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Discounted cost model for condition-based maintenance optimization

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ABSTRACT

This paper presents methods to evaluate the reliability and optimize the maintenance of engineering systems that are damaged by shocks or transients arriving randomly in time and overall degradation is modeled as a cumulative stochastic point process. The paper presents a conceptually clear and comprehensive derivation of formulas for computing the discounted cost associated with a maintenance policy combining both condition-based and age-based criteria for preventive maintenance. The proposed discounted cost model provides a more realistic basis for optimizing the maintenance policies than those based on the asymptotic, non-discounted cost rate criterion.

1. Introduction

Critical engineering systems in nuclear power plants, such as the reactor fuel core and piping systems, experience degradation due to stresses and unfavorable environment produced by transients or shocks in the reactor. For example, unplanned shutdowns and excursions to poor chemistry conditions result in degradation of components through corrosion, wear and fatigue of material. To control the risk due to failure of critical engineering systems in the plant, maintenance and replacements of degraded components are routinely performed. Because of uncertainty associated with the occurrence of shocks and damage produced by them, theory of stochastic processes plays a key role in estimating reliability and developing cost-effective maintenance strategies.

The failure of a system or structure occurs when its strength drops below a threshold that is necessary for resisting the applied stresses. This paper investigates the reliability of a structure that suffers damage due to shocks arriving randomly in time. Technically the total damage experienced by a system can be modeled as a sum of damage increments produced by individual shocks. To incorporate uncertainties, shocks are modeled as a stochastic point process and the damage produced by each shock is modeled as a positive random variable. In essence, the cumulative damage is modeled as a compound point process [1].

The theory of stochastic processes and its applications to reliability analysis have been discussed in several monographs [2–6]. Mercer [7] developed a stochastic model of wear (degradation) as a cumulative process in which shocks arrive as a Poisson process. A more generalized formulation of the first passage time or reliability function due to damage modeled as a compound renewal process was presented by Morey [8]. Kahle and Wendt [9] modeled shocks as a doubly stochastic Poisson process. Ebrahimi [10] proposed a cumulative damage model based on the Poisson shot noise process. Finkelstein [11] presented a non-homogeneous Poisson process model of shocks and considered the effect of population heterogeneity.

The cumulative damage models are popularly applied to the optimization of maintenance policies using the condition or age based criteria. Nakagawa [12] formulated a preventive maintenance policy, and an age-based policy was analyzed by Boland and Proschan [13]. Later several other policies were investigated by Nakagawa and co-workers [14–16]. Aven [17] presented an efficient method for optimizing the cost rate. An in-depth discussion of inspection and maintenance optimization models is presented in a recent monograph [18]. Grall et al. [19,20] analyzed condition-based maintenance policies by modeling the damage as a gamma process.

Previous studies mostly adopted asymptotic cost rate criterion for optimizing maintenance policies. However, the optimization of discounted cost is more pertinent to practical applications. Our

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experience also suggests that practical applications of compound point processes are limited due to lack of clarity about the mathematical derivations of cost rate and life expectancy. The primary objective of this paper is to present a conceptually clear derivation of discounted cost criterion for optimizing the maintenance of systems subject to stochastic cumulative damage. The proposed derivation is general and it can be reduced to special cases of homogeneous or non-homogeneous Poisson processes, or renewal process.

This paper is organized as follows. In Section 2, stochastic models of degradation and maintenance are presented. Section 3 describes a mathematical framework to evaluate the total expected discounted cost of maintenance. Expressions are derived for three specific maintenance policies. Illustrative examples are presented in Section 4 and conclusions are summarized in Section 5.

2. Stochastic model for degradation

In this paper the degradation is modeled as a stochastic cumulative damage process, where the system suffers damage due to shocks produced by transients (extreme of pressure, temperature and chemical environment). In this model occurrences of shocks are random in time and the damage produced by each shock is also a random variable. The total damage at time t is a sum or cumulation of damage increments produced by all j shocks occurred up to this time. In this section we present some well-known results, see for example Tijms [6] or Nakagawa [18].

Random occurrences of shocks over time, $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_j, \ldots$, are taken as points in a stochastic point process on $[0,\infty)$, as shown in Fig. 1. The total number of shocks in the interval $(0,t]$ is denoted by $\mathcal{N}(t)$ and $\mathcal{N}(0)\equiv 0$.

Define the probability of occurrence of j shocks in $(0,t]$ as

$$H_j(t) = P(\mathcal{N}(t)=j), \tag{1}$$

and the expected number of shocks as

$$R(t) = E(\mathcal{N}(t)). \tag{2}$$

In a given time interval $(0,t]$, the probability associated with the number of shocks (j) is related with that of the time of occurrence of the j th shock ($\mathcal{S}_j$) as

$$F_j(t) = P(\mathcal{S}_j \le t) = P(\mathcal{N}(t)\ge j) = \sum_{i=j}^{\infty} H_i(t). \tag{3}$$

Using this, Eq. (1) can also be rewritten as

$$H_j(t) = P(\mathcal{N}(t)\ge j) - P(\mathcal{N}(t)\ge j+1) = F_j(t) - F_{j+1}(t). \tag{4}$$

Note that $F_j(t)$ depends on the distribution of the time between the shocks.

A shock produces a random amount of damage Y, and its cumulative distribution function is denoted as

$$G(x) = P(Y \le x). \tag{5}$$

The damage occurred at the j th shock is denoted as Y_j.

The evaluation of cumulative damage is based on two key assumptions: (1) damage increments, $Y_1, Y_2, \ldots$, are independent and identically distributed (*iid*), and (2) the damage increments $(Y_j)_{j\ge 1}$ and the shock process $\{\mathcal{N}(t): t\ge 0\}$ are independent.

The total damage caused by j shocks is given as

$$D_j = \sum_{i=1}^{j} Y_j, \quad j \ge 1, \tag{6}$$

and $D_0 \equiv 0$. The distribution of D_j is obtained from the convolution of $G(x)$ as

$$P(D_j \le x) = G^{(j)}(x), \tag{7}$$

where

$$G^{(j+1)}(x) = \int_0^x G^{(j)}(x-y)\,\mathrm{d}G(y) = \int_0^x G(x-y)\,\mathrm{d}G^{(j)}(y). \tag{8}$$

Note that $G^{(0)}(x)=1$, $x \ge 0$. The total damage, $Z(t)$, at time t, however, depends on the number of shocks $\mathcal{N}(t)$ occurred in this interval, i.e.,

$$Z(t) = \sum_{j=1}^{\mathcal{N}(t)} Y_i = D_{\mathcal{N}(t)}. \tag{9}$$

Using the total probability theorem and independence between the sequence $Y_1, Y_2, \ldots$ and $\mathcal{N}(t)$, we can write for $x>0$

$$P(Z(t)>x) = \sum_{j=1}^{\infty} P(D_j > x, \mathcal{N}(t)=j) = \sum_{j=1}^{\infty}(1-G^{(j)}(x))H_j(t). \tag{10}$$

Using the facts that $\sum_{j=0}^{\infty} H_j(t)=1$ and $G^{(0)}(x)=1$, the distribution of the total damage can be written as

$$P(Z(t)\le x) = H_0(t) + \sum_{j=1}^{\infty} G^{(j)}(x)H_j(t) = \sum_{j=0}^{\infty} G^{(j)}(x)H_j(t). \tag{11}$$

Substituting $H_0(t)=1-F_1(t)$ and $H_j(t)=F_j(t)-F_{j+1}(t)$ from Eq. (4), it can be written as

$$P(Z(t)\le x) = 1 - \sum_{j=1}^{\infty}[G^{(j-1)}(x) - G^{(j)}(x)]F_j(t). \tag{12}$$

This is a fundamental expression that can be used to compute the system reliability. Suppose damage exceeding a limit z_F causes the component failure, Eq. (12) provides $P(Z(t)\le z_F)$ which is synonymous with the reliability function.

