu)^{*k/n}.$$

Now let $x > 0$ and let $(r_n)_n$ be a sequence of rational numbers such that $r_n \to x$ when $n \to \infty$. Since $(\lambda \circ \mu)^{*r_n} \to (\lambda \circ \mu)^{*x}$ and

$$\{\lambda^{r_n} \circ \mu : n \in \mathbb{N}\} = \left\{(\lambda \circ \mu)^{*r_n} : n \in \mathbb{N}\right\}$$

then this family of measures is tight. Lemma 2 in [57] implies that also the family $\{\lambda^{r_n} : n \in \mathbb{N}\}$ is tight, so there exists a subsequence $\lambda^{r_{n_k}}$ weakly convergent to a probability measure which we call λ^x. Since $\lambda^x \circ \mu = (\lambda \circ \mu)^{*x}$ then uniqueness of the measure λ^x follows from the uniqueness of $(\lambda \circ \mu)^{*x}$, Remak 1 in [57] and our assumptions.

To see 3) let $r_n \to 0$, $r_n > 0$. Since

$$\lambda^{r_n} \circ \mu = (\lambda \circ \mu)^{r_n} \Rightarrow \delta_0 = \delta_0 \circ \mu,$$

then $\{(\lambda \circ \mu)^{r_n} : n \in \mathbb{N}\}$ is tight, and by Lemma 5.2.2 the set $\{\lambda^{r_n} : n \in \mathbb{N}\}$ is also tight. Let $\{r'_n\}$ be the subsequence of $\{r_n\}$ such that $\lambda^{r'_n}$ converges weakly to some probability measure λ^0. Then we have

$$\lambda^{r'_n} \circ \mu \Rightarrow \lambda^0 \circ \mu,$$

and therefore $\lambda^0 \circ \mu = \delta_0 \circ \mu$. If μ is not symmetric then Remark 5.3.1 implies that $\lambda^0 = \delta_0$. If μ is symmetric then by our assumptions λ and λ^{r_n} are concentrated on $[0, \infty)$, thus also λ^0 is concentrated on $[0, \infty)$. Since by Remark 5.3.1 the symmetrization of the mixing measure is uniquely determined in this case we also conclude that $\lambda^0 = \delta_0$.

□

5.6 Elliptically contoured vectors

The investigations of elliptically contoured distributions started in 1938 with the paper of Schoenberg "Metric spaces and complete monotonic functions" (see [64]). This paper is devoted to the study of random vectors which are invariant under isometries in $\mathbb{R}^n$ and in ℓ_2. Later on this concept was generalized to the elliptically contoured random vectors, which are images under linear operators of random vectors which are invariant under isometries. In this section we recall only some basic properties of elliptically contoured random vectors. For further information we refer the reader to the following review papers: *Elliptically symmetric distributions: A review and bibliography* by M. A. Chmielewski (see [13], 1981), which treats the problem mainly from the statistical point of view; and *Pseudo-isotropic distributions* by C. L. Scheffer and J. Misiewicz (see [52], 1990), where emphasis is put on the theory of measure. Both papers contain rich bibliographies.

Definition 5.6.1. *A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is elliptically contoured if there exists a symmetric positive definite $n \times n$-matrix $\Re$ and a function $\varphi \colon \mathbb{R} \to \mathbb{R}$ such that*

$$\varphi_{\mathbf{X}}(\xi) = \varphi\left(\langle \xi, \Re\xi \rangle^{1/2}\right), \qquad \forall \xi \in \mathbb{R}^n.$$

Remark 5.6.1. If $\Re = I$, i.e. if $\langle \xi, \Re\xi \rangle = \sum_{k=1}^n \xi_k^2$, then the corresponding elliptically contoured random vectors are also known in the literature under names such as rotationally invariant, spherically generated or spherically contoured random vectors (see Askey [2], Box [9], Gualtierotti [22], Huang and Cambanis [26], Kelker [31], [32], Kingman [33] [34], Letac [42]).

Notice that the function *varphi* appearing in the definition of an elliptically contoured random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is equal up to a scale parameter to the characteristic function of the first coordinate X_1 of this vector. Indeed, we see that

$$\varphi_{X_1}(t) = \varphi_{\mathbf{X}}((t, 0, \ldots, 0)) = \varphi\left(\sqrt{r_{1,1}t^2}\right) = \varphi\left(\sqrt{r_{1,1}}t\right),$$

where the last equality follows from the symmetry of φ. From now on we will use the notation $\mathcal{EC}(\varphi, \Re, n)$ for the distribution of elliptically contoured random vector $\mathbf{X} = (X_1, \ldots, X_n)$ with function $\langle \xi, \Re\xi \rangle$ and $\mathbf{E}\exp\{itX_1\} = \varphi(t)$. The following lemma has been proved by Crawford (see [14]) in 1977, originally for absolutely continuous distributions:

Lemma 5.6.1. *Let $\mathbf{X} = (X_1, \ldots, X_n)$ be elliptically contoured with distribution $\mathcal{EC}(\varphi, \Re, n)$, $\Re = B^T B$, and let C be a nonsingular, $n \times n$-matrix. If $\mathbf{Y} = B^{-1}C\mathbf{X}$, then $\mathbf{Y}$ is elliptically contoured with distribution $\mathcal{EC}(\varphi, C^T C, n)$.*

As a corollary one has that a random vector $\mathbf{X}$ on $\mathbb{R}^n$ is elliptically contoured if and only if there exists a non-degenerate linear operator $B : \mathbb{R}^n \longmapsto \mathbb{R}^n$ such that $B^{-1}\mathbf{X}$ is rotationally invariant. The next crucial result was proven in 1938 by Schoenberg (see [64] [66]).

Theorem 5.6.1. *A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is rotationally invariant if and only if there exists a nonnegative random variable Θ such that*

$$\mathbf{X} \stackrel{d}{=} (U_1, \ldots, U_n)\,\Theta,$$

where the random vector $\mathbf{U}^{(n)} = (U_1, \ldots, U_n)$ is independent of Θ and has a uniform distribution on the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : \sum_{k=1}^n x_k^2 = 1\}$.

Proof. It is enough to define $\Theta = \|\mathbf{X}\|_2$, check that Θ and $\mathbf{X}/\|\mathbf{X}\|_2$ are independent and notice that $\mathbf{X}/\|\mathbf{X}\|_2$ have the same distribution as $\mathbf{U}^{(n)}$. □

The so called *Box-Muller method* (see [10]) of computer simulation for standard two-dimensional normal $N(0, I)$ random vector $\mathbf{X} = (X_1, X_2)$ is based on Theorem 5.6.1. It follows from Theorem 5.6.1 that $\mathbf{X} \stackrel{d}{=} \mathbf{U}^{(2)}\sqrt{X_1^2 + X_2^2}$, where $\mathbf{U}^{(2)}$ is uniformly distributed on the unit sphere in $\mathbb{R}^2$ and independent of $\sqrt{X_1^2 + X_2^2}$. The probability distribution ω_2 of $\mathbf{U}^{(2)}$ and the distribution of $(\cos\theta, \sin\theta)$ with θ uniform on the interval $[0, 2\pi]$, are identical. It is easy to calculate that

$$X_1^2 + X_2^2 \stackrel{d}{=} -2\ln(Q),$$

where Q independent of Θ is uniformly distributed on the interval $[0,1]$. This finally leads to the following Box-Muller statement:

$$\left.\begin{array}{l}\cos\theta\sqrt{-2\ln Q},\\ \sin\theta\sqrt{-2\ln Q}\end{array}\right\}\quad \text{are i.i.d. with } N(0,1) \text{ distribution.}$$

In the same way we obtain that

$$\left.\begin{array}{l}\cos\theta_1\cos\theta_2\sqrt{2Z}\\ \sin\theta_1\cos\theta_2\sqrt{2Z}\\ \sin\theta_2\sqrt{2Z}\end{array}\right\}\quad \text{are i.i.d. with } N(0,1) \text{ distribution,}$$

where Z, θ_1, θ_2 are independent, Z has distribution $\Gamma(\frac{1}{2}, 1)$, θ_1 has uniform distribution on $[0, 2\pi]$ and θ_2 uniform distribution on $[-\pi/2, \pi/2]$. Instead of the variable $\sqrt{2Z}$ we can also take a variable W with the density function $\sqrt{2/\pi}x^2e^{-x^2/2}$ for $x > 0$. The same construction can be done for $(X_1, \ldots, X_n)$ with X_i's i.i.d. $N(0,1)$ random variables for every $n \in \mathbb{N}$. It is a simple consequence of Theorem 5.6.1; namely we have

$$(X_1, \ldots, X_n) = (U_1, \ldots, U_n) \cdot \sqrt{S^2(n)},$$

where $S^2(n) \stackrel{d}{=} X_1^2 + \ldots + X_n^2$ is independent of the random vector $\mathbf{U}^{(n)} = (U_1, \ldots, U_n)$ with uniform distribution on the unit sphere in $\mathbb{R}^n$. It is evident that the distribution of $\mathbf{U}^{(n)}$, as supported in S^{n-1} cannot be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$. The distribution of $\mathbf{U}^{(n)}$ is of sign-symmetric Dirichlet type with parameters $(2, \ldots, 2; 1, \ldots, 1)$, i.e. the following conditions hold:

(i) $(U_1, \ldots, U_n)$ is a sign-symmetric random vector;

(ii) $\sum_{k=1}^n U_k^2 = 1$ with probability one;

(iii) the joint density function of $\big(U_1, \ldots, U_{n-1}\big)$ is

$$\frac{\Gamma(n/2)}{\Gamma(1/2)^n}\left(1 - \sum_{k=1}^{n-1} u_k^2\right)_+^{-1/2},$$

where $(a)_+ = \max\{a, 0\}$.

It is easy to see that the joint density function of the first k components $(U_1, \ldots, U_k)$, $k < n$, of the random vector $\mathbf{U}^{(n)}$ is of the form:

$$\frac{\Gamma(n/2)}{\Gamma((n-k)/2)\Gamma(1/2)^k}\left(1 - \sum_{1}^{k} u_j^2\right)_+^{\frac{n-k}{2}-1},$$

for details see [23]. Now, we have the following:

Theorem 5.6.2. *The marginal density function of* $(X_1, \dots, X_k)$, $k < n$, *of the rotationally invariant random vector* $\mathbf{X} = (X_1, \dots, X_n) = \mathbf{U}^{(n)}\Theta$ *admits the following representation:*

$$f_k(\mathbf{x}) = \frac{\Gamma(n/2)}{\Gamma((n-k)/2)\Gamma(1/2)^k} \int_0^\infty r^{-k} \left(1 - r^{-2} \sum_1^k x_j^2\right)_+^{\frac{n-k}{2}-1} \lambda(dr),$$

$\mathbf{x} \in \mathbb{R}^k$, *where* λ *is the distribution of the random variable* Θ. *If the random vector* $(X_1, \dots, X_n)$ *is elliptically contoured with representation* $B\mathbf{U}^{(n)}\Theta$ *then the density of its* k*-dimensional projection* $(X_1, \dots, X_k)$, $k < n$, *is of the form*

$$f_k(x) = \frac{\Gamma(n/2)\, |\Re_k|^{-1/2}}{\Gamma((n-k)/2)\Gamma(1/2)^k} \int_0^\infty r^{-k} \left(1 - r^{-2} < x, \Re_k^{-1} x >\right)_+^{\frac{n-k}{2}-1} \lambda(dr),$$

$x \in \mathbb{R}^k$, *where the* $k \times k$*-matrix* $\Re_k$ *is built from the first* k *rows and columns of the matrix* $\Re = B^T B$.

Remark 5.6.2. The formula of the density function of the k-dimensional projection $(X_1, \dots, X_k)$, $k < n$, for the elliptically contoured random vector $\mathbf{X} = B\mathbf{U}^{(n)}\Theta$ can be also written in the following way:

$$f_k(\mathbf{x}) = |\Re_k|^{-1/2}\, f\Big(< \mathbf{x}, \Re_k^{-1}\mathbf{x} >\Big),$$

where $f : [0, \infty) \mapsto [0, \infty)$ is $\frac{n-k}{2}$-times monotonic function. More about α-times monotonic functions can be found in the paper of Williamson [68]. In this book it is enough to know that g is α-times monotonic function if it admits the representation

$$g(r) = \int_0^\infty \Big(1 - ru\Big)_+^{\alpha-1} dF(u),$$

with F non-decreasing, non-negative function. In the case $\alpha \in \mathbb{N}$ the function g is α-times monotonic if and only if it is α-times differentiable and $(-1)^k g^{(k)}(t) \geq 0$ for every $0 \leq k \leq \alpha$

Evidently all one-dimensional projections of $\mathbf{U}^{(n)}$ are the same up to a scale parameter and

$$\begin{aligned}
\mathbf{E}\exp\left\{i \sum_{k=1}^n \xi_k U_k\right\} &= \mathbf{E}\exp\left\{i \Big(\sum_{k=1}^n \xi_k^2\Big)^{1/2} U_1\right\} \\
&= \frac{\Gamma(n/2)}{\Gamma((n-1)/2)\Gamma(1/2)} \int_{-1}^1 \cos\Big(\|\xi\|_2 \cdot u\Big)\, \big(1 - u^2\big)^{(n-3)/2} du \\
&\equiv \Omega_n\Big(\|\xi\|_2\Big).
\end{aligned}$$

The function Ω_n can also be written in the following way:

$$\begin{aligned}\Omega_n(r) &= \frac{2\Gamma(n/2)}{\Gamma((n-1)/2)\Gamma(1/2)}\int_0^{\pi/2}\cos\Big(r\sin\varphi\Big)\cos^{n-2}(\varphi)\,d\varphi \\ &= \Gamma\Big(\frac{n}{2}\Big)\Big(\frac{2}{r}\Big)^{\frac{n}{2}-1}J_{\frac{n-2}{2}}(r),\end{aligned}$$

where $J_\nu(r)$ is a Bessel function; i.e. a cylindrical function of the first kind, thus it is a solution of the following differential equation (for details see e.g. [20]):

$$\frac{d^2J_\nu(r)}{dr^2}+\frac{1}{r}\frac{dJ_\nu(r)}{dr}+\left(1-\frac{\nu^2}{r^2}\right)J_\nu(r)=0.$$

This implies that

$$\frac{d^2}{dr^2}\Omega_n(r)+\frac{n-1}{r}\frac{d}{dr}\Omega_n(r)+\Omega_n(r)=0.$$

Now, we have the following:

Theorem 5.6.3 (Schoenberg [66]). *If* $\mathbf{X}=(X_1,\ldots,X_n)$ *is an elliptically contoured random vector with representation* $\mathbf{X}\overset{d}{=}B\mathbf{U}^{(n)}\sqrt{\Theta}$, $\Re=B^TB$, *then*

$$\mathbf{E}\exp\Big\{i\sum_{k=1}^{n}\xi_kX_k\Big\}=\int_0^\infty\Omega_n\Big(\big(<\xi,\Re\xi>\big)^{1/2}r\Big)\,\lambda(dr),$$

where λ *is the distribution of the random variable* Θ.

We can see that not every symmetric positive definite function φ on $\mathbb{R}$ with $\varphi(0)=1$ has the property that $\varphi(\|\cdot\|_2)$ is a characteristic function of an elliptically contoured random vector. In 1973 Askey [2] proved the following

Theorem 5.6.4. *Let* $n\geq 2$ *and let* $\varphi:[0,\infty)\mapsto\mathbb{R}$ *be continuous and such that*

1) $\varphi(0)=1$, $\lim_{t\to\infty}\varphi(t)=0$;
2) $(-1)^k\varphi^(k)(t)\geq 0$ *is convex for* $k=[\frac{n}{2}]$.

Then, for every positive definite $n\times n$*-matrix* $\Re$, $\varphi\big((<\xi,\Re\xi>)^{1/2}\big)$ *is the characteristic function of some elliptically contoured random vector.*

In 1982 Hardin published a paper (see [24]) in which he studied the linear regression property for elliptically contoured processes. He showed that one of the possible definitions of linearity of regression is just equivalent to the property of processes being elliptically contoured. Namely, he considered the following definition:

Definition 5.6.2. *Let $\mathcal{X} = \{X_t : t \in \mathbf{T}\} \subset L_1(\Omega, P)$ be a stochastic process and let $\mathcal{L}(\mathcal{X})$ be the real vector space of all finite linear combinations $\sum \xi_k X_{t_k}$. We say that $\mathcal{X}$ has the* **linear regression property** *if all regressions in $\mathcal{L}(\mathcal{X})$ are linear, that is*

$$\mathbf{E}\big(X_0|X_1,\ldots,X_n\big) \in \mathcal{L}(\{X-1,\ldots,X_n\}) \quad \textit{if} \quad X_0, X_1, \ldots, X_n \in \mathcal{L}(\mathcal{X}).$$

The main result in Hardin's paper is based on the following lemma:

Lemma 5.6.2 (Hardin [24]). *Let $\mathcal{X} = \{X_t : t \in \mathbf{T}\} \subset L_1(\Omega, P)$ be an elliptically contoured process, i.e. process having all finite dimensional distributions elliptically contoured. Then*

1. *there exists an inner product $< \cdot, \cdot >$ on $\mathcal{L}(\mathcal{X})$ such that*

$$\big(\mathbf{E}|X|\big)^2 = < X, X > \qquad \textit{for all} \qquad X \in \mathcal{L}(\mathcal{X}),$$

and

$$\mathbf{E}\big(X|Y\big) = \frac{< X, Y >}{< Y, Y >} \cdot Y \qquad \textit{for all} \qquad X, Y \neq 0 \textit{ in } \mathcal{L}(\mathcal{X});$$

2. *if $X, Y \in \mathcal{L}(\mathcal{X})$, $\mathbf{E}|X| = \mathbf{E}|Y| = 1$, and $\mathbf{E}(X|Y) = 0$ then the random vector (X, Y) is rotationally invariant.*

Theorem 5.6.5 (Hardin [24]). *Let $\mathcal{X} = \{X_t : t \in \mathbf{T}\}$ be a stochastic process satisfying at least one of the following two conditions:*

a) $$\mathcal{X} \subset L_2(\Omega, P) \qquad \textit{and} \qquad \dim\big(\mathcal{L}(\mathcal{X})\big) \geq 2;$$

b) $$\mathcal{X} \subset L_1(\Omega, P) \qquad \textit{and} \qquad \dim\big(\mathcal{L}(\mathcal{X})\big) \geq 3.$$

Then $\mathcal{X}$ has the linear regression property if and only if $\mathcal{X}$ is an elliptically contoured process.

Of course, every elliptically contoured process for which regressions are defined has the linear regression property - in this implication the dimension condition is not needed. For the opposite implication when $\mathcal{X} \subset L_1(\Omega, P)$ the condition $\dim\big(\mathcal{L}(\mathcal{X})\big) \geq 3$ is essential. Hardin has given an example ($\mathcal{X} = (X, Y)$, where X, Y are i.i.d. random variables with a symmetric α-stable distribution, $\alpha < 2$) of a two-dimensional process which has the linear regression property but which is not elliptically contoured. See also [47], Th. 6.1.1, and [30], Th. 1.4.

By the Fourier Inversion Formula we can get that if a rotationally invariant random vector $\mathbf{X} = \mathbf{U}^{(n)}\Theta$ with representation $\mathcal{EC}(\varphi, I, n)$ has an integrable characteristic function $\varphi\big(\|\xi\|_2\big)$, $\xi \in \mathbb{R}^n$, then its density function can be written as

$$f(x) = \frac{\Gamma\big((n-1)/2\big)\Gamma\big(1/2\big)}{\Gamma(n/2)} \int_0^\infty r^{n-1}\varphi(r)\Omega_n\Big(\|x\|_2 \cdot r\Big)\, dr.$$

Let us also notice here that $\varphi(r)$ is the characteristic function of the random variable $\Theta \cdot U_1$.

Now we are ready to prove the main result of this section, stating that for every $n \in \mathbb{N}$ the distribution ω_n of the random vector $\mathbf{U}^{(n)}$ which is uniform on the unit sphere $S_{n-1} \subset \mathbb{R}^n$ is weakly stable. This is the property, which makes elliptically contoured distributions so useful in applications.

Theorem 5.6.6. *For every $n \in \mathbb{N}$ the random vector $\mathbf{U}^{(n)} = (U_{1,n}, \ldots, U_{n,n})$ with uniform distribution ω_n on the unit sphere $S_{n-1} \subset \mathbb{R}^n$ is weakly stable. For every $n \times n$-matrix A the random vector $A\mathbf{U}^{(n)}$ is weakly stable. Moreover for every $a, b \in \mathbb{R}$*

$$a\mathbf{U}_1^{(n)} + b\mathbf{U}_2^{(n)} \stackrel{d}{=} \mathbf{U}^{(n)} \left\| a\mathbf{U}_1^{(n)} + b\mathbf{U}_2^{(n)} \right\|_2,$$

where $\mathbf{U}_1^{(n)}, \mathbf{U}_2^{(n)}$ and $\mathbf{U}^{(n)}$ are independent, identically distributed.

Proof. In view of Theorem 5.6.3 we have that

$$\mathbf{E}\exp\left\{i\langle\xi, \mathbf{U}^{(n)}\rangle\right\} = \mathbf{E}\exp\left\{i\sum_{k=1}^{n} \xi_k U_k^n\right\} = \Omega\left(\|\xi\|_2\right).$$

This implies that

$$\mathbf{E}\exp\left\{i\langle\xi, a\mathbf{U}_1^{(n)} + b\mathbf{U}_2^{(n)}\rangle\right\} = \Omega\left(|a|\|\xi\|_2\right)\Omega\left(|b|\|\xi\|_2\right),$$

so the right hand side is a function dependent only on $\|\xi\|_2$ (a, b are only some parameters here). This means by definition that $a\mathbf{U}_1^{(n)} + b\mathbf{U}_2^{(n)}$ is rotationally invariant, thus it follows from Theorem 5.6.1 that

$$a\mathbf{U}_1^{(n)} + b\mathbf{U}_2^{(n)} \stackrel{d}{=} Q\mathbf{U}^{(n)}$$

for some random variable $Q = Q_{a,b}$ independent of $\mathbf{U}^{(n)}$. From the proof of Theorem 5.6.1 it follows also that $\|a\mathbf{U}_1^{(n)} + b\mathbf{U}_2^{(n)}\|_2$ can be chosen as Q. To finish the proof it is enough to notice that

$$a\left(A\mathbf{U}_1^{(n)}\right) + b\left(A\mathbf{U}_2^{(n)}\right) = A\left(a\mathbf{U}_1^{(n)} + b\mathbf{U}_2^{(n)}\right) \stackrel{d}{=} A\left(Q\mathbf{U}^{(n)}\right) = Q\left(A\mathbf{U}^{(n)}\right).$$

□

The set $\mathcal{M}(\omega_n)$ is well known as the set of all rotationally invariant distributions on $\mathbb{R}^n$. The set $\mathcal{M}(\omega_{k,n})$ is a convex and closed subset of $\mathcal{M}(\omega_k)$. If $n = k+2$, then $\omega_{k,n}$ is the uniform distribution on the unit ball $B_k \subset \mathbb{R}^k$. In particular, we obtain that $\mathcal{M}(\omega_{1,3})$ is the set of symmetric unimodal probability measures on $\mathbb{R}$. Notice also that $\omega_{1,3}$ is uniform on the interval $[-1, 1]$. This property was used in [39] to define elliptically contoured copula.

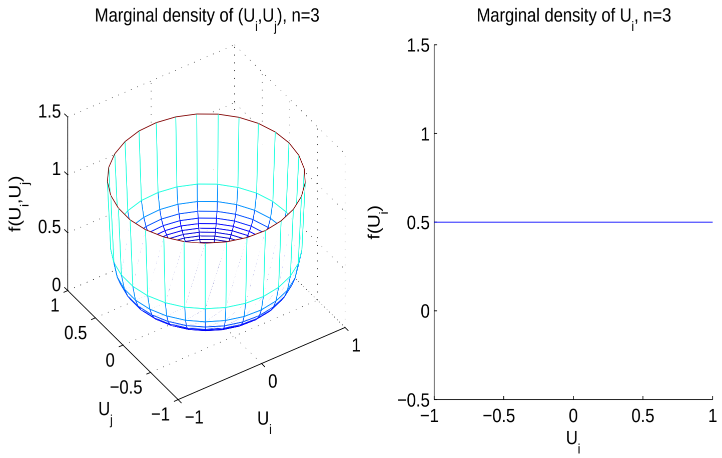


Figure 5.1:

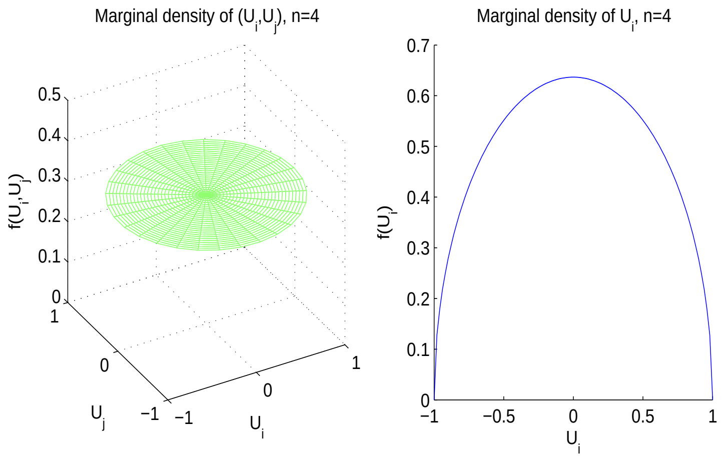


Figure 5.2:

5.7 Cambanis, Keener and Simons distributions

We say that the distribution μ on $\mathbb{R}^n$ is ℓ_1-symmetric (sometimes the name ℓ_1-pseudo-isotropic is used here) if the characteristic function of μ has the following form

$$\widehat{\mu}(\xi) = \varphi\left(\|\xi\|_1\right),$$

for some function φ, where $\|\xi\|_1 = |\xi_1| + \ldots + |\xi_n|$. Since an ℓ_1 symmetric distribution is evidently symmetric then putting $\xi = (\xi_1, 0, \ldots, 0)$ we see that $\varphi(|\xi_1|)$ is the characteristic function of X_1 for $\mathbf{X} = (X_1, \ldots, X_n)$ having the distribution μ. This means that the random vector $\mathbf{X}$ is ℓ_1-symmetric (ℓ_1-pseudo-isotropic) if for every $\xi \in \mathbb{R}^n$ the following equation holds

$$< \xi, \mathbf{X} >= \sum_{k=1}^{n} \xi_k X_k \stackrel{d}{=} \|\xi\|_1 \cdot X_1.$$

Let $\mathcal{M}_1(n)$ denotes the set of all non-degenerate ℓ_1-symmetric distributions on $\mathbb{R}^n$. It is not difficult to see that $\mathcal{M}_1(n)$ is convex and weakly closed. For a long time the extreme points of this set were unknown.
In 1983 S. Cambanis, R. Keener and G. Simons [12] found all the extreme points for ℓ_1-dependent distributions on $\mathbb{R}^n$. This result was based on the following, surprisingly general, definite integral identity:

$$\int_0^{\pi/2} f\left(\frac{s^2}{\sin\theta} + \frac{t^2}{\cos\Theta}\right) d\theta = \int_0^{\pi/2} f\left(\frac{(|s|+|t|)^2}{\sin\theta}\right) d\theta,$$

which holds for each $s, t \in \mathbb{R}$ and every function f for which the integrals make sense. The S. Cambanis, R. Keener and G. Simons proof of this identity was not very difficult but complex. In 2002 K. Oleszkiewicz gave (but never published) a very simple geometrical proof of this identity and with his permission we present it here.

Consider the unit sphere in $\mathbb{R}^2$, and assume that the chords EF and DG are parallel, the points a, B, C, O, P are situated as in Figure 5.3. Let $CP \perp AE$, $|OP| = x$ and $\angle DPC = \angle CPB = \alpha$. Notice that the length of the chord EF determines the length of the curve $|\widehat{EF}|$ and it depends only on $\angle PEF = \frac{\pi}{2} - \alpha$. We obtain

$$|\widehat{BD}| = |\widehat{BE}| - |\widehat{DE}| = |\widehat{EG}| - |\widehat{FG}| = |\widehat{EF}|,$$

which means that $|\widehat{BD}|$ depends only on α and not on x.
Let $s, t \in \mathbb{R}$ and consider the sphere $S \subset \mathbb{R}^2$ with the radius $r = |s| + |t|$. We assume that the random variable θ is uniformly distributed on the interval $[0, 2\pi]$ and the point H is the projection of $X = r(\cos\theta, \sin\theta)$ on the diameter AE - see the Figure 5.4. Of course the random vector X is uniformly distributed on S and

$$\mathbf{P}\left\{|\text{ctg}\angle XOH| < u\right\} = \mathbf{P}\left\{|\text{ctg}\angle XPH| < u\right\}$$

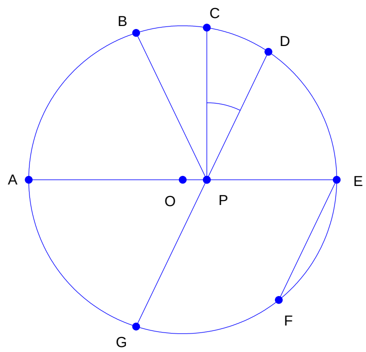


Figure 5.3:

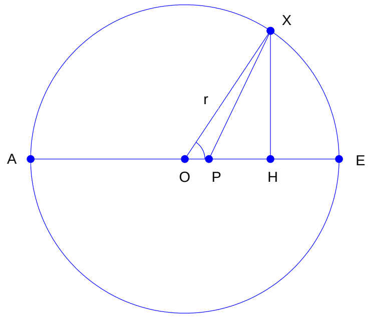


Figure 5.4:

for all $u \in \mathbb{R}$. This equality follows from the property that the points of the sphere S, for which the inequality $|\text{ctg}\angle XOH| < u$ holds, and the points of the sphere for which the inequality $|\text{ctg}\angle XPH| < u$ holds, form subsets of S with the same Lebesgue measure. This implies that

$$\left|\frac{\cos\theta}{\sin\theta}\right| \stackrel{d}{=} \left|\frac{r\cos\theta - x}{r\sin\theta}\right|.$$

The same argument can be used to show that $\left|\frac{r\cos\theta - x}{r\sin\theta}\right| \stackrel{d}{=} \left|\frac{r\cos(2\theta)-x}{r\sin(2\theta)}\right|$ and as a consequence we have the following

$$\left(\frac{r\cos(2\theta) - x}{r\sin(2\theta)}\right)^2 + 1 \stackrel{d}{=} \left(\frac{\cos\theta}{\sin\theta}\right)^2 + 1.$$

This implies that

$$\frac{(r-x)^2}{\sin^2\theta} + \frac{(r+x)^2}{\cos^2\theta} \stackrel{d}{=} \frac{4r^2}{\sin^2\theta}.$$

Now it is enough to substitute $2r = |s| + |t|$ and $2x = |s| - |t|$ to obtain that

$$\frac{t^2}{\sin^2\theta} + \frac{s^2}{\cos^2\theta} \stackrel{d}{=} \frac{(|t| + |s|)^2}{\sin^2\theta}.$$

The result of Cambanis, Keener and Simons can be formulated in the following way

Theorem 5.7.1. *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *with distribution* μ *on* $\mathbb{R}^n$ *be non-degenerate. Then the following conditions are equivalent:*

1. $\mu \in \mathcal{M}_1(n)$;
2. *There exists a nonnegative variable* Θ *such that*

$$\mathbf{X} \stackrel{d}{=} \left(\frac{U_1}{\sqrt{D_1}}, \ldots, \frac{U_n}{\sqrt{D_n}} \right) \cdot \Theta,$$

where $\mathbf{U}^n = (U_1, \ldots, U_n)$ *has uniform distribution on the unit sphere in* $\mathbb{R}^n$, $\mathbf{D} = (D_1, \ldots, D_n)$ *has Dirichlet distribution with parameters* $(\frac{1}{2}, \ldots, \frac{1}{2})$, $\mathbf{U}^n$, $\mathbf{D}$ *and* Θ *are independent.*

3. *The characteristic function of the measure* μ *can be written in the form* $\widehat{\mu}(\xi) = \varphi(\|\xi\|_1)$, *where*

$$\varphi(r) = \int_0^\infty \varphi_n(rx)\lambda(dx),$$

for λ *being the distribution of random variable* Θ *and*

$$\varphi_n(r) = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_1^\infty \Omega_n(ur^2)u^{-n/2}(u-1)^{(n-3)/2}du,$$

for $\Omega_n(\|\xi\|_2^2)$ *the characteristic function of the vector* $\mathbf{U}^n$.

4. μ *is absolutely continuous with the density function*

$$f_n(\mathbf{x}) = \int_0^\infty r^{-n} g_n(\mathbf{x}r^{-1})\lambda(dr),$$

where

$$g_n(\mathbf{x}) = \frac{\left[\Gamma(\frac{n}{2})\right]^2}{(n-2)!\pi^n} \sum_{k=1}^n \frac{(x_k^2-1)_+^{n-2}}{\Pi_{j=1, j\neq k}^n (x_j^2 - x_k^2)}, \qquad |x_j| \neq |x_k| \text{ for } j \neq k.$$

The next theorem was given by G. Mazurkiewicz in 2005 in the paper *Weakly stable distributions and magic distribution of S. Cambanis, R. Keener and G. Simons* [?]. Why *magic*? First, because of the surprising integral identity which was the crucial point for their construction. Secondly, because of the resemblance to the magic square, where sums of all elements in every row, every column and every diagonal are constant. Here we have that every one-dimensional marginal has the same distribution up to a scale parameter, and this scale parameter is defined as the sum of absolute values of coefficients in the linear combination defining the projection.

Theorem 5.7.2. *For every $n, \alpha \in \mathbb{N}$, $n - 4\alpha \geq 4$, the marginal densities Π_{n-k} of the density g_n are given by the following:*
For $k = 4\alpha$

$$\Pi_{n-k}(x_1, \ldots, x_{n-k}) = \left[\sum_{m=1}^{n-k} \frac{(x_m^2 - 1)_+^{n-2}}{\Pi_{j=1, j\neq m}^{n-k}(x_m^2 - x_j^2)|x_m|^k} - D_{2\alpha-1}^{n-k}\right] \frac{\left[\Gamma(\frac{n}{2})\right]^2}{(n-2)!\pi^{n-k}}.$$

For $k = 4\alpha + 1$

$$\begin{aligned}\Pi_{n-k}(x_1, \ldots, x_{n-k}) &= \left[\sum_{m=1}^{n-k} \frac{(x_m^2 - 1)^{n-2}}{\Pi_{j=1, j\neq m}^{n-k}(x_m^2 - x_j^2)|x_m|^k} \ln\left|\frac{1 + |x_m|}{1 - |x_m|}\right|\right. \\ &+ \left.\sum_{i=1}^{2\alpha} \frac{2}{2i-1}\left({}_{4\alpha}C_{2i}^{n-k} - D_{2\alpha-i}^{n-k}\right)\right] \frac{\left[\Gamma(\frac{n}{2})\right]^2}{(n-2)!\pi^{n-k+1}}.\end{aligned}$$

For $k = 4\alpha + 2$

$$\begin{aligned}\Pi_{n-k}(x_1, \ldots, x_{n-k}) &= \left[\sum_{m=1}^{n-k} \frac{(x_m^2 - 1)_-^{n-2}}{\Pi_{j=1, j\neq m}^{n-k}(x_m^2 - x_j^2)|x_m|^k} \ln\left|\frac{1 + |x_m|}{1 - |x_m|}\right|\right. \\ &+ \left.{}_{4\alpha+1}C_1^{n-k}\right] \frac{\left[\Gamma(\frac{n}{2})\right]^2}{(n-2)!\pi^{n-k}}.\end{aligned}$$

For $k = 4\alpha + 3$

$$\begin{aligned}\Pi_{n-k}(x_1, \ldots, x_{n-k}) &= \left[-\sum_{m=1}^{n-k} \frac{(x_m^2 - 1)^{n-2}}{\Pi_{j=1, j\neq m}^{n-k}(x_m^2 - x_j^2)|x_m|^k} \ln\left|\frac{1 + |x_m|}{1 - |x_m|}\right|\right. \\ &- \left.\sum_{i=1}^{2\alpha+1} \frac{2}{2i-1}\left({}_{4\alpha+2}C_{2i}^{n-k} - D_{2\alpha+1-i}^{n-k}\right)\right] \frac{\left[\Gamma(\frac{n}{2})\right]^2}{(n-2)!\pi^{n-k+1}}.\end{aligned}$$

In this theorem we use the following notation:

$$S_i^m = \begin{cases} \sum_{\binom{m}{i}} \left(x_{\pi(1)} \ldots x_{\pi(i)}\right)^2 & \text{if } i \in \{1, \ldots, m\}, \\ 0 & \text{if } i = 0 \text{ or } i > m, \end{cases}$$

where $(\pi(1), \ldots, \pi(i)) = \pi \in \binom{m}{i}$ denotes any choice of i elements from the set $\{1, \ldots, m\}$.

$$D_j^m = \begin{cases} 1 & j = 0, \\ (-1)^j\binom{n-2}{j} - \sum_{i=0}^{j-i}(-1)^{j-i}S_{j-i}^m D_i^m & j = 1, \ldots, t, \\ 0 & j < 0, \end{cases}$$

where $\omega = n - m - 2$, $t = \omega - \left[\frac{\gamma-1}{2}\right]$, here $\gamma = k$ with the condition $m, \omega \in \mathbb{N}$.

$$S_i^{j,m} = \begin{cases} \sum_{\binom{m-1}{i}} \left(x_{\pi_j(1)} \ldots x_{\pi_j(i)}\right)^2 & j = 1, 2, \ldots, m, \\ 0 & i = 0 \text{ or } i > m, \end{cases}$$

where $(\pi_j(1), \ldots, \pi_j(i)) = \pi_j \in \binom{m-1}{i}$ denotes any choice of i elements from the set $\{1, \ldots, m\} \setminus \{j\}$.

$$
{}_{\gamma+1}A_j^m = \begin{cases} \frac{(x_j^2-1)^{n-2}}{|x_j|^{\gamma+1}\prod_{i=1,i\neq j}^m (x_j^2-x_i^2)} & j = 1, 2, \ldots, m, \\ \frac{(x_j^2-1)^{n-2}}{|x_j|^{\gamma+1}} & j = m = 1. \end{cases}
$$

The values of ${}_\gamma C_j^m$ are given by the following recursive formulas

$$
\begin{aligned}
{}_\gamma C_j^m &= \sum_{i=1}^{\left[\frac{j-1}{2}\right]} S_i^m \, {}_\gamma C_{j-2i}^m + 2(-1)^{\left[\frac{j-1}{2}\right]} \sum_{i=1}^m {}_{\gamma+1}A_i^m S_{\left[\frac{j-1}{2}\right]}^{i,j} \cdot G_j(|x_i|) \\
&+ (-1)^{t+\left[\frac{j-1}{2}\right]} \binom{n-2}{t+\left[\frac{j-1}{2}\right]} - \sum_{i=0} (-1)^{t+\left[\frac{j-1}{2}\right]-i} S_{t+\left[\frac{j-1}{2}\right]-i}^m D_i^m,
\end{aligned}
$$

for $G_j(r) = r$ if j is even, and $G_j(r) = 1$ if j is odd, $j = 1, \ldots, \gamma - 1$, and

$$
{}_\gamma C_\gamma^m = \frac{(-1)^\omega}{\prod_{i=1}^m x_i^2}.
$$

Yes, this theorem looks very complicated and unpleasant. Fortunately the computer programs have no special difficulties in calculating these densities and with presenting graphs of two-dimensional marginal densities. We give such graphs on the next few pages in order to justify our opinion that they are just beautiful.

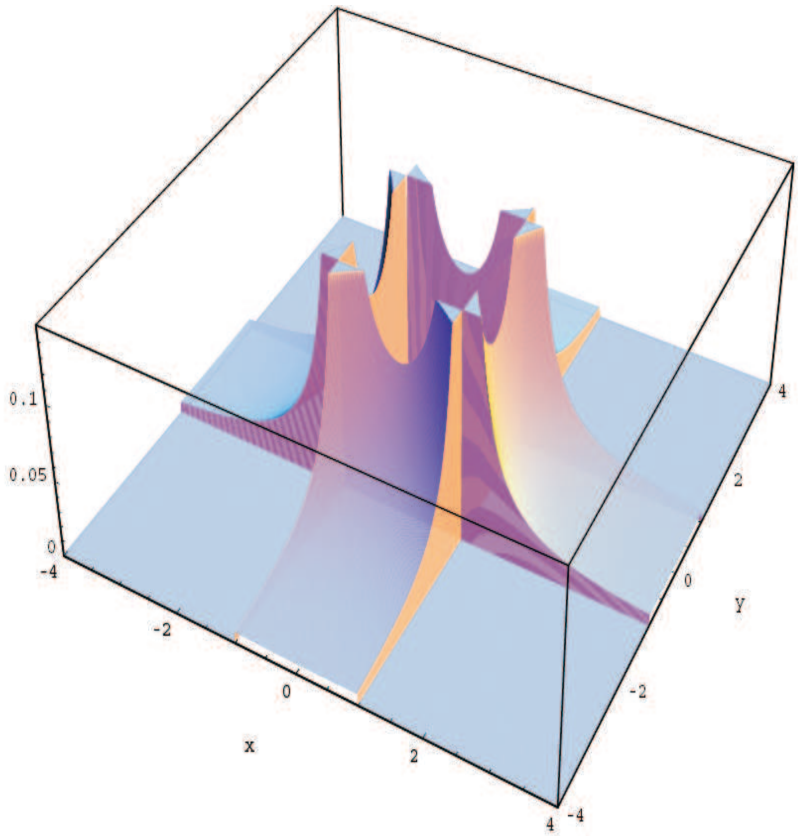


Figure 5.5: two-dimensional density $g_2(x, y)$

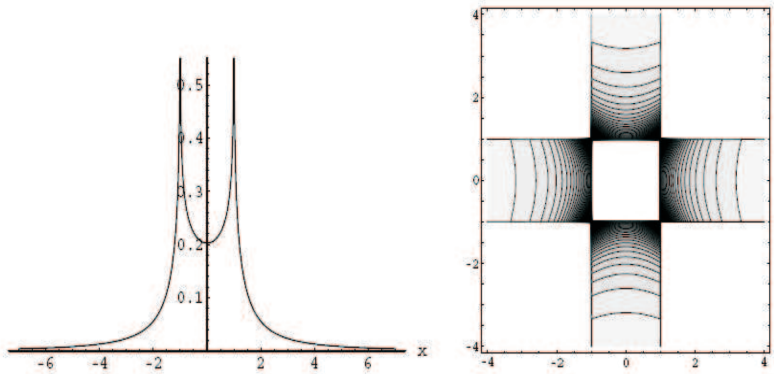


Figure 5.6: one-dimensional marginal $\Pi_{2-1}(x)$ and level curves for density $g_2(x, y)$

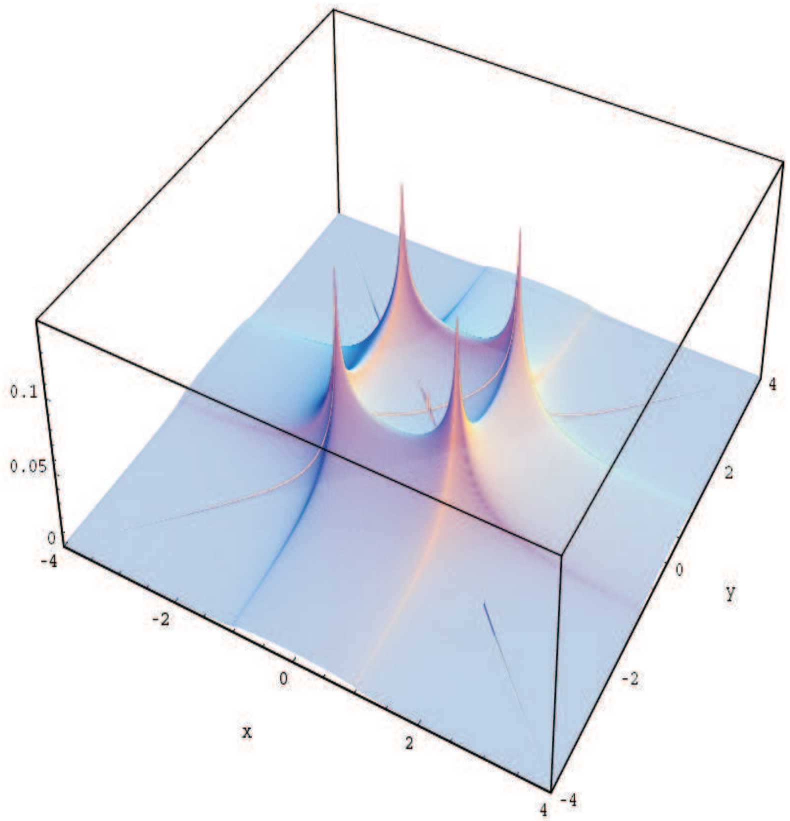
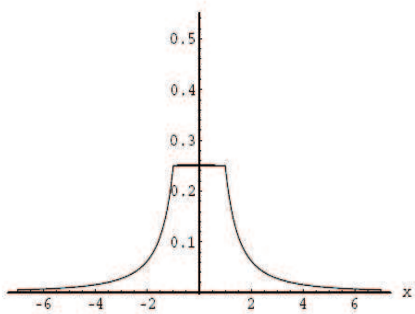


Figure 5.7: two-dimensional marginal $\Pi_{3-1}(x, y)$ for density $g_3(x, y, z)$



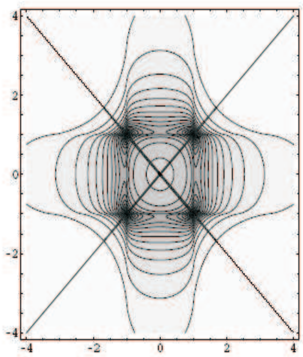


Figure 5.8: one-dimensional marginal $\Pi_{3-2}(x, y)$ and level curves of $\Pi_{3-1}(x, y)$ for density $g_3(x, y, z)$

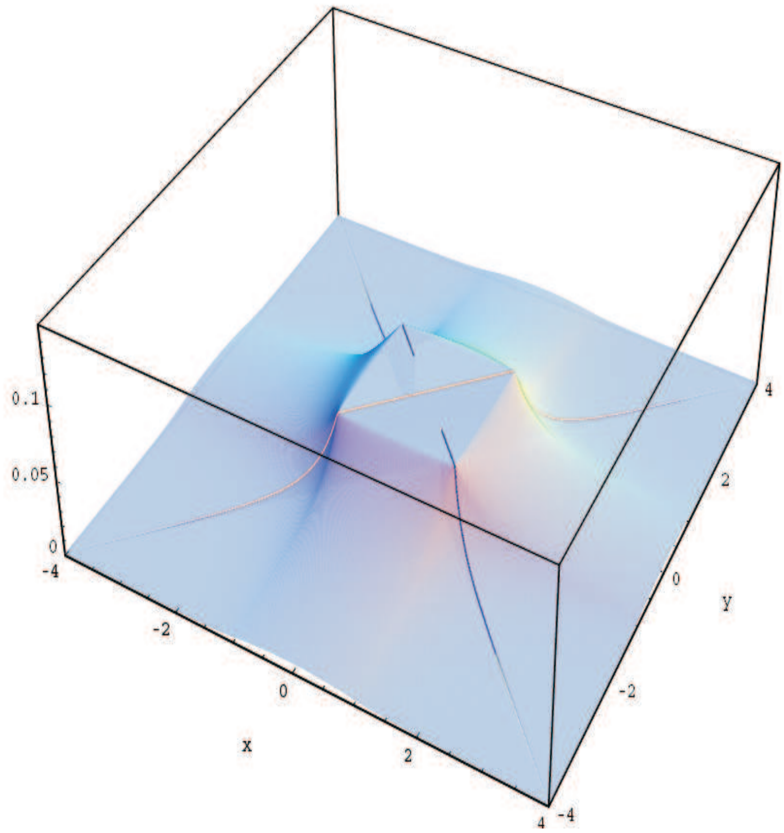


Figure 5.9: two-dimensional marginal $\Pi_{4-2}(x, y)$ for density $g_4(x, y, z, u)$

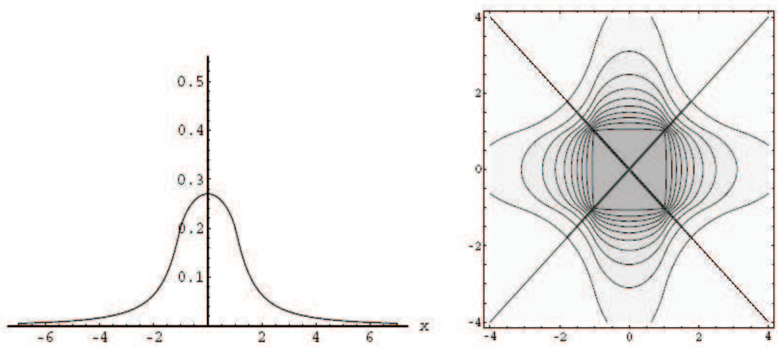


Figure 5.10: one-dimensional marginal $\Pi_{4-3}(x)$ and level curves of $\Pi_{4-2}(x, y)$ for density $g_4(x, y, z, u)$

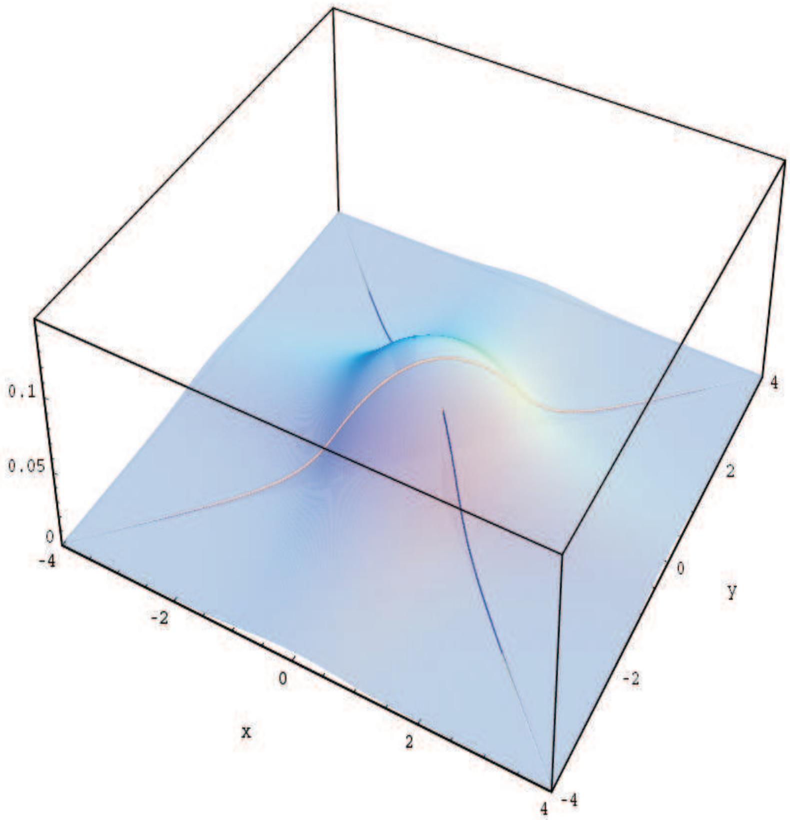


Figure 5.11: two-dimensional marginal $\Pi_{6-4}(x, y)$ for density $g_6(x, y, z, u, w, t)$

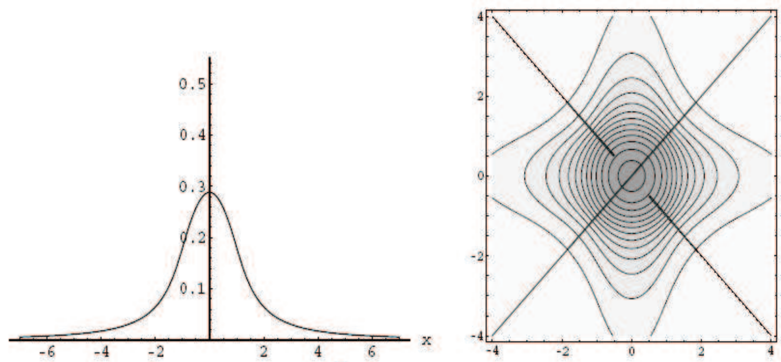


Figure 5.12: one-dimensional marginal $\Pi_{6-5}(x)$ and level curves of $\Pi_{6-4}(x, y)$ for density $g_3(x, y, z, u, w, t)$

In spite of the very complicated formulas for multidimensional densities of ℓ_1-symmetric distributions, simulating the corresponding random vectors is very simple. Before we give the description of the method of simulating let us recall some basic facts on Dirichlet and generalized Dirichlet distributions.

Definition 5.7.1. *A random vector* $(D_1, \dots, D_n)$ *has a Dirichlet distribution with parameters* $\beta_1, \dots, \beta_n$, $\beta_i > 0$ *for* $i = 1, \dots, n$ *(notation* $D(\beta_1, \dots, \beta_n)$*) if*

(i) $\sum_{i=1}^n |D_i| = 1$ *almost surely;*

(ii) the joint density function of $(D_1, \dots, D_{n-1})$ *is*

$$\frac{\Gamma(p_n)}{\Gamma(\beta_n)} \prod_{i=1}^{n-1} \frac{1}{\Gamma(\beta_i)} d_i^{\beta_i - 1} \left(1 - \sum_{i=1}^{n-1} d_i\right)_+^{\beta_n - 1},$$

where $p_n = \sum_{i=1}^n \beta_i$.

It is easy to see that the random vector $(\cos^2\theta, \sin^2\theta)$ with θ uniformly distributed on the interval $[0, \frac{\pi}{2}]$ has the $D(\frac{1}{2}, \frac{1}{2})$ distribution. There is a very well known procedure for constructing the Dirichlet distribution using Gamma random variables (for details see e.g. [35]). We describe it here in the following proposition.

Proposition 5.7.1. *Let* $Z_1, \dots, Z_n$ *be independent positive valued random variables with distributions* $\Gamma(\beta_i, 1)$ *respectively, i.e. the random variable* Z_i *has the density function*

$$\frac{1}{\Gamma(\beta_i)} z_i^{\beta_i - 1} \exp\{-z_i\} \qquad \text{for } z_i > 0.$$

Then the random vector

$$\left(\frac{Z_1}{\sum_{i=1}^n Z_i}, \dots, \frac{Z_n}{\sum_{i=1}^n Z_i}\right)$$

has the Dirichlet distribution $D(\beta_1, \dots, \beta_n)$.

Now we shall have no problem with simulating a random vector $(D_1, \dots, D_n)$ with the Dirichlet distribution $D(\frac{1}{2}, \dots, \frac{1}{2})$ appearing in the condition 2 of Theorem 5.7.1. It turns out however that we do. The computer simulation of $\Gamma(\frac{1}{2}, 1)$ distribution appeared to be good enough to simulate $(D_1, \dots, D_n)$, but the sample $(D_1, \dots, D_n)$ obtained this way is not good enough to produce samples of the random vector

$$\left(\frac{U_1}{\sqrt{D_1}}, \dots, \frac{U_n}{\sqrt{D_n}}\right).$$

Much better results can be obtained if we use trigonometric representation in

$\mathbb{R}^2$ and $\mathbb{R}^3$:

$$\left(\sqrt{D_1}, \sqrt{D_2}\right) \stackrel{d}{=} (\cos\theta, \sin\theta) \qquad \theta \sim \frac{2}{\pi}\mathbf{1}_{[0,\pi/2]};$$

$$\left(\sqrt{D_1}, \sqrt{D_2}, \sqrt{D_3}\right) \stackrel{d}{=} (\cos\theta\cos Q, \sin\theta\cos Q, \sin Q) \qquad \theta, Q \sim \frac{2}{\pi}\mathbf{1}_{[0,\pi/2]}.$$

In the case of higher dimensional spaces we shall use the following representation

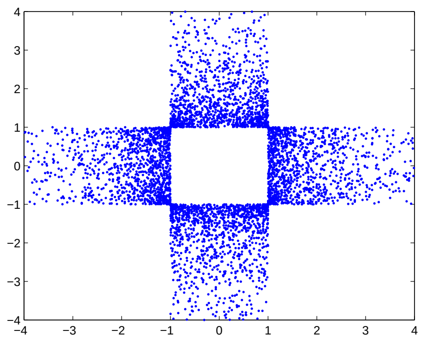


Figure 5.13: $n = 2$

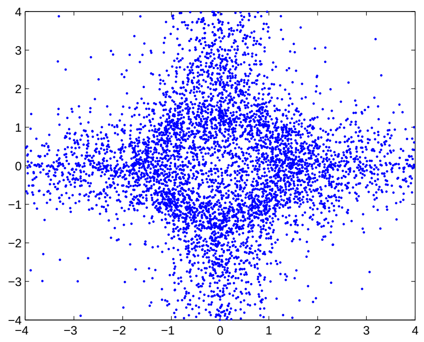


Figure 5.14: $n = 3$

$$\left(\sqrt{D_1}, \dots, \sqrt{D_n}\right) \stackrel{d}{=} \left(\frac{|W_1|}{\sqrt{\sum^n W_i^2}}, \dots, \frac{|W_n|}{\sqrt{\sum^n W_i^2}}\right),$$

where $W_1, \dots, W_n$ are independent identically distributed $N(0,1)$ Gaussian random variables.

Simulating the random vector $\mathbf{U}^n$ uniformly distributed on the unit sphere in $\mathbb{R}^n$ can be easily done using trigonometric representation for $n = 2$ and $n = 3$. In higher dimensional spaces however this construction is becoming rather complicated, thus we will use rather the idea of sign-symmetric Dirichlet type distributions (for details see [23]).

Definition 5.7.2. *A random vector $(B_1, \dots, B_n)$ has a sign-symmetric Dirichlet-type distribution with parameters $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ (notation $D(\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_n)$) if*

1. *B_n is a symmetric random variable;*
2. *$\sum_{i=1}^n |B_i|^{\alpha_i} = 1$ almost surely;*
3. *the joint density function of $(B_1, \dots, B_{n-1})$ is*

$$\frac{\Gamma(q_n)}{\Gamma(\beta_n/\alpha_n)} \prod_{i=1}^{n-1} \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} |b_i|^{\beta_i - 1} \left(1 - \sum_{i=1}^{n-1} |b_i|^{\alpha_i}\right)_+^{\frac{\beta_n}{\alpha_n} - 1},$$

where $q_k = \sum_{i=1}^k \beta_i/\alpha_i$.

If $\alpha_i \equiv 2$ and $\beta_i \equiv 1$ then the random vector $(B_1, \ldots, B_n)$ has a uniform distribution on the unit sphere in $\mathbb{R}^n$, i.e. $\mathbf{U}^n$ has generalized Dirichlet-type distribution $D(2, \ldots, 2; 1, \ldots, 1)$. In the following proposition we describe the method of simulating Dirichlet-type distribution.

Proposition 5.7.2. *Let $W_1, \ldots, W_n$ be independent real valued random variables such that the random variable W_i has the density function*

$$\frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} |w_i|^{\beta_i - 1} \exp\{-|w_i|^{\alpha_i}\}.$$

Then the random vector

$$\left(\frac{W_1}{(\sum_{i=1}^n |W_i|^{\alpha_i})^{1/\alpha_1}}, \ldots, \frac{W_n}{(\sum_{i=1}^n |W_i|^{\alpha_i})^{1/\alpha_n}} \right)$$

has the Dirichlet-type distribution $D(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$.

Notice that the random vector $\mathbf{U}^{(n)} = (U_1, \ldots, U_n)$ with uniform distribution on the unit sphere in $\mathbb{R}^n$ has the sign-symmetric Dirichlet-type distribution $D(2, \ldots, 2; 1, \ldots, 1)$ thus it can be sampled using Proposition 5.7.2. In this case we have that the corresponding random variables W_i are Gaussian, thus

$$(U_1, \ldots, U_n) \stackrel{d}{=} \left(\frac{W_1}{\sqrt{\sum^n W_i^2}}, \ldots, \frac{W_n}{\sqrt{\sum^n W_i^2}} \right),$$

where $W_1, \ldots, W_n$ are independent identically distributed $N(0, 1)$ Gaussian random variables.

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