

Mathematical Models for Reliability Data

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Mathematical Models for Reliability Data

PROEFSCHRIFT

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Chapter 1

Introduction

1.1 History and concept approach

Recently, a study of manuscripts in the Israeli National Library brought to light some unknown theories of Sir Isaac Newton on the Apocalypse. This dread event he predicted for the year 2060, based on his attempts to decode the Bible. Perhaps this is the event of human existence that has been prophesized most often throughout history, from the Delphic Oracle to Nostradamus to present day sooth-sayers. Some "visionaries" have proposed the year 2000 as the end of life on Earth, while others pushed the date further away. Take a young scientist who survived the year 2000 and the neighbouring time points. What can he expect for the year 2060? Should he wait and play the role of observer, if Karma is indulgent to him, or should he try to predict another time limit given the facts and the time history? Waiting will give him nothing: dead or a failed carrier, but is he certain that his work will not be in vain?

Predicting the time at which a particular event will occur. This is what reliability theory is all about. As human generally try to avoid unpleasant surprises rather than pleasant ones, they rather try to predict the first than the second, which is reflected in the application of the reliability theory. Of course reliability theory and its applications developed especially in the 20th century, side by side with the technological revolution. This revolution was significantly accelerated by the Second World War, the Korean War, the Cold War, the First Gulf War, and the permanent stress on military preparedness. These developments were also reflected in the civilian market, at the beginning by the car industry and then by areas (as the nuclear

and chemical industry) with greater consequences in cost, time wasted, psychological effect of inconvenience, and in certain instances personal and national security. The implications of the complexity are extensive. In order to estimate the successful operation of a complex system at a certain time, one must be sure that all its subsystems, components and parts shall function successfully. This interdependency introduces probability theory, which in its simplest form states that the probability of a successful operation of the system is the probability that a sufficient number of its subsystems operate successfully.

One of the most important parts of reliability analysis is the availability of a well-organized system for reliability data collection, processing and reporting. When setting up any form of data base, we are faced with considerations as the following: What kind of data are to be collected and why do we want them? Or perhaps a better way of putting the question is: What types of data are meaningful and how do we intend to analyze the data? The answers to these questions must be governed by the cost and difficulty of gathering the information versus the risk. Data can be selected and sorted both automatically and by hand. Reducing or sorting data by hand is important to limit its coverage considerably, and even greater care must be taken when judging what pertinent data should be collected.

The next step is to analyze data. Modern Reliability Data Bases (RDBs) are designed to meet the needs of diverse users, including component designers, reliability analysts and maintenance engineers. To meet these needs, RDBs distinguish a variety of ways in which a component's service sojourn may be terminated. Until quite recently, this data was analyzed from the viewpoint of independent competing risks. Such independence is often quite implausible, for example when degraded failures related to preventive maintenance compete with critical failure. The maintenance crew tries to prevent critical failures and to lose as little useful service time as possible; hence it creates a dependence between these competing risks. We have recently learned how to use simple models of dependent competing risks to identify survival functions and hence to analyze competing risk data. This type of analysis requires new statistical tests, and/or adaptations of existing tests. Competing risk theory is introduced in this thesis to model this behavior of reliability data.

The first competing risks reliability data bases can be traced back to the beginning of the 17th century, when city halls around England started to keep weekly registers on the number of human deaths and their causes. However, the theory of competing risk was developed later in the 18th century, and is related to Daniel Bernoulli's attempt

to separate the risk of dying from smallpox from that of other causes. Returning to the present, the theory of competing risks finds its main applicability in statistical inference, medicine and reliability theory and even in politics and social science. For example, members of parliament can retire, be defeated, run for higher office, or die in office; Supreme Court members can either die or retire; marriages can end in death, separation or divorce.

Assume we have k competing risks, X_1, \dots, X_k , and denote by $\wedge X_i$ the minimum of X_1, \dots, X_k . In a competing risk context, we observe the shortest of X_i , and observe which it is. In other words we observe $Z = (\wedge X_i, 1_{\wedge X_i = X_j}, j = 1, \dots, k)$. We say that risk X_j is cured if it is eliminated without disturbing the distributions of the others risks. Mathematically, curing risk j corresponds to observing $Z_{(j)} = (\wedge_{i \neq j} X_i, 1_{\wedge X_i = X_h}, h = 1, \dots, k, h \neq j)$, where the distribution of $Z_{(j)}$ is obtained from the distribution of Z by integrating out over variable X_j .

In many cases we can reduce the problem to the analysis of two competing risk classes, described by two random variables X and Y , where we call Y the censoring variable. Usually X will be the minimum of several variables which compete to terminate a service sojourn of the component. Hence we observed the least of X and Y ,

$$Z = [\min(X, Y), 1_{\{X < Y\}}]$$

and observe which it is.

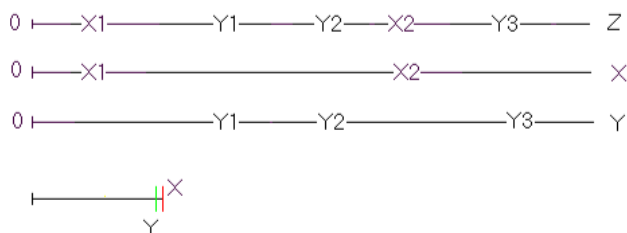


Figure 1. Competing risks representation

Competing risks data may be described as a colored point process, where each point event is described by a number of properties, and where a coloring is a grouping of properties into mutually exclusive and exhaustive classes. For example, a maintenance engineer is interested in degraded and incipient failures, as they are associated with preventive maintenance. He also wants to choose the least expensive maintenance action: repair actions and adjustment actions are favored over replace actions. Critical failures are of primary interest in risk and reliability calculations and a component designer is interested in the particular component function that will be lost, and in the failure mechanisms as he wishes to prevent the failure of the most expensive components of the system.