We conclude this section with a formula for the mean value of the first time τ_B that the total damage exceeds a level B

$$\tau_B = \min\{t : Z(t) > B\}. \tag{13}$$

So $\{\tau_B > t\} = \{Z(t)\le B\}$ and

$$E(\tau_B) = \sum_{j=0}^{\infty} G^{(j)}(B)\int_0^{\infty} H_j(t)\,\mathrm{d}t. \tag{14}$$

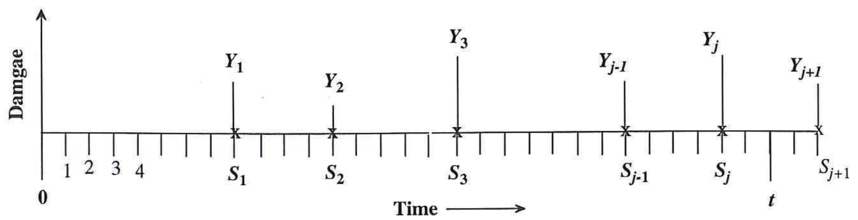

Fig. 1. A schematic of the stochastic shock process causing random damage.

3. Maintenance model

In practice, reliability of safety-critical systems in a nuclear plant is maintained by implementing preventive maintenance and replacement programs. In the proposed cumulative damage model, the failure can take place at time t when a shock occurring at t causes the total damage $Z(t)$ to exceed a critical threshold z_F. The failure prompts a corrective maintenance (CM) action involving the system renewal through replacement or complete overhaul (as good as new repair). We will study a preventive maintenance (PM) plan based on: (1) a condition-based strategy in which the system is renewed preventively as soon as $Z(t)$ exceeds a maintenance threshold value z_M, $z_M < z_F$, (2) an age-based strategy in which the system is replaced at certain age a, irrespective of its condition. It is assumed here that system's degradation is continuously monitored. The probabilities of occurrence of any such maintenance actions at time t need to be evaluated, considering the stochastic nature of the shock process and randomness associated with damage increments. This section presents the derivation of these probability terms.

3.1. Formulation

First we investigate three basic disjoint events that can take place at the time of occurrence ($\mathcal{S}_j$) of any j th shock, as shown in Fig. 2.

Define an event, A_j, that the total damage exceeds the PM threshold, z_M, at the j th shock ($j = 1,2,\ldots$) as

$$A_j = \{D_{j-1} \le z_M < D_j\}. \tag{15}$$

The event A_j consists of two disjoint events, namely, the occurrence of PM or CM action, and they are defined as

$$A_j^{PM} = A_j \cap \{D_j \le z_F\} \quad \text{and} \quad A_j^{CM} = A_j \cap \{D_j > z_F\}.$$

Finally, the event that no maintenance action is needed up to time $\mathcal{S}_j$ is defined as

$$B_j = \{D_j \le z_M\} = \bigcup_{i=j+1}^{\infty} A_i. \tag{16}$$

For the sake of conciseness, denote the probabilities of these events by $\alpha_j = P(A_j)$, $\beta_j = P(A_j^{PM})$, $\gamma_j = P(A_j^{CM})$ and $\pi_j = P(B_j)$. Now these probabilities can be derived in terms of the distribution of damage increments $G(x)$ as follows:

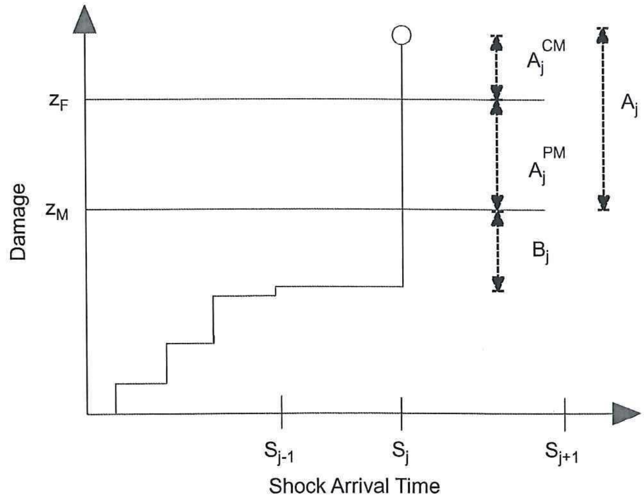

Fig. 2. Events related to the maintenance model.

$$\alpha_j = G^{(j-1)}(z_M) - G^{(j)}(z_M), \tag{17}$$

$$\beta_j = \int_0^{z_M} [G(z_F - x) - G(z_M - x)]\, dG^{(j-1)}(x), \tag{18}$$

$$\gamma_j = \alpha_j - \beta_j, \tag{19}$$

$$\pi_j = 1 - \sum_{i=1}^{j} \alpha_i = G^{(j)}(z_M). \tag{20}$$

The derivation of these probabilities is given in detail in Appendix A.

Note that $\alpha_j = \beta_j + \gamma_j$. Furthermore, for any $n \ge 1$, the collections $\{A_1,\ldots,A_n,B_n\}$ and $\{A_1^{PM},A_1^{CM},\ldots,A_n^{PM},A_n^{CM},B_n\}$ are both finite partitions of the sample space, so that we can write

$$\sum_{i=1}^{n} \alpha_i + \pi_n = \sum_{i=1}^{n} \beta_i + \sum_{i=1}^{n} \gamma_i + \pi_n = 1. \tag{21}$$

4. Probabilities associated with maintenance actions

The proposed model involves an interaction of age-based and condition-based preventive maintenance criteria. We start at time 0 when a new system is put into service. The system will be replaced at age a, should it survive up to this age. On the other hand, a corrective or preventive maintenance action before age a will be required if the cumulative damage at this time, $Z(a)$, exceeds the PM threshold z_M. The probability associated with this event, P_{CP}, can be derived using Eqs. (10) and (20) as

$$P_{CP} = P(Z(a) > z_M) = 1 - \sum_{j=0}^{\infty} G^{(j)}(z_M) H_j(a) = 1 - \sum_{j=0}^{\infty} \pi_j H_j(a).$$

Defining $\mathcal{A}_j = \sum_{i=1}^{j} \alpha_j$, it follows from Eq. (20) that

$$P_{CP} = \sum_{j=1}^{\infty} \mathcal{A}_j H_j(a). \tag{22}$$

Using Eqs. (12) and (17), we can also rewrite

$$P_{CP} = \sum_{j=1}^{\infty} \alpha_j F_j(a). \tag{23}$$

Let P_F be the probability that a corrective maintenance will be performed before age a.

$$P_F = \sum_{j=1}^{\infty} P(A_j^{CM}, \mathcal{S}_j < a) = \sum_{j=1}^{\infty} \gamma_j F_j(a). \tag{24}$$

Using Eq. (3), we can express P_F also in terms of the probability distribution of the total number of shocks as follows:

$$P_F = \sum_{j=1}^{\infty} \gamma_j F_j(a) = \sum_{j=1}^{\infty} \gamma_j \sum_{i=j}^{\infty} H_i(a).$$

Interchanging the order of the double sum, we get

$$P_F = \sum_{i=1}^{\infty} \mathcal{C}_i H_i(a), \tag{25}$$

where $\mathcal{C}_i = \sum_{j=1}^{i} \gamma_j$. Similarly, the probability of preventive maintenance before a can be obtained as

$$P_M = \sum_{j=1}^{\infty} P(A_j^{PM}, \mathcal{S}_j < a) = \sum_{j=1}^{\infty} \beta_j F_j(a) = \sum_{i=1}^{\infty} \mathcal{B}_i H_i(a), \tag{26}$$

where $\mathcal{B}_i = \sum_{j=1}^{i} \beta_j$. Finally, the probability of (age-based) replacement at age a is obtained as

$$P_A = 1 - P_{CP} = 1 - \sum_{j=1}^{\infty} \alpha_j F_j(a) = 1 - \sum_{i=1}^{\infty} \mathcal{A}_i H_i(a). \tag{27}$$

The results of this section are useful in computing the cost associated with a particular maintenance policy.

5. Discounted maintenance cost

Let T denote the length of renewal cycle and C the cost associated with the renewal. After the first renewal at time T_1, a new cycle starts, and it survives the duration T_2 and so on, as shown in Fig. 3. The renewal cost varies depending on the maintenance actions. We assume cost c_F for CM, c_A for replacement at age a and c_M for PM before age a. A random vector of renewal cycles and associated costs, (T_m, C_m) is an *iid* sequence generated by random variables T and C with the following joint distribution if $\mathcal{N}(a) = n$, $n \geq 1$:

$$(T, C) = \begin{cases} (\mathcal{S}_j, c_M) & \text{on } A_j^{PM}, \quad j \leq n, \\ (\mathcal{S}_j, c_F) & \text{on } A_j^{CM}, \quad j \leq n, \\ (a, c_A) & \text{on } B_n. \end{cases} \tag{28}$$

If no shocks occurred up to time a, i.e. $\mathcal{N}(a) = 0$, then $(T, C) = (a, c_A)$.

The total cost over a time interval $(0, t]$ is a sum of costs incurred over $M(t)$ number of completed renewal cycles, given as

$$K(t) = \sum_{j=1}^{M(t)} C_j. \tag{29}$$

So $M(t)$ is the renewal process associated with the times $U_j = \sum_{i=1}^{j} T_i, j = 1, 2, \ldots,$ at which new cycles start.

Note that $K(t)$ is a random function involving random variables $M(t)$ and C, and its distribution is very difficult to evaluate. Therefore it is convenient to work with asymptotic formulas for long term expected cost. A well-known result is that the long-term or asymptotic cost rate Q (i.e., cost per unit time) is given as [6]

$$Q = \lim_{t \to \infty} \frac{1}{t} K(t) = \frac{E(C)}{E(T)}. \tag{30}$$

Note that $E(T) \leq a$ denotes the expected length of renewal cycle and $E(C)$ is the expected cost over a renewal cycle.

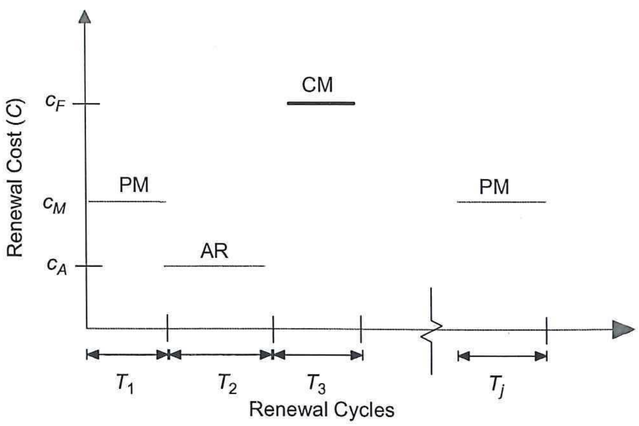

Fig. 3. Illustration of renewal cycles and costs (PM, preventive maintenance; CM, corrective maintenance; AR, age-based replacement).

Similarly, the discounted cost can be evaluated over a time horizon $(0, t]$ considering an exponential discounting factor with interest rate r per unit time as

$$K(t, r) = \sum_{j=1}^{M(t)} e^{-rU_j} C_j, \tag{31}$$

where $U_j = T_1 + \cdots + T_j$ denotes the time of j th renewal.

The long-term expected equivalent average cost per unit time is given as [21]:

$$Q(r) = \lim_{t \to \infty} rE(K(t, r)) = \frac{rE(Ce^{-rT})}{1 - E(e^{-rT})}. \tag{32}$$

Note that

$$\lim_{r \downarrow 0} Q(r) = \lim_{r \downarrow 0} \frac{rE(Ce^{-rT})}{1 - E(e^{-rT})} = \frac{E(C)}{E(T)} = Q.$$

6. Key formulas

In this section we present the formulas for the asymptotic cost rate Q and the long-term expected equivalent average discounted cost $Q(r)$ per unit time. Before the analysis of a general case, we discuss a simple case of a condition-based maintenance strategy without age replacement to illustrate the key ideas. Our numerical examples in Section 6 indicate that this simplified strategy could prove quite effective.

6.1. Without age replacement

First we will consider the simpler condition-based maintenance strategy in which the system is renewed preventively as soon as $Z(t)$ exceeds a maintenance threshold value z_M, $z_M < z_F$, without age replacement. In this case a renewal cycle ends only if the total damage exceeds the PM level z_M. We have

$$(T, C) = \begin{cases} (\mathcal{S}_j, c_M) & \text{on } A_j^{PM}, \\ (\mathcal{S}_j, c_F) & \text{on } A_j^{CM}. \end{cases}$$

In order to derive the cost rate using Eq. (30) or (32), a number of expectations have to be evaluated:

$$E(Ce^{-rT}) = \sum_{j=1}^{\infty} (c_M \beta_j + c_F \gamma_j) E(e^{-r\mathcal{S}_j}),$$

and substituting $C \equiv 1$, we get

$$E(e^{-rT}) = \sum_{j=1}^{\infty} \alpha_j E(e^{-r\mathcal{S}_j}),$$

and substitution of $r = 0$ yields

$$E(C) = \sum_{j=1}^{\infty} (c_M \beta_j + c_F \gamma_j).$$

For the expected value of the cycle length we get

$$E(T) = \sum_{j=1}^{\infty} \alpha_j E(\mathcal{S}_j).$$

It follows that

$$Q = \frac{\sum_{j=1}^{\infty} (c_M \beta_j + c_F \gamma_j)}{\sum_{j=1}^{\infty} \alpha_j E(\mathcal{S}_j)} \tag{33}$$

and

$$Q(r) = \frac{r \sum_{j=1}^{\infty} (c_M \beta_j + c_F \gamma_j) E(e^{-r\mathcal{S}_j})}{1 - \sum_{j=1}^{\infty} \alpha_j E(e^{-r\mathcal{S}_j})}. \tag{34}$$

These formulas for Q and $Q(r)$ contain the information of the shock process in terms of the occurrence times of the shocks. Especially when the shock process is a renewal process we can simplify formulas (33) and (34). In this case $\mathcal{S}_j = X_1 + \cdots + X_j$ where (X_j) is the iid sequence of inter-occurrence times and we can substitute $E(\mathcal{S}_j) = jm$ and $E(e^{-r\mathcal{S}_j}) = \omega^j$, where $m = E(X_1)$ and $\omega = E(e^{-rX_1})$.

We now continue with expressions for Q and $Q(r)$ in terms of the probability distribution of the total number of shocks. Using partial integration and Eq. (3) we get

$$E(e^{-r\mathcal{S}_j}) = \int_0^\infty e^{-rx}\,\mathrm{d}F_j(x) = r\sum_{i=j}^\infty \int_0^\infty H_i(x)e^{-rx}\,\mathrm{d}x$$

and

$$E(\mathcal{S}_j) = \sum_{i=j}^\infty \int_0^\infty H_i(x)\,\mathrm{d}x.$$

Note that, interchanging the order of the double sum, we can write for example

$$\sum_{j=1}^\infty \alpha_j E(e^{-r\mathcal{S}_j}) = \sum_{j=1}^\infty \alpha_j r \sum_{i=j}^\infty \int_0^\infty H_i(x)e^{-rx}\,\mathrm{d}x = r\sum_{i=1}^\infty \mathcal{A}_i \int_0^\infty H_i(x)e^{-rx}\,\mathrm{d}x.$$

It follows that

$$Q = \frac{\sum_{j=1}^\infty (c_M\beta_j + c_F\gamma_j)}{\sum_{i=1}^\infty \mathcal{A}_i \int_0^\infty H_i(x)\,\mathrm{d}x} \tag{35}$$

and

$$Q(r) = \frac{r^2\sum_{i=1}^\infty (c_M\mathcal{B}_i + c_F\mathcal{C}_i)\int_0^\infty H_i(x)e^{-rx}\,\mathrm{d}x}{1 - r\sum_{i=1}^\infty \mathcal{A}_i \int_0^\infty H_i(x)e^{-rx}\,\mathrm{d}x}. \tag{36}$$

6.2. *With age replacement*

We will present now the results for condition-based PM with age replacement. The derivations can be found in Appendix B.

6.2.1. *No discounting*

As discussed before, there are three possible ways for the renewal of a structure at time T: PM when $z_M \le Z(T) < z_F$, replacement when $T = a$, and CM when $Z(T) \ge z_F$. The expected length of the renewal cycle is given as

$$E(T) = \sum_{j=0}^\infty \pi_j \int_0^a H_j(x)\,\mathrm{d}x. \tag{37}$$

A complete derivation is presented in Appendix B.

The expected cost incurred in a renewal cycle is simply a product of probabilities of the three possible renewal actions with the associated costs:

$$E(C) = c_AP_A + c_MP_M + c_FP_F = c_A + \sum_{j=1}^\infty [c_M\beta_j + c_F\gamma_j - c_A\alpha_j]F_j(a). \tag{38}$$

These probabilities are derived in Section 2.2.3. From now on we will assume that

$$c_0 = c_M = c_A \quad \text{and} \quad c_F = c_0 + \delta_F. \tag{39}$$

Substituting Eq. (25), the following expression is obtained:

$$E(C) = c_0 + \delta_F P_F = c_0 + \delta_F \sum_{i=1}^\infty C_i H_i(a). \tag{40}$$

Substituting this expression for $E(C)$ and Eq. (37) in Eq. (30), the asymptotic cost rate Q is obtained as

$$Q = \frac{c_0 + \delta_F \sum_{i=1}^\infty C_i H_i(a)}{\sum_{j=0}^\infty \pi_j \int_0^a H_j(x)\,\mathrm{d}x}. \tag{41}$$

It is also possible to express Q in terms of the distribution functions F_j of the occurrence times $\mathcal{S}_j$ of the shocks. Using Eq. (20), we can write the enumerator as

$$\sum_{j=0}^\infty \pi_j \int_0^a H_j(x)\,\mathrm{d}x = \int_0^a H_0(x)\,\mathrm{d}x + \sum_{j=1}^\infty \left(1 - \sum_{i=1}^j \alpha_i\right)\int_0^a H_j(x)\,\mathrm{d}x.$$

Since $\sum_j H_j(x) \equiv 1$, we get, interchanging the double sum and using Eq. (3)

$$\sum_{j=0}^\infty \pi_j \int_0^a H_j(x)\,\mathrm{d}x = a - \sum_{i=1}^\infty \alpha_i \int_0^a F_i(x)\,\mathrm{d}x.$$

The enumerator can be written as

$$c_0 + \delta_F \sum_{i=1}^\infty C_i H_i(a) = c_0 + \delta_F \sum_{i=1}^\infty \gamma_i F_i(a).$$

These expressions are useful if the shock process $\mathcal{N}$ is a renewal process.

6.2.2. *With discounting*

A general expression for expected discounted cost in a renewal cycle T is derived in Appendix B as formula (42)

$$E(Ce^{-rT}) = e^{-ra}\left(c_0 + \delta_F \sum_{n=1}^\infty C_nH_n(a)\right) + \sum_{n=1}^\infty (c_0\mathcal{A}_n + \delta_F\mathcal{C}_n)\int_0^a H_n(x)re^{-rx}\,\mathrm{d}x. \tag{42}$$

From this expression, the expected (discounted) length of the renewal cycle can be obtained by substituting $C = 1$

$$E(e^{-rT}) = 1 - \sum_{n=0}^\infty \pi_n \int_0^a H_n(x)re^{-rx}\,\mathrm{d}x. \tag{43}$$

It follows that the long-term expected equivalent average discounted cost per unit time is given as

$$Q(r) = \frac{e^{-ra}(c_0 + \delta_F\sum_{n=1}^\infty C_nH_n(a)) + r\sum_{n=1}^\infty (c_0\mathcal{A}_n + \delta_F\mathcal{C}_n)\int_0^a H_n(x)e^{-rx}\,\mathrm{d}x}{\sum_{n=0}^\infty \pi_n \int_0^a H_n(x)e^{-rx}\,\mathrm{d}x}. \tag{44}$$

In the case that the shock process $\mathcal{N}$ is a renewal process, it is useful to have an expression for $Q(r)$ in terms of the times $\mathcal{S}_j$ at which the shocks occur. We refer to Appendix B for the expressions. Also here Laplace transforms can be used for the calculations.

All the expressions derived in this section are completely general, without making any specific assumptions about the form of the point process $\mathcal{N}$ that describes the random arrival of shocks in time.

7. Applications

In Section 3 we have derived formulas for the cost of maintenance of systems that deteriorate as a consequence of the cumulative damage of shocks occurring randomly in time. The damage per shock was also assumed to be random, but independent of the shock process. Monitoring is continuous and if the level of damage exceeds some critical threshold z_F the system needs a major repair (CM). A preventive maintenance (PM) will be performed if the total damage to the system exceeds the intermediate level $z_M < z_F$ and a planned replacement takes place at age a. After repair the system is good as new. To get simpler expressions, we assume as in Section 3, that the cost c_0 for PM is the same as for age replacement. The cost for CM is $c_0 + \delta_F$. In this section, a specific case is analyzed and results are presented in explicit analytical form. We will assume that the damage increments are exponentially distributed. For the shock process we will consider a homogeneous Poisson process (HPP) and also an example of non-homogeneous Poisson process (NHPP). Results will be presented for the maintenance cost rate considering both discounted and non-discounted cases.

7.1. Damage process

The damage increment per shock follows an exponential distributed with mean $1/\lambda > 0$, and its distribution is given as

$$G(x) = 1 - e^{-\lambda x}, \quad x \geq 0, \tag{45}$$

The distribution of the sum D_k of k damage increments is then obtained as

$$P(D_k \leq x) = G^{(k)}(x) = 1 - \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!} e^{-\lambda x}, \quad k = 1, 2, \ldots . \tag{46}$$

We continue with the calculation of the probabilities α_j, β_j, γ_j and π_j defined in Section 3.1. Note first that for any $\zeta \geq z_M$, we have by partial integration and the definition of G

$$\int_0^{z_M} [1 - G(\zeta - x)]\, dG^{(j-1)}(x) = e^{-\lambda(\zeta - z_M)} G^{(j-1)}(z_M) - \int_0^{z_M} G^{(j-1)}(x) \lambda e^{-\lambda(\zeta - x)}\, dx.$$

Substituting Eq. (46), it follows after simplification that

$$\int_0^{z_M} [1 - G(\zeta - x)]\, dG^{(j-1)}(x) = \frac{(\lambda z_M)^{j-1}}{(j-1)!} e^{-\lambda \zeta}.$$

So

$$\beta_j = \int_0^{z_M} [G(z_F - x) - G(z_M - x)]\, dG^{(j-1)}(x) = \frac{(\lambda z_M)^{j-1}}{(j-1)!} (e^{-\lambda z_M} - e^{-\lambda z_F}) \tag{47}$$

and

$$\alpha_j = G^{(j-1)}(z_M) - G^{(j)}(z_M) = \frac{(\lambda z_M)^{j-1}}{(j-1)!} e^{-\lambda z_M}, \tag{48}$$

$$\gamma_j = \alpha_j - \beta_j = \frac{(\lambda z_M)^{j-1}}{(j-1)!} e^{-\lambda z_F}, \tag{49}$$

$$\pi_j = 1 - \sum_{i=0}^{j-1} \frac{(\lambda z_M)^i}{i!} e^{-\lambda z_M}. \tag{50}$$

It follows that $\mathcal{B}_n = (1 - e^{-\lambda(z_F - z_M)}) \mathcal{A}_n$, $\mathcal{C}_n = e^{-\lambda(z_F - z_M)} \mathcal{A}_n$ and $\pi_n = 1 - \mathcal{A}_n$. Define the sums

$$\Sigma_1 = \sum_{n=1}^{\infty} \mathcal{A}_n H_n(a),$$

$$\Sigma_2 = \sum_{n=1}^{\infty} \mathcal{A}_n \int_0^a H_n(x)\, dx,$$

$$\Sigma_3 = \sum_{n=1}^{\infty} \mathcal{A}_n \int_0^a H_n(x) e^{-rx}\, dx.$$

It follows from Eqs. (40) and (37) that

$$E(C) = c_0 + \delta_F e^{-\lambda(z_F - z_M)} \Sigma_1$$

and

$$E(T) = \sum_{j=0}^{\infty} \pi_j \int_0^a H_j(x)\, dx = \sum_{j=0}^{\infty} (1 - \mathcal{A}_j) \int_0^a H_j(x)\, dx = a - \Sigma_2.$$

Hence, the asymptotic non-discounted cost rate can be given as

$$Q = \frac{c_0 + \delta_F e^{-\lambda(z_F - z_M)} \Sigma_1}{a - \Sigma_2}. \tag{51}$$

To get the expected equivalent average discounted cost per unit time $Q(r)$, we first note that

$$\sum_{n=0}^{\infty} \pi_n \int_0^a H_n(x) e^{-rx}\, dx = \sum_{n=0}^{\infty} (1 - \mathcal{A}_n) \int_0^a H_n(x) e^{-rx}\, dx = \frac{1 - e^{-ra}}{r} - \Sigma_3.$$

Substituting this in Eq. (44) we get

$$Q(r) = r \frac{e^{-ra}(c_0 + \delta_F e^{-\lambda(z_F - z_M)} \Sigma_1) + r(c_0 + \delta_F e^{-\lambda(z_F - z_M)}) \Sigma_3}{1 - e^{-ra} - r\Sigma_3}. \tag{52}$$

For a condition based maintenance plan without age replacement, the formulas (33) and (34) can de simplified to

$$Q = \frac{c_0 + \delta_F e^{-\lambda(z_F - z_M)}}{m(\lambda z_M + 1)} \tag{53}$$

and

$$Q(r) = r \frac{(c_0 + \delta_F e^{-\lambda(z_F - z_M)}) \omega e^{-\lambda z_M(1-\omega)}}{1 - \omega e^{-\lambda z_M(1-\omega)}}. \tag{54}$$

where $m = E(X_1)$ is the mean of the inter-occurrence time X_1 and $\omega = E(e^{-rX_1})$ the moment-generating function evaluated in the discount rate r.

7.2. Shock process

We will consider three examples of shock processes: (1) a homogeneous Poisson process with intensity μ and (2) a non-homogeneous Poisson process with intensity $\mu(t) = 2t$ and (3) a renewal process with Weibull distributed inter-occurrence times. It is clear from Section 7.1 that for a calculation of the cost we have to calculate the sums Σ_1, Σ_2 and Σ_3. Define

$$H_j = H_j(a),$$

$$I_j = \int_0^a H_j(x)\, dx,$$

$$int_j = \int_0^a H_j(x) e^{-rx}\, dx.$$

For a homogeneous Poisson process with intensity μ we have

$$H_j(t) = \frac{(\mu t)^j e^{-\mu t}}{j!}, \quad j = 0, 1, 2, \ldots . \tag{55}$$

The calculation of the sums Σ_1, Σ_2 and Σ_3 is based on the following recursion. For $j = 1, 2, \ldots$

$$H_j = \frac{\mu a}{j} H_{j-1},$$

$$I_j = -\frac{a}{j} H_{j-1} + I_{j-1},$$

$$int_j = -\frac{\mu a e^{-ra}}{(\mu + r) j} H_{j-1} + \frac{\mu}{\mu + r} I_{j-1}$$

and

$$H_0 = e^{-\mu a},$$

$$I_0 = \frac{1}{\mu}(1 - e^{-\mu a})$$

$$int_0 = \frac{1}{\mu + r}(1 - e^{-(\mu + r)a}).$$

Based on this recursion we can set up an algorithm that calculates the sums Σ_1, Σ_2 and Σ_3 and the cost.

For a nonhomogeneous Poisson process with intensity $\mu(t) = 2t$ we have

$$H_j(t) = \frac{1}{j!} t^{2j} e^{-t^2}, \quad j = 0, 1, 2, \ldots . \tag{56}$$

As for a HPP, we can set up an recursion to calculate the sums Σ_1, Σ_2 and Σ_3 and the cost.

In the end, we also consider a more general case in which shocks arrive as a renewal process with Weibull distributed inter-occurrence times with probability density

$$f(t) = \alpha\left(\frac{t}{\alpha}\right)^{\alpha-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\alpha}\right]. \tag{57}$$

In this case, we compute the costs of a condition-based maintenance plan without age replacement, and compare the costs with and without discounting.

7.3. Results

This section presents a number of numerical examples to illustrate the optimization of cost rate in terms of the replacement age a and the PM level z_M. In all the calculations, we use the following parameter values. The parameter λ of the damage increment distribution is taken to be $\lambda = 0.5$. The failure level $z_F = 30$. The cost parameters are chosen as $c_0 = 20$ and $\delta_F = 80$ and the discount rate is $r = 0.05$. The intensity of the HPP is chosen as $\mu = 4.06$. For this choice of μ, the expected value of the time τ_{30} to exceed the failure level $z_F = 30$ is equal to 3.9, for both the HPP and the NHPP shock processes.

7.3.1. Optimization of the replacement age a for a specified PM level z_M

Suppose that the threshold for PM is specified as $z_M = 28$ and the objective is to find an optimal value of the replacement age (a) that would minimize the cost rate. Fig. 4 contains plots of the cost rate as a function of the replacement age a, for HPP and an NHPP shock processes. The optimal replacement age (a_{OPT}) through the discounted analysis is found as 2.8 for HPP case and 2.1 for the NHPP shock process. In this example, results with and without discounting are relatively close.

7.3.2. Simultaneous optimization of replacement age a and PM level z_M

The cost rate can be optimized with respect to both variables $z_M \in [0; z_F]$ and $a \in [0.1; 15]$. The results of two-dimensional optimization can be presented in a more concise form. For example, first we select a $z_M \in [0; z_F]$ value and compute the cost for all possible values of the replacement age $a \in [0.1; 15]$, and then select the minimum cost rate. Fig. 5 shows one dimensional variation of these minimal cost rates as a function of the PM level z_M for both cases of HPP and NHPP shock processes. The global minimum discounted cost rate over is 6.8 for $z_M = 22.6$, and in case of NHPP, it is 5.5 for $z_M = 21.3$.

In this optimization, the relationship between the discounted cost rate and replacement age for the optimum value of z_M is shown in Fig. 6. The first plot shows the impact of the replacement age in HPP case with an optimal $z_M = 22.6$, and the second plot shows results for the NHPP case ($z_M = 21.3$). In both cases we see that the minimum is attained almost after the replacement age of $a = 5$, and after that there is no influence of the replacement age on the cost. It seems to suggests that age replacement could be deleted from the cost optimization. In the next subsection we will compare maintenance strategies with and without age replacement.

7.3.3. Optimal PM level for a maintenance strategy without age replacement

Motivated by the results of Fig. 6, we compute cost rate as a function of z_M only, and ignore the replacement age from the analysis. Results presented in Fig. 7 show that in HPP case the minimum discounted cost is 6.8 for $z_M = 22.5$, whereas in NHPP case the result is 5.5 for $z_M = 21.3$. The optimal PM level in both the cases (HPP and NHPP shock processes) is the same as that we obtained considering the age-based replacement strategy in Section 7.3.2. Thus the comparison of results shown in Figs. 6 and 7 confirms that for condition-based maintenance with continuous monitoring, age-based replacement is not contributing to minimization of the cost. This may also be due to the asymptotic cost criterion that we are using in the analysis. For a finite time horizon, it is possible that replacement age may have more pronounced effect in the minimization of the cost rate.

7.3.4. Optimum replacement age without PM

The consideration PM can be excluded from the analysis by setting the PM level z_M equal to the failure level z_F. Now the cost rate is a function of only the replacement age, as shown by the results presented in Fig. 8. Although an optimal age exists ($a \approx = 2.5-3$) for both HP and NHPP cases, the associate cost rate is much higher (almost double) than that obtained from the optimization based on PM level (see results in Fig. 7).

7.3.5. Is discounting important?

Results presented in Figs. 4–8 suggest that results obtained from with and without discounting are in close agreement. We wish to assert that it is not a general conclusion. The consideration of discounting is very important in practical maintenance optimization problem. The reason is that in practical situations money is borrowed from Banks to implement the maintenance programs. Therefore, deferring the maintenance cost to future results in a lower net present value, which should be accounted for in the optimization process.

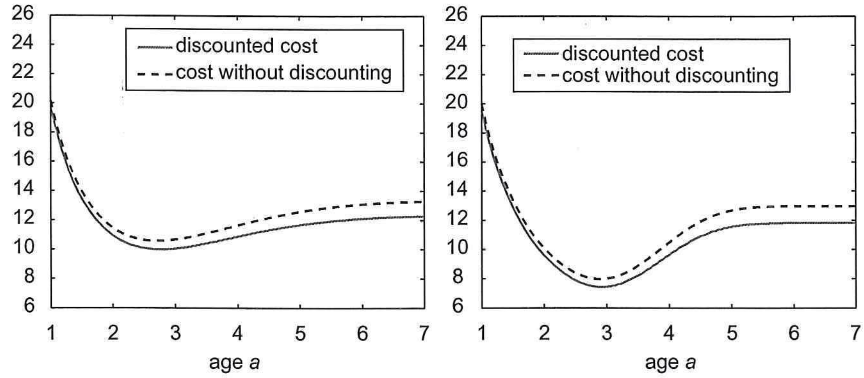

Fig. 4. Cost rate as a function of the replacement age a, left HPP, right NHPP shock process.

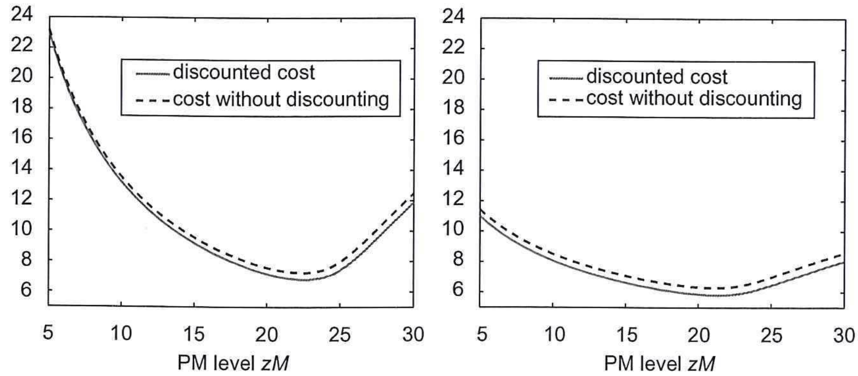

Fig. 5. Minimum cost rate over the replacement age a as a function of the PM level z_M, left HPP, right NHPP shock process.

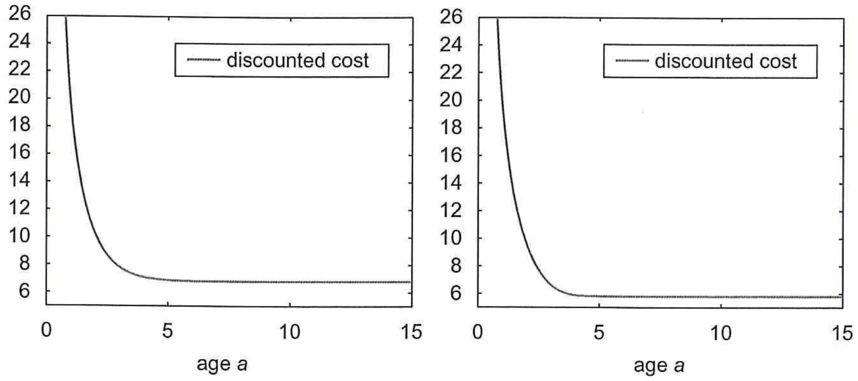

Fig. 6. Discounted cost rate as a function of the replacement age a for optimal value of z_M, left HPP with $z_M=22.6$, right NHPP shock process with $z_M=21.3$.

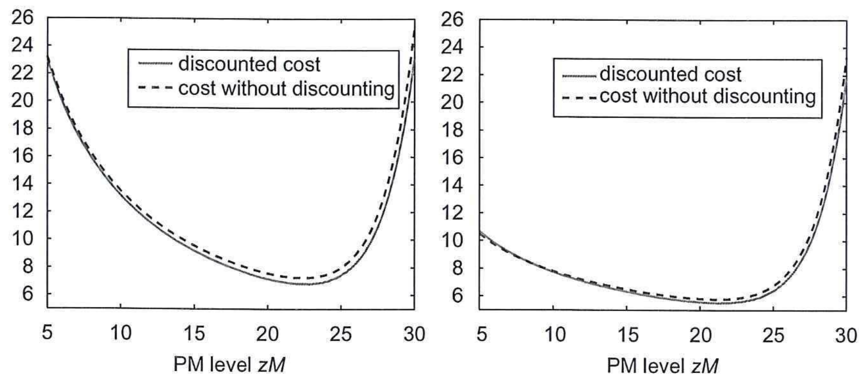

Fig. 7. Cost rate as a function of the PM level z_M without considering age replacement, left HPP, right NHPP shock process.

The effect of discounting depends on the magnitude of the intensity (rate) of shock process. To illustrate this point, we consider an example in which the shock process is modeled as a renewal process with Weibull inter-occurrence times. We choose the following parameters in (57): the scale parameter $\theta=3$ and $\alpha=2$. This results in the mean time of the first occurrence, $m=2.9867$ unit of time. We compute both discounted and non-discounted cost rate as a function of the PM level z_M. Results plotted in Fig. 9 show the importance of discounting, since the discounted minimum cost rate is approximately half of that obtained without discounting.

8. Conclusions

The theory of stochastic point processes and the renewal theorem have been fundamental to the development of risk-based maintenance models for critical engineering systems. The paper presents a comprehensive analytical formulation of the condition and age based maintenance policies for systems subject to random shocks occurring randomly in time. Explicit expressions are derived for asymptotic cost rate with and without discounting. The proposed framework allows to determine preventive maintenance (PM) level and age of replacement that would minimize the maintenance cost rate.

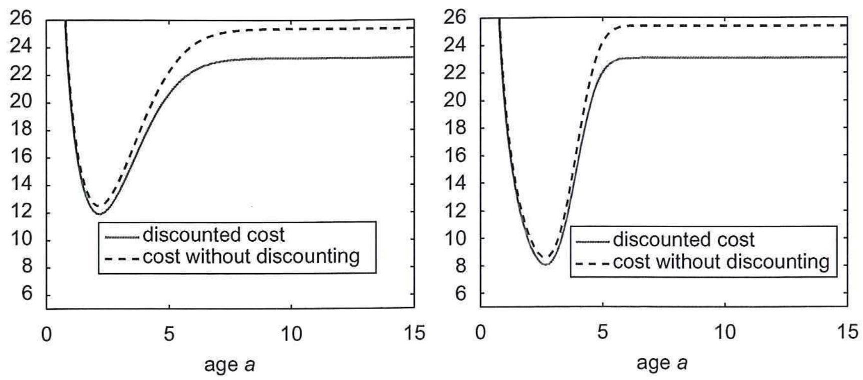

Fig. 8. Cost rate as a function of the age a considering only age replacement, left HPP, right NHPP shock process.

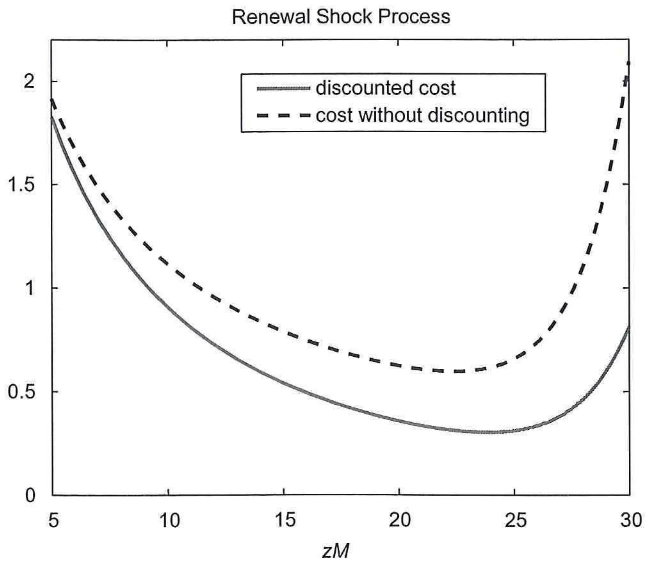

Fig. 9. Cost rate as a function of the PM level z_M without age replacement for a renewal shock process with Weibull inter-occurrence times.

Numerical examples of maintenance cost optimization are presented in the paper to illustrate the proposed model. Shock processes are modeled as homogeneous (HPP) and non-homogeneous Poisson (NHPP) processes, and the renewal process. The damage increments are taken as exponentially distributed. The optimization variables are the PM damage threshold (z_M) and the replacement age (a). For a specified z_M, an optimal a can be found that would minimize the cost rate. It is interesting to note that the global minimum of the cost rate is rather insensitive to the age of replacement (a), and it is the same as the minimum cost associated with a condition-based strategy only. This suggests that a condition-based strategy is likely to supersede the age-based replacement for minimizing the cost rate.

The following key conclusions are drawn:

- The derivation of discounted cost rate is a main contribution of this paper. This provides more practical solutions to the optimization of maintenance policies than those based on the asymptotic (non-discounted) cost rate criterion.
- This formulation can be extended to include dependent damage increments, such as the damage following a Markov chain.
- The proposed formulation can incorporate a general point process as a shock process. This topic will presented in a future paper.

Acknowledgments

This paper is dedicated to the memory of our esteemed friend and colleague Professor Jan van Noortwijk, who recently passed away at a very young age.

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Appendix A

First we explain the derivation of $P(A_j)=\alpha_j = G^{(j-1)}(z_M) - G^{(j)}(z_M)$. Using the total probability theorem, we can write

$$P(D_{j-1} \le z_M) = P(D_{j-1} \le z_M, D_j \le z_M) + P(D_{j-1} \le z_M, D_j > z_M). \tag{58}$$

Since $\{D_j \le z_M\}$ is contained within the event $\{D_{j-1} \le z_M\}$, it reduces the first probability term in Eq. (58) to $P(D_j \le z_M)$. It is clear from Eq. (15) that the event in the second probability term is the same as $\{D_{j-1} \le z_M < D_j\} = A_j$. Thus,

$$P(D_{j-1} \le z_M) = P(D_j \le z_M) + P(A_j).$$

Using Eq. (7), we can simplify

$$G^{(j-1)}(z_M) = G^{(j)}(z_M) + P(A_j),$$

which leads to

$$P(A_j) = \alpha_j = G^{(j-1)}(z_M) - G^{(j)}(z_M).$$

When the PM threshold is exceeded in shocks j to $(j-1)$, there are two possible mutually exclusive events: either the total damage is below z_F requiring a PM action, or the total damage exceeds level z_F leading to a CM action. It implies that $\alpha_j = \beta_j + \gamma_j$.

To calculate β_j, first note that

$$A_j^{PM} = \{D_{j-1} \le z_M, z_M < D_{j-1} + Y_j \le z_F\}. \tag{59}$$

It follows that

$$\beta_j = P(D_{j-1} \le z_M, z_M < D_{j-1} + Y_j \le z_F) \tag{60}$$

which can be evaluated as

$$\beta_j = \int_0^{z_M} P(z_M < x + Y_j \le z_F)\, dG^{(j-1)}(x) = \int_0^{z_M} [G(z_F - x) - G(z_M - x)]\, dG^{(j-1)}(x). \tag{61}$$

In the same way we can evaluate γ_j as

$$\gamma_j = P(D_{j-1} \le z_M, D_{j-1} + Y_j > z_F). \tag{62}$$

As before, it can be rewritten as

$$\gamma_j = \int_0^{z_M} [1 - G(z_F - x)]\, dG^{(j-1)}(x). \tag{63}$$

Finally, the probability $P(B_j) = \pi_j$ of the event that the total damage after the j th shock is still below z_M is derived from Eq. (16) as

$$\pi_j = P(D_j \le z_M) = P\left(\bigcup_{i=j+1}^{\infty} A_i\right) = 1 - \sum_{i=1}^{j} P(A_i) = 1 - \sum_{i=1}^{j} \alpha_i. \tag{64}$$

Substituting for $P(A_i) = \alpha_i$ from Eq. (59) and further simplification leads to

$$\pi_j = 1 - \sum_{i=1}^{j} [G^{(i-1)}(z_M) - G^{(i)}(z_M)] = G^{(j)}(z_M). \tag{65}$$

Appendix B

This appendix presents the derivations of a key result (42). First the expected discounted cost expended can be using the total probability theorem as

$$E[Ce^{-rT}] = \sum_{n=0}^{\infty} E[Ce^{-rT}; \mathcal{N}(a) = n]. \tag{66}$$

For $n = 0$ we have a simple form

$$E[Ce^{-rT}; \mathcal{N}(a) = 0] = c_A e^{-ra} H_0(a).$$

For $n \ge 1$ we split the expectation over the disjoint events $\{A_1^{PM}, A_1^{CM}, \ldots, A_n^{PM}, A_n^{CM}, B_n\}$. Using definition (28) of (T, C), we get

$$E[Ce^{-rT}; \mathcal{N}(a) = n] = c_M \sum_{j=1}^{n} \beta_j E[e^{-rS_j}; \mathcal{N}(a) = n] + c_F \sum_{j=1}^{n} \gamma_j E[e^{-rS_j}; \mathcal{N}(a) = n] + c_A e^{-ra} \pi_n H_n(a). \tag{67}$$

Substitution of Eq. (67) in Eq. (66), results in two terms involving double sum. The first such term can be simplified as follows:

$$\sum_{n=1}^{\infty} \sum_{j=1}^{n} \beta_j E[e^{-rS_j}; \mathcal{N}(a) = n] = \sum_{j=1}^{\infty} \beta_j \sum_{n=j}^{\infty} E[e^{-rS_j}; \mathcal{N}(a) = n] = \sum_{j=1}^{\infty} \beta_j E[e^{-rS_j}; \mathcal{N}(a) \ge j] = \sum_{j=1}^{\infty} \beta_j E[e^{-rS_j}; S_j \le a].$$

The first step is obtained by interchanging the order of double sum from n and j to j and n. The second step is obtained from the fact that the inner sum $n = j - \infty$ implies the condition $\mathcal{N}(a) \ge j$. This condition further implies that the time of occurrence of j th shock should be within the interval $(0, a]$, which leads to the final step in the above equation. The second double term sum is simplified similarly. The third sum can also be expressed in terms of the times $\mathcal{S}_j$ at which the shocks occur as follows:

$$\sum_{n=1}^{\infty} \pi_n H_n(a) = \sum_{n=1}^{\infty} \left(1 - \sum_{j=1}^{i} \alpha_j\right) P(\mathcal{N}(a) = n) = P(\mathcal{S}_1 \le a) - \sum_{j=1}^{\infty} \alpha_j P(\mathcal{S}_j \le a).$$

Finally, we obtain

$$E[Ce^{-rT}] = c_A e^{-ra}\left(P(\mathcal{S}_1 \le a) - \sum_{j=1}^{\infty} \alpha_j P(\mathcal{S}_j \le a)\right) + \sum_{j=1}^{\infty} (c_M \beta_j + c_F \gamma_j) E[e^{-rS_j}; \mathcal{S}_j \le a]. \tag{68}$$

Taking in this formula $c_A = c_M = c_F = 1$ and $\beta_j + \gamma_j = \alpha_j$, we get after simplification

$$E[e^{-rT}] = e^{-ra}\left(P(\mathcal{S}_1 \le a) - \sum_{j=1}^{\infty} \alpha_j P(\mathcal{S}_j \le a)\right) + \sum_{j=1}^{\infty} \alpha_j E[e^{-rS_j}; \mathcal{S}_j \le a]. \tag{69}$$

It follows that the long-term expected equivalent average discounted cost is given as

$$Q(r) = r\frac{c_A e^{-ra}(P(\mathcal{S}_1 \le a) - \sum_{j=1}^{\infty} \alpha_j P(\mathcal{S}_j \le a)) + \sum_{j=1}^{\infty} (c_M \beta_j + c_F \gamma_j) E[e^{-rS_j}; \mathcal{S}_j \le a]}{1 - e^{-ra}(P(\mathcal{S}_1 \le a) - \sum_{j=1}^{\infty} \alpha_j P(\mathcal{S}_j \le a)) - \sum_{j=1}^{\infty} \alpha_j E[e^{-rS_j}; \mathcal{S}_j \le a]}.$$

Taking $c_0 = c_A = c_M$ and $c_F = c_0 + \delta_F$, we can write $Q(r)$ as

$$r\frac{c_0 e^{-ra}(P(\mathcal{S}_1 \le a) - \sum_{j=1}^{\infty} \alpha_j P(\mathcal{S}_j \le a)) + \sum_{j=1}^{\infty} (c_0 \alpha_j + \delta_F \gamma_j) E[e^{-rS_j}; \mathcal{S}_j \le a]}{1 - e^{-ra}(P(\mathcal{S}_1 \le a) - \sum_{j=1}^{\infty} \alpha_j P(\mathcal{S}_j \le a)) - \sum_{j=1}^{\infty} \alpha_j E[e^{-rS_j}; \mathcal{S}_j \le a]}. \tag{70}$$

We continue with the derivation of an expression for $Q(r)$ in terms of the probability distribution of the total number of shocks. The term $E[e^{-rS_j}; \mathcal{S}_j \le a]$ can be evaluated in terms of its distribution F_j of $\mathcal{S}_j$ as

$$E[e^{-rS_j}; \mathcal{S}_j \le a] = \int_0^a e^{-rx}\, dF_j(x) = e^{-ra} F_j(a) + \int_0^a F_j(x) r e^{-rx}\, dx,$$

Using Eq. (3), $F_j(t)$ can be expressed in terms of probabilities $H_n(t)$ as

$$E[e^{-rS_j}; \mathcal{S}_j \le a] = \sum_{n=j}^{\infty} h_n,$$

where

$$h_n = e^{-ra} H_n(a) + \int_0^a H_n(x) r e^{-rx}\, dx.$$

By substituting this last expression in (68) and interchanging the order of summation, we obtain

$$E[Ce^{-rT}] = c_A e^{-ra} \sum_{n=0}^{\infty} \pi_n H_n(a) + \sum_{n=1}^{\infty} (c_M \mathcal{B}_n + c_F \mathcal{C}_n) h_n,$$

where $\mathcal{B}_n = \sum_{j=1}^{n} \beta_j$ and $\mathcal{C}_n = \sum_{j=1}^{n} \gamma_j$. Re-arranging the terms, we can also write

$$E[Ce^{-rT}] = c_A e^{-ra} H_0(a) + e^{-ra} \sum_{n=1}^{\infty} (c_A \pi_n + c_M \mathcal{B}_n + c_F \mathcal{C}_n) H_n(a) + \sum_{n=1}^{\infty} (c_M \mathcal{B}_n + c_F \mathcal{C}_n) \int_0^a H_n(x) r e^{-rx}\, dx. \tag{71}$$

This formula can be simplified by taking $c_0 = c_A = c_M$ and $c_F = c_0 + \delta_F$,

$$E[Ce^{-rT}] = e^{-ra}\left(c_0 + \delta_F \sum_{n=1}^{\infty} \mathcal{C}_n H_n(a)\right) + \sum_{n=1}^{\infty} (c_0 \mathcal{B}_n + c_F \mathcal{C}_n)$$

$$\int 0aH_n(x)re^{-rx}\,dx.$$

Substituting $C=1$ in (71) and noting that

$$\sum_{n=1}^{\infty}\int_0^a H_n(x)re^{-rx}\,dx=\int_0^a(1-H_0(x))re^{-rx}\,dx=1-e^{-ra}-\int_0^a H_0(x)re^{-rx}\,dx.$$

The first step in the above equation is obtained by interchanging the order of summation and integration and recognizing that $\sum_{n=1}^{\infty}H_n(x)=1-H_0(x)$. In this manner, (43) is derived as

$$E[e^{-rT}]=e^{-ra}+\sum_{n=1}^{\infty}(1-\pi_n)\int_0^a H_n(x)re^{-rx}\,dx=1-\sum_{n=0}^{\infty}\pi_n\int_0^a H_n(x)re^{-rx}\,dx. \tag{72}$$

It follows that

$$Q(r)=r\frac{e^{-ra}(c_0+\delta_F\sum_{n=1}^{\infty}C_nH_n(a))+\sum_{n=1}^{\infty}(c_0B_n+c_FC_n)\int_0^a H_n(x)re^{-rx}\,dx}{1-\sum_{n=0}^{\infty}\pi_n\int_0^a H_n(x)\,dx}. \tag{73}$$

The expected length of the renewal cycle can be derived by differentiating Eq. (72) with respect to r and then substituting $r=0$:

$$E[T]=\sum_{n=0}^{\infty}\pi_n\int_0^a H_n(x)\,dx.$$

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