Let's consider the case of the maintenance engineer. A very good maintenance team will try to minimize the repair (replacement) cost over a long time interval. Since the repair (replacement) cost of a critical failure (corresponding to X corrective maintenance) is much higher than the cost of a degraded failure (corresponding to Y preventive maintenance), the maintenance team will try to avoid critical failure. Also, the maintenance team will try not to lose too much of the lifetime of the component because of the increased number of repairs (cost) over a long time interval. This entails that preventive maintenance should be highly correlated to failure (Figure 1). Ideally, the component is preventively maintained at time t if and only if it would otherwise have failed shortly after time t . This situation is captured in the Random Signs Model developed by Cooke (1996 [20]): consider a component subject to right censoring, where X denotes the time at which a component would expire if not censored, then the event that the component's life be censored is independent of the age of X at which the component would expire, but given that the component is censored, the time at which it is censored may depend on X . Not every set of censored observations is consistent with a random signs model. Cooke (1996 [20]) proved that if the random signs model holds, then the conditional subsurvival function for X dominates the conditional subsurvival function of preventive maintenance and they are equal for independent exponential model and the conditionally independent model.

Cooke and Bedford (2002 [21]) presented different models of dependent competing risks with an application for pressure relief valves data from one Swedish nuclear station operating two identical reactors. Like most modern RDBs, this data base was designed to serve the interest of at least three types of engineers: the maintenance engineer interested in measuring and optimising maintenance performance, the design

engineer interested in optimising component performance, and the risk analyst wishing to predict the reliability of the complex system in which the component operates. They showed that models of dependent competing risks made it easier to meet the needs of these users. This involves selecting an appropriate competing risk model on the basis of empirical subsurvival functions. In Cooke and Bedford (2002 [21]) this selection was simply made graphically. An example is shown in Figure 1., where “alarm” and “unintended discovery” are events that maintenance personnel would try to avoid. There are 4 such events and 248 other events. The conditional subsurvival function of “alarm” and “unintended discovery” [CSSF1] dominates the conditional subsurvival function of “other” [CSSF2], hence a random signs model seems to describe the data, but no evidence is given that this model or another one actually fits the data.

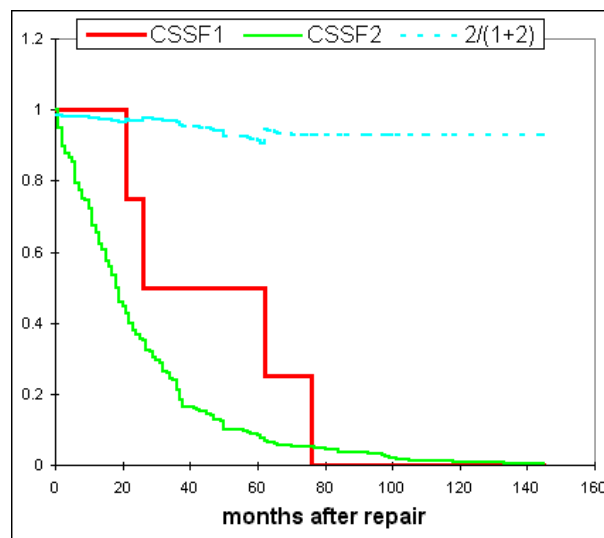


Figure 2. Graphical interpretation of data on pressure relief valves

In addition to this, two other main operations may be performed on the data: superposition and pooling. Time histories having the same beginning and end points may be superposed. The set of event times of the superposition is the union of the times of the superposed processes. In general, data are superposed in order to obtain a renewal process. If the maintenance team returns components to service as good as new, then all time histories of the components should be superposed. The pooled data are considered multiple realizations of the same random variable or stochastic process. When time histories are pooled, these are considered to be realizations of the same (colored) point process. In general, pooling is performed on

identical independent point processes in order to obtain better statistical estimates of the inter-event distribution. To perform these operations on data, a set of questions should be used that can only be answered by statistical test:

1. *Are the time histories homogeneous and independent?* Independence will fail if the events for the time histories of the components tend to cluster in calendar time. If homogeneity fails, the uncolored events should not be considered to be realizations of the same point process. If homogeneity holds, then the number of events up to time t for the components should not differ significantly.
2. *Is the coloring stationary?* The pooled process is now considered to be colored. The coloring is stationary if the proportion of “red” and “green” points does not vary significantly over calendar time.
3. *Is the process a “color blind” competing risk?* The process is color blind if the distribution of the i -th event is independent of the color of the previous event. Color blindness implies that the processes obtained by splicing together all inter-event times beginning with color j , $j = 1, \dots, n$ are homogeneous.
4. *Is the uncolored process a stationary competing risk?*
5. *Is the uncolored process a renewal competing risk?*

1.2 Outline of the thesis

This Ph.D. thesis deals with both aspects of competing risk analysis: probabilistic and statistic analysis. The second chapter introduces the basic concepts of the theory of competing risks from the probability point of view, and presents a number of independent and dependent competing risk models (see Paulsen et al. [62], Bedford and Cooke [7], Bunea et al. [15]).

For risk and reliability analysis we ultimately need life distributions and the estimation of the life distribution from time histories calls for renewal processes. However, we are also interested in trends, types of failure, failure modes, failure effects, maintenance operations, etc., and this leads us to competing risk renewal processes.

Chapter 3 presents a set of statistical tests used to validate the analysis scheme of the data. An analysis scheme is needed in order to build the reliability model. This

scheme consists of a structured set of assumptions about the data set, together with statistical sets to validate these assumptions. Typically, each assumption must be validated with "accepted" before the next assumption can be dealt with (see Bunea et al. [16], Dorrepaal [32]).

The next chapter presents the modality of rejecting a particular independent model (exponential) against the alternative of a dependent model (random signs model), even when a small number of events is available (see Bunea et al. [14]). An algorithm for calculating the test statistic is also given. The performance of the probabilistic model we propose is illustrated with the Gas Generator data used by Langseth (1999 [52]). This is a subset of Phase IV of the Gas Turbine data set from the Offshore Reliability Database (OREDA 1997 [61]). Only the Gas Generator subsystem was included in the study. We chose to analyze data from a single offshore installation only to ensure maximum homogeneity of the data sample.

Further, to check the performance of different probabilistic models, we discuss a data set from two identical compressor units at an ammonia plant of Norsk Hydro, covering the observation period from 1968 to 1989 (Erlingsen[34]). The competing risk models available in the reliability literature are especially developed for the nuclear sector, where strict regulations are imposed. Consequently these models are often not appropriate for the various fields of the compressor unit data. Due to the fact that the compressor unit consists of several heterogeneous sub-components, we therefore introduce another competing risk model, called the "mixture of exponentials model", to interpret the competing risks of different failure modes (see Bunea et al. [15]). In order to increase the available observation for the theoretical processes, and hence to reduce the uncertainty in model estimation, we also performed the statistical analysis on this data (see Bunea et al [16]).

Chapter 5 is based on (Bunea and Bedford [13]). In this chapter, we test the effect of model uncertainty on the problem of optimizing maintenance. We assume that data is available which contains censors from an existing preventive maintenance (PM) program, and use this data to estimate an optimal age replacement PM program. We take three model classes of competing risks. The independent model is used as the most extreme pessimistic model of existing PM. The other extreme model is used for the most optimistic model of existing PM. The dependent competing risk model is used for the general case and the dependence between competing risks is given by a copula. The minimally informative copula with respect to the uniform distribution and Archimedean copula are studied. The latter will be used to approximate the

first, as there are numerical difficulties in working with the minimally informative copula when there is strong dependence between risks. The results show that model uncertainty leads to substantial uncertainty in the estimation of optimal maintenance intervals and to excessive costs. The results also show little difference among different families of copulas, regarding the optimal replacement time (see Bunea and Bedford [12]).

A well-known mathematical tool to analyze data from nuclear power facilities is the 2-stage Bayesian model. In Chapter 6, we review this mathematical model, its underlying assumptions and supporting arguments. Furthermore, we will verify the software implementations of the major German effort to collect data from nuclear facilities - the ZEDB database - and compare the results. Lastly, the relevance of new developments is assessed, and the viability of the two-stage Bayesian approach is discussed (see Cooke et al. [23]).

Chapter 2

Competing risk theory-probabilistic approach

2.1 Introduction

The first competing risk reliability data bases can be traced back in the beginning of the 17th century, when city halls around England kept the weekly registers with the number of human deaths and the causes which generate them. However, the theory of competing risk has its origins later in the 18th century, and it is related to Daniel Bernoullis' attempt to separate the risk of dying from smallpox and other causes. Back to the present days, the competing risk theory finds its main applicability in statistical inference, medicine and reliability theory and even in politics and social science. E.g., members of parliament can retire, be defeated, run for higher office, or die in office; Supreme Court members can either die or retire; marriages can end in death, separation or divorce.

In this chapter we introduce the basic mathematical formalism for describing competing risks. The goal is to extract information about the failure rates of competing risks, sometimes called naked failure rates. The theory of independent and dependent competing risks is developed in ([28] [21] [31] [70]).

2.2 Concepts and Identifiability

Assume we have k competing risks, X_1, \dots, X_k , and denote by $\wedge X_i$ the minimum of X_1, \dots, X_k . In a competing risk context, we observe the shortest of X_i , and observe which it is. In other words we observe $Z = (\wedge X_i, 1_{\wedge X_i = X_j}, j = 1, \dots, k)$. We say that risk X_j is cured, if it is eliminated without disturbing the distributions of the others risks. Mathematically, curing risk j corresponds to observing $Z_{(j)} = (\wedge_{i \neq j} X_i, 1_{\wedge X_i = X_h}, h = 1, \dots, k, h \neq j)$, where the distribution of $Z_{(j)}$ is obtained from the distribution of Z by integrating out over variable X_j .

In many cases we can reduce the problem to the analysis of two competing risks classes, describe by two random variables X and Y , and we call Y the censoring variable. Usually X will be the minimum of several variables which compete to terminate a service sojourn of the component. Hence we observed the least of X and Y ,

$$Z = [\min(X, Y), 1_{\{X < Y\}}]$$

and observe which it is. For simplicity we assume that $P(X = Y) = 0$.

If $F_X = Pr\{X < t\}$ is the cumulative distribution of X , then $S_X = 1 - F_X$ is called the survival or reliability function of X . If F_X has a density then the failure rate of X is

$$r_X(t) = f_X(t)/S_X(t) = -(dS_X(t)/dt)/S_X(t).$$

Since

$$d[\log(S_X)] = dS_X/S_X,$$

we have

$$S_X(t) = \exp\left\{-\int_0^t r_X(s)ds\right\}.$$

Competing risk data will only allow us to estimate the *sub-survival functions* (Peterson 1976 [63]),

$$S_X^*(t) = Pr\{X > t, X < Y\}$$

and

$$S_Y^*(t) = Pr\{Y > t, Y < X\}$$

but not the true survival functions of X and Y . Hence we are not able to estimate the underlying failure distribution for X without making additional, non-testable, model

assumptions. If $S_X^*(t)$ and $S_Y^*(t)$ are continuous at 0 then $S_X^*(0) = Pr\{X < Y\}$ and $S_Y^*(0) = Pr\{Y < X\}$.

Definition 2.2.1. Real functions S_1^* and S_2^* on $[0, \infty)$ for a (continuous) subsurvival pair if:

1. S_1^* and S_2^* are non-negative, non-increasing (continuous, continuous from the right of zero), $S_1(t)^* < 1$, $S_2(t)^* < 1$
2. $\lim_{t \rightarrow \infty} S_1^*(t) = 0$, $\lim_{t \rightarrow \infty} S_2^*(t) = 0$
3. $S_1^*(0) + S_2^*(0) = 1$

Clearly, $S_X^*(t)$ and $S_Y^*(t)$ form a subsurvival pair. If $\lim_{t \rightarrow \infty} S_X^*(t) > 0$, then we say that X has an atom at infinity. This means that there is a nonzero probability that a component with life distribution X never expires.

The conditional subsurvival function is the subsurvival function, conditioned on the event that the failure mode in question is manifested. Assuming continuity of S_X^* and S_Y^* at zero:

$$CS_X^*(t) = Pr\{X > t, X < Y | X < Y\} = S_X^*(t)/S_X^*(0),$$

$$CS_Y^*(t) = Pr\{Y > t, Y < X | Y < X\} = S_Y^*(t)/S_Y^*(0).$$

Closely related to the notion of the subsurvival functions is the probability of censoring beyond time t ,

$$\Phi(t) = Pr\{Y < X | Y \wedge X > t\} = \frac{S_Y^*(t)}{S_X^*(t) + S_Y^*(t)}.$$

This function seems to have some diagnostic value, enabling us to choose the competing risk model which fits the data. Note that for continuous subsurvival functions $\Phi(0) = Pr\{Y < X\} = S_Y^*(0)$.

If we have data from Z , then we can calculate the empirical subsurvival functions; these contain all the information in the data, that means, any parameter that can be estimated from the data can be written as a function of the empirical subsurvival functions. The empirical subsurvival functions and the conditional subsurvival functions are defined as (Dorrepaal 1996 [32]):

$$\begin{aligned}\widehat{S}_X^*(t) &= \frac{\text{number of } X \text{ events after time } t}{\text{total number of events}} \\ \widehat{S}_Y^*(t) &= \frac{\text{number of } Y \text{ events after time } t}{\text{total number of events}} \\ \widehat{CS}_X^*(t) &= \frac{\text{number of } X \text{ events after time } t}{\text{total number of } X \text{ events}} \\ \widehat{CS}_Y^*(t) &= \frac{\text{number of } Y \text{ events after time } t}{\text{total number of } Y \text{ events}}\end{aligned}$$

The subdistribution functions for X and Y are defined as:

$$\begin{aligned}F_X^*(t) &= Pr\{X \leq t, X \leq Y\} = S_X^*(0) - S_X^*(t), \\ F_Y^*(t) &= Pr\{Y \leq t, Y \leq X\} = S_Y^*(0) - S_Y^*(t).\end{aligned}$$

Peterson (1976 [63]) derived bounds on the survival function S_X by noting that

$$Pr\{X \leq t, X \leq Y\} \leq Pr\{X < t\} \leq Pr\{X \wedge Y \leq t\},$$

which entails

$$1 - F_X^*(t) \geq S_X(t) \geq S_X^*(t) + S_Y^*(t).$$

Note that the quantities on the left and the right sides are observable.

Deshpande (1990 [30]), Aras and Deshpande (1992 [1]) and others have emphasized an alternative approach of competing risk in terms of the observable random pair (Z, δ) , where Z is the minimum of several lifetime distributions and δ identifies the minimum. Simplifying to the case of two competing risks, we have the equivalent definitions of the subsurvival and subdistribution functions: $S_i(t) = Pr\{Z > t, \delta = i\}$, $i = 0, 1$ and $F_i(t) = Pr\{Z \leq t, \delta = i\}$, $i = 0, 1$. The survival function of Z is given by

$$S_Z(t) = S_0(t) + S_1(t)$$

and the distribution function is given by

$$F_Z(t) = F_0(t) + F_1(t).$$

This approach introduces diagnostic functions:

$$\Phi_1(t) = Pr\{\delta = 1 | Z \geq t\} = S_1(t-)/S(t-)$$

and

$$\Phi_0^*(t) = Pr\{\delta = 0 | Z < t\} = F_0(t-)/F(t-),$$

whenever $S(t-) > 0$ and $F(t-) > 0$. Equivalently, we can define

$$\Phi_0(t) = Pr\{\delta = 0 | Z \geq t\} = 1 - \Phi_1(t),$$

and

$$\Phi_1^*(t) = Pr\{\delta = 1|Z < t\} = 1 - \Phi_0^*(t).$$

Note that $\Phi_1(t)$ is identical with $\Phi(t)$ used in the previous competing risk approach. One can easily verify that $\Phi_1(t) = \Phi_1^*(t) = \Phi$ for all $t > 0$ is equivalent to the independence of Z and δ . This simplifies the study of the competing risks considerably. If Z and δ are independent then $S_i(t) = S(t)Pr\{\delta = i\}$. Hence, the failure time and the failure types or the risk of failure can be studied separately.

The following definitions present some dependence structures that can be used for Z and δ .

Definition 2.2.2. X_2 is Right Tail Increasing in X_1 , $RTI(X_2|X_1)$, if $Pr\{X_2 > t_2|X_1 > t_1\}$ is increasing in t_1 for all t_2 .

Definition 2.2.3. X_2 is Left Tail Decreasing in X_1 , $LTD(X_2|X_1)$, if $Pr\{X_2 \leq t_2|X_1 \leq t_1\}$ is decreasing in t_1 for all t_2 .

Definition 2.2.4. X_1 and X_2 are Positively Quadrant Dependent, $PQD(X_1, X_2)$, if $Pr\{X_1 > t_1, X_2 > t_2\} \geq Pr\{X_1 > t_1\}Pr\{X_2 > t_2\}$, for all t_1, t_2 or equivalently $Pr\{X_1 \leq t_1, X_2 \leq t_2\} \leq Pr\{X_1 \leq t_1\}Pr\{X_2 \leq t_2\}$, for all t_1, t_2 .

Definition 2.2.5. A function $K(s, t)$ is Totally Positive of Order 2, TP2, if

$$K(s_1, t_1)K(s_2, t_2) \geq K(s_2, t_1)K(s_1, t_2)$$

for all $s_1 < s_2, t_1 < t_2$.

Note that, $RTI(X_2|X_1)$ and $LTD(X_2|X_1)$ both imply $PQD(X_1, X_2)$ but there is no hierarchy between $RTI(X_2|X_1)$ and $LTD(X_2|X_1)$.

Dewan et al (2002 [31]) presented the following relations between the monotonicity of $\Phi_1(t)$ and $\Phi_0^*(t)$, and the dependence structure of Z and δ :

1. Independence of Z and δ is equivalent to

(a) $\Phi_1(t) = \phi = Pr\{\delta = 1\}$, for all $t > 0$, a constant

- (b) $\Phi_0^*(t) = 1 - \delta = Pr\{\delta = 0\}$, for all $t > 0$, a constant.
2. $PQD(\delta, Z)$ is equivalent to
- (a) $\Phi_1(t) \geq \Phi_1(0) = \Phi$, for all $t > 0$,
- (b) $\Phi_0^*(t) \geq \Phi_0^*(\infty)$, for all $t > 0$.
3. $RTI(\delta|Z)$ is equivalent to $\Phi_1(t)$ is increasing for all $t > 0$
4. Subsurvival functions $S_i(t)$ being TP2 is equivalent to $\Phi_1(t)t$ is increasing for all $t > 0$
5. $LTD(\delta|Z)$ is equivalent to $\Phi_0^*(t)$ is decreasing for all $t > 0$
6. Subdistribution functions $F_i(t)$ being TP2 is equivalent to $\Phi_1(t)$ is decreasing for all $t > 0$

Note that (3) and (4) are equivalent and both imply (2). Similary, (5) and (6) are equivalent and both imply (2), but there is no relationship between (3) and (5).

The above results bring out the fact that the various kinds of dependence structure between Z and δ can be expressed in terms of various shapes of $\Phi_1(t)$ and $\Phi_0^*(t)$, which are observable from the data.

2.3 Naked and Observed failure rates

In this section we revert to k competing risks, X_1, \dots, X_k . Recall that in a competing risk context we observed $Z = [\min(X_1, \dots, X_k), 1_{\min(X_1, \dots, X_k)=X_j}, j = 1, \dots, k]$. The marginal failure rate of X_i , r_{X_i} , is the rate which would be observed if we could observe X_i without the observation being censored by earlier occurrences of $X_j, j \neq i$. We say that risk X_j is cured, if it is eliminated without disturbing the distributions of the other risks. Mathematically, curing risk j corresponds to integrating out over variable X_j . If we could cure all risks other than i , then we should observe the failure rate r_{X_i} . This is what we call the naked failure rate for X_i .

When the competing risks are not removed, we observe a different rate of failure for X_i . The observed failure rate for X_i is defined as:

$$obr_i = \frac{Pr\{\min(X_1, \dots, X_k) = X_i, X_i \in (t, t + dt) | \min(X_1, \dots, X_k) > t\}}{dt} = \frac{-\frac{dS_i^*(t)}{dt}}{\sum_{j=1}^k S_j^*(t)}.$$

If the competing risks are independent, then

$$Pr\{\min(X_1, \dots, X_k) > t\} = \prod_{i=1}^k S_i(t) = \sum_{i=1}^k S_i^*(t).$$

This is the basis of the identity of observed and naked failure rates for independent competing risks:

Theorem 2.3.1. (Cooke 1996 [20]) *If competing risks X_1, \dots, X_k are independent, with differentiable (sub)survival functions, then*

$$r_{X_i}(t) = obr_i(t), \quad i = 1, \dots, k.$$

The most important result for the observed processes when we use the observed failure rates is (Paulsen et al 1996 [62]):

Theorem 2.3.2. *The subsurvival functions of a competing risk variable with k competing risks may be expressed in terms of the observed failure rates:*

$$S_i^*(t) = \int_t^\infty obr_i(u) \exp\left\{-\int_0^u \sum_{j=1}^k obr_j(v) dv\right\} du$$

Let Z be the competing risk renewal process generated by independent copies of

$$Z = [\min(X_1, \dots, X_k), 1_{\min(X_1, \dots, X_k)=X_j}, j = 1, \dots, k],$$

where the inception of observation is random, $Pr\{Z = 0\} = 0$ and $E(\min(X_1, \dots, X_k))$ exists. Then the equilibrium observed rate of occurrence of failure type j is defined as:

$$obrocof_j = \frac{Pr\{X_j = \min(X_1, \dots, X_k)\}}{E(\min(X_1, \dots, X_k))}.$$

Using the results of the previous theorems we can write this as

$$obrocof_j = \frac{S_j^*(0)}{\sum_{i=1}^k \int_0^\infty S_i^*(u) du} = \frac{\int_0^\infty obr_j(u) \exp\left\{-\int_0^u \sum_{i=1}^k obr_i(v) dv\right\} du}{\int_0^\infty \exp\left\{-\int_0^u \sum_{j=1}^k obr_j(v) dv\right\} du}.$$

Remark 2.3.1. If $obr_j(t) = \lambda_j$ (constant) for $j = 1, \dots, k$, then:

1. $S_j^*(t) = \frac{\lambda_j}{\sum_{i=1}^k \lambda_i} \exp\{-\sum_{i=1}^k \lambda_i t\}$
2. $Pr\{X_j = \min(X_1, \dots, X_k)\} = S_j^*(0) = \frac{\lambda_j}{\sum_{i=1}^k \lambda_i}$
3. $\int_0^\infty S_j^*(t) dt = \frac{\lambda_j}{\{\sum_{i=1}^k \lambda_i\}^2}$
4. $obrocof_j = \lambda_j$

The above formulas are known for independent exponential competing risks. In the case of two competing risks the following results is obtained:

Theorem 2.3.3. *Let Z_i be i.i.d. copies of $Z = [\min(X, Y), 1_{\{X < Y\}}]$, then*

1. $obr_X(t) = \frac{S_X^*(t)}{S_Y^*(t)} obr_Y(t) + \frac{S_X^*(t)}{S_X^*(t) + S_Y^*(t)} \frac{d(S_X^*(t)/S_Y^*(t))}{dt}$
2. $obr_X(t) = \frac{dS_X^*(t)}{S_X^*(t) + S_Y^*(t)} = \frac{S_X^*(0)}{S_X^*(t) + S_Y^*(t)} \frac{dS_X^*(t)}{S_X^*(0)}$

From the above we see that if the subsurvival functions are proportional $S_X^*(t)/S_Y^*(t) = \alpha$, α independent of time, then $obr_X(t) = obr_Y(t) = \alpha$. If the risks are independent, then this result is transferred to the naked failure rates. The second statement relates the failure rate of the conditional subsurvival function to the observed failure rate.

2.4 Independent Competing risk models

The most frequently made assumption in the literature is that of probabilistic independence between X and Y . We have

$$S_X^*(t) + S_Y^*(t) = Pr\{X > t, Y > t\} = Pr\{X > t\}Pr\{Y > t\} = S_X(t)S_Y(t),$$

hence

$$S_X^*(t) = \int S_Y(t) dS_X(t) \quad \text{and} \quad S_Y^*(t) = \int S_X(t) dS_Y(t).$$

Using the above results Cooke (1996 [20]) showed that if the competing risks X and Y are independent with differentiable survival functions, then the failure rate is equal with the observed failure rate

$$r_X(t) = obr_X(t).$$

Now, the underlying marginal distributions of X and Y can be identified in terms of the observable subsurvival functions,

$$S_X(t) = \exp\left(\int_0^t \frac{dS_X^*(s)}{S_X^*(s) + S_Y^*(s)}\right). \quad (2.4.1)$$

Tsiatis (1975 [72]), Peterson (1977 [64]) and later Weide and Bedford (1998 [74]) stated the main result for independent competing risks. This theorem generalizes for k competing modes.

Theorem 2.4.1.

1. *Let X and Y be independent life variables, with F_X and F_Y continuous. Let X' and Y' be independent life variables such that $S_X^* = S_{X'}^*$, and $S_Y^* = S_{Y'}^*$; then $F_X = F_{X'}$ and $F_Y = F_{Y'}$.*
2. *If S_1^* and S_2^* are a subsurvival pair and are continuous, then there exist independent life variables X and Y such that $S_X^* = S_1^*$ and $S_Y^* = S_2^*$, and at most one of X, Y has an atom at infinity.*

Assuming independence of X and Y we can determine uniquely the survival functions; X and Y are said to be identifiable from the censored data. Of course, X and Y may not actually be independent, and in this case the survival function given by the above theorem would not be correct. Moreover, the independence assumption can never be tested by the censored observation since any censored observation can be explained by an independent model. However, we can test for a specific independent model as independent exponential model, which will be presented in next section.

2.4.1 Independent exponential model

Although competing risk data can always be explained by an independent model, this does not mean that any censored observations can be explained by a model with exponential life variables. Cooke (1996 [20]) derived a very sharp criterion for independence and exponentiality in terms of the subsurvival functions:

Theorem 2.4.2. *Let X and Y be independent life variables, then any two of the following imply the others:*

- $S_X(t) = \exp(-\lambda t)$
- $S_Y(t) = \exp(-\gamma t)$
- $S_X^*(t) = \frac{\lambda}{\lambda + \gamma} \exp(-(\lambda + \gamma)t)$
- $S_Y^*(t) = \frac{\gamma}{\lambda + \gamma} \exp(-(\lambda + \gamma)t)$

Remark 2.4.1. If X and Y are independent exponential life variables with failure rates λ and γ , then the conditional subsurvival functions of X and Y are equal and exponential distributed with failure rate $\lambda + \gamma$ and the probability of censoring beyond time t is constant:

$$\begin{aligned} S_X^*(t)/S_X^*(0) &= S_Y^*(t)/S_Y^*(0) = \exp(-(\lambda + \gamma)t) \\ \Phi(t) &= \frac{\gamma}{\lambda + \gamma}. \end{aligned}$$

A few results are known for independent competing risks when only one of the variables is exponential.

Theorem 2.4.3 (Zheng and Klein 1994 [75], Bedford and Meilijson 1995 [8]).

Let $(S_X^(t), S_Y^*(t))$ be a subsurvival pair, and suppose that in the (unique) independent model determined by $(S_X^*(t), S_Y^*(t))$, X is exponentially distributed with failure rate λ_I . Consider the set of all joint distributions consistent with $(S_X^*(t), S_Y^*(t))$ in which X is exponentially distributed, and let Λ denote the set of all corresponding failure rates. Then $\lambda_I = \min\{\Lambda\}$.*

Proof. Let $Pr\{X = Y\} = 0$. Then the first order derivative of the subdistribution function of X is:

$$[F_X^*(t)]' = \frac{dF_X^*(t)}{dt} = \frac{Pr\{X \in [t, t + \delta], X < Y\}}{\delta} = \frac{Pr\{X \in [t, t + \delta], Y > t\} + 0(\delta)}{\delta}.$$

Assuming independence between X and Y , we have:

$$[F_X^*(t)]' = \frac{Pr\{X \in [t, t + \delta]\}}{\delta} Pr\{Y > t\} = [F_X^I(t)]' S_Y(t),$$

hence

$$[F_X^*(t)]'|_{t=0} = [F_X^I(t)]'|_{t=0}.$$

Given that $F_X(t) - F_X^*(t)$ is increasing in t (see Theorem 2.7.1), hence

$$[F_X(t)]' - [F_X^*(t)]' \geq 0,$$

and

$$[F_X(t)]'|_{t=0} - [F_X^*(t)]'|_{t=0} \geq 0.$$

For every possible distribution of X , we have

$$[F_X^I(t)]'|_{t=0} \leq [F_X(t)]'|_{t=0},$$

which gives

$$\lambda_I = \min\{\Lambda\}$$

□

Theorem 2.4.4 (Cooke 1993 [18]). *Let X and Y be independent with subsurvival functions $S_X^*(t)$ and $S_Y^*(t)$ respectively, strictly decreasing. Let $S_X(t) = \exp(-\lambda t)$, and let S_Y be continuous and strictly monotone. If Y has a decreasing failure rate (increasing failure rate) - $r_Y(t)$ and $\frac{d}{dt} \frac{S_X(t)}{S_X(0)}|_0 \geq (\leq) \frac{d}{dt} \frac{S_Y(t)}{S_Y(0)}|_0$ then*

$$\frac{S_X(t)}{S_X(0)} \geq (\leq) \frac{S_Y(t)}{S_Y(0)}.$$

Proof. Let $g_X = \frac{d}{dt} \frac{S_X(t)}{S_X(0)}$ and $g_Y = \frac{d}{dt} \frac{S_Y(t)}{S_Y(0)}$ be the density functions of conditional distribution functions of X and Y . By assuming independence, the failure rates of both risks are equal to the observed failure rates. Hence

$$\lambda = -\frac{dS_X}{S_X} = -\frac{dS_X^*}{S_X^* + S_Y^*}$$

and

$$r_Y = -\frac{dS_Y}{S_Y} = -\frac{dS_Y^*}{S_X^* + S_Y^*} = \lambda \frac{dS_Y^*}{dS_X^*} = \lambda \frac{g_Y}{g_X}.$$

Hence $r_Y \searrow (\nearrow) \Leftrightarrow \frac{g_Y}{g_X} \searrow (\nearrow)$. For $t > 0$

$$\frac{g_Y}{g_X} \searrow (\nearrow) \Leftrightarrow \frac{dg_X/dt}{g_X} > (<) \frac{dg_Y/dt}{g_Y}.$$

Given $g_X < 0$ and $g_Y < 0$ we get

$$r_Y \searrow (\nearrow) \Leftrightarrow d \ln(g_X) > (<) d \ln(g_Y).$$

But also $g_X(0) \geq (\leq) g_Y(0)$ and the proof is completed. \square

2.4.2 Conditional independent model

Another model from which we have identifiability is conditional independent model. This model considers the competing risk variables, X and Y , as sharing a common quantity V , and as being independent given V :

$$X = V + W, \quad Y = V + U,$$

where V, U, W are mutually independent. Hokstadt (1997 [33]) derived explicit expressions for the case that V, U, W are exponential distributed:

Theorem 2.4.5. *Let V, U, W be independent with $S_V(t) = e^{-\lambda_V t}$, $S_U(t) = e^{-\lambda_U t}$, $S_W(t) = e^{-\lambda_W t}$, then*

- $S_X^*(t) = \frac{\lambda_V \lambda_W e^{-(\lambda_U + \lambda_W)t}}{(\lambda_U + \lambda_W)(\lambda_V - \lambda_W - \lambda_U)} - \frac{\lambda_W e^{-\lambda_V t}}{\lambda_V - \lambda_W - \lambda_U}$

- $S_Y^*(t) = \frac{\lambda_V \lambda_U e^{-(\lambda_U + \lambda_W)t}}{(\lambda_U + \lambda_W)(\lambda_V - \lambda_W - \lambda_U)} - \frac{\lambda_U e^{-\lambda_V t}}{\lambda_V - \lambda_W - \lambda_U}$
- $S_X^*(t) + S_Y^*(t) = \frac{\lambda_V e^{-(\lambda_U + \lambda_W)t}}{\lambda_V - \lambda_W - \lambda_U} - \frac{(\lambda_W + \lambda_U)e^{-\lambda_V t}}{\lambda_V - \lambda_W - \lambda_U}$
- $S_X^*(t)/S_X^*(0) = S_Y^*(t)/S_Y^*(0) = S_X^*(t) + S_Y^*(t)$
- *If V has an arbitrary distribution such that $P(V \geq 0) = 1$, and V is independent of U and W , then*

$$S_X^*(t)/S_X^*(0) = S_Y^*(t)/S_Y^*(0)$$

As in the case of Independent Exponential Competing Risks we have equal conditional subsurvival functions and the probability of censoring beyond time t is constant: $\Phi(t) = \frac{\lambda_U}{\lambda_U + \lambda_W}$.

Further, it is easy to see that $X \wedge Y$ is the sum of the V and $U \wedge W$. Hence the expectation of $X \wedge Y$ is:

$$1/\lambda_V + 1/(\lambda_U + \lambda_W).$$

A calculation gives the variance of $X \wedge Y$:

$$1/\lambda_V^2 + 1/(\lambda_U + \lambda_W)^2.$$

The ratio of the naked over the observed rocof's for for X is found to be:

$$(\lambda_V + \lambda_U + \lambda_W)/(\lambda_V + \lambda_W).$$

Together with $S_X^*(0)$, these give three equations for estimating the three parameters $\lambda_V, \lambda_U, \lambda_W$, from which the model is identifiable from the subsurvival functions.

2.4.3 Mixture of exponentials model

Let the survival function of the life time - $S_X(t)$ be a mixture of two exponential distributions with parameters λ_1, λ_2 and the mixing coefficient p , and the censoring survival function - $S_Y(t)$ is an exponential with parameter λ_y ,

$$S_X(t) = p \exp\{-\lambda_1 t\} + (1 - p) \exp\{-\lambda_2 t\},$$

and

$$S_Y(t) = \exp\{-\lambda_y t\}.$$

Theorem 2.4.6 (Bunea et al 2002 [15]). *Let X and Y be independent life variables with the above distributions, then:*

- $S_X^*(t) = p \frac{\lambda_1}{\lambda_y + \lambda_1} \exp\{-(\lambda_y + \lambda_1)t\} + (1 - p) \frac{\lambda_2}{\lambda_y + \lambda_2} \exp\{-(\lambda_y + \lambda_2)t\}$
- $S_Y^*(t) = p \frac{\lambda_y}{\lambda_y + \lambda_1} \exp\{-(\lambda_y + \lambda_1)t\} + (1 - p) \frac{\lambda_y}{\lambda_y + \lambda_2} \exp\{-(\lambda_y + \lambda_2)t\}$
- $\frac{S_X^*(t)}{S_X^*(0)} = \frac{\exp\{-(\lambda_y + \lambda_1)t\} + \frac{1-p}{p} \frac{\lambda_2}{\lambda_1} \frac{\lambda_y + \lambda_1}{\lambda_y + \lambda_2} \exp\{-(\lambda_y + \lambda_2)t\}}{1 + \frac{1-p}{p} \frac{\lambda_2}{\lambda_1} \frac{\lambda_y + \lambda_1}{\lambda_y + \lambda_2}}$
- $\frac{S_Y^*(t)}{S_Y^*(0)} = \frac{\exp\{-(\lambda_y + \lambda_1)t\} + \frac{1-p}{p} \frac{\lambda_y + \lambda_1}{\lambda_y + \lambda_2} \exp\{-(\lambda_y + \lambda_2)t\}}{1 + \frac{1-p}{p} \frac{\lambda_y + \lambda_1}{\lambda_y + \lambda_2}}$
- $\frac{S_X^*(t)}{S_X^*(0)} \leq \frac{S_Y^*(t)}{S_Y^*(0)}$
- $\Phi(t)$ is minimum at the origin, and is continuously increasing

Proof. The first four statements follow directly from the assumption of independence and the functional form of the distribution functions for X and Y . Given independence of X and Y , we have:

$$S_X^*(t) + S_Y^*(t) = Pr\{X > t, Y > t\} = Pr\{X > t\}Pr\{Y > t\} = S_X(t)S_Y(t),$$

hence

$$S_X^*(t) = \int S_Y(t) dS_X(t) \quad \text{and} \quad S_Y^*(t) = \int S_X(t) dS_Y(t).$$

After integration the desired formulas are obtained.

To prove the fifth statement, we can rewrite the conditional subsurvival functions in compact form:

$$\frac{S_X^*(t)}{S_X^*(0)} = \frac{A + \frac{\lambda_2}{\lambda_1} BC}{1 + \frac{\lambda_2}{\lambda_1} B} \quad \text{and} \quad \frac{S_Y^*(t)}{S_Y^*(0)} = \frac{A + BC}{1 + B}.$$

Hence,

$$\frac{S_X^*(t)}{S_X^*(0)} \quad \text{vs} \quad \frac{S_Y^*(t)}{S_Y^*(0)}$$

is equivalent with

$$\begin{aligned} \frac{A + \frac{\lambda_2}{\lambda_1} BC}{1 + \frac{\lambda_2}{\lambda_1} B} & \text{ vs } \frac{A + BC}{1 + B}, \\ A\left(1 - \frac{\lambda_2}{\lambda_1}\right) & \text{ vs } C\left(1 - \frac{\lambda_2}{\lambda_1}\right), \\ \exp\{-\lambda_1 t\}(\lambda_1 - \lambda_2) & \text{ vs } \exp\{-\lambda_2 t\}(\lambda_1 - \lambda_2). \end{aligned}$$

If $\lambda_1 \geq \lambda_2$ then,

$$\exp\{-\lambda_1 t\}(\lambda_1 - \lambda_2) \leq \exp\{-\lambda_2 t\}(\lambda_1 - \lambda_2).$$

If $\lambda_1 < \lambda_2$ then,

$$\exp\{-\lambda_1 t\}(\lambda_1 - \lambda_2) < \exp\{-\lambda_2 t\}(\lambda_1 - \lambda_2).$$

Hence,

$$\frac{S_X^*(t)}{S_X^*(0)} \leq \frac{S_Y^*(t)}{S_Y^*(0)}.$$

One can verify that $S_X^*(0) + S_Y^*(0) = 1$. Indeed,

$$S_X^*(0) = p \frac{\lambda_1}{\lambda_y + \lambda_1} + (1-p) \frac{\lambda_2}{\lambda_y + \lambda_2},$$

and

$$S_Y^*(0) = p \frac{\lambda_y}{\lambda_y + \lambda_1} + (1-p) \frac{\lambda_y}{\lambda_y + \lambda_2},$$

and the equality required can be easily obtained.

Using the fact that $S_X^*(0) + S_Y^*(0) = 1$ and $\frac{S_X^*(t)}{S_X^*(0)} \leq \frac{S_Y^*(t)}{S_Y^*(0)}$ it follows immediately that $\Phi(t)$ is minimum at the origin. More it can be shown, after some arduous calculations, that $\Phi(t)$ is continuously increasing. \square

The derivative of $\Phi(t)$ is:

$$\varphi(t) = \frac{d\Phi(t)}{dt} = C \frac{pS_1(t) \cdot (1-p)S_2(t)}{(pS_1(t) + (1-p)S_2(t))^2},$$

where $C = \frac{\lambda_y(\lambda_1 - \lambda_2)^2}{(\lambda_y + \lambda_1)(\lambda_y + \lambda_2)}$ and $S_1 = \exp\{-\lambda_1 t\}$, $S_2 = \exp\{-\lambda_2 t\}$.

The value of φ at the origin is:

$$\varphi(0) = p(1 - p) \frac{\lambda_y(\lambda_1 - \lambda_2)^2}{(\lambda_y + \lambda_1)(\lambda_y + \lambda_2)}.$$

Recalling that Z is the minimum of two variables and the indicator of which variable is smaller, the survival function of Z becomes:

$$\begin{aligned} P(Z > t) &= P(\min(X, Y) > t) = P(X > t, Y > T) = P(X > t)P(Y > t) = \\ &= p \exp\{-(\lambda_1 + \lambda_y)t\} + (1 - p) \exp\{-(\lambda_2 + \lambda_y)t\}, \end{aligned}$$

which is also a mixture of two exponential distributions with parameters $\lambda_1 + \lambda_y$, respectively $\lambda_2 + \lambda_y$.

The expectation and variance of Z are:

$$\begin{aligned} E(Z) &= \int_0^\infty z f_z(z) dz = \int_0^\infty P(Z > t) dt = \\ &= p \frac{1}{\lambda_y + \lambda_1} + (1 - p) \frac{1}{\lambda_y + \lambda_2}, \end{aligned}$$

respectively

$$\frac{1}{(\lambda_y + \lambda_2)^2} + 2p \frac{1}{\lambda_y + \lambda_1} \left(\frac{1}{\lambda_y + \lambda_1} - \frac{1}{\lambda_y + \lambda_2} \right) - p^2 \left(\frac{1}{\lambda_y + \lambda_1} - \frac{1}{\lambda_y + \lambda_2} \right)^2$$

Using the above equations and $S_Y^*(0)$ we can obtain an estimation for the unknown parameters. For example $\lambda_y = S_Y^*(0)/E(Z)$ and the values of p , λ_1 and λ_2 can be obtained numerically. Note that the same solution for λ_y is obtained using the maximum likelihood method.

2.5 Dependent competing risk models

2.5.1 Random Signs model

Perhaps the simplest dependent competing risk model which leads to identifiable marginal distributions is random sign censoring (Cooke 1996 [20]). Consider a component subject to right censoring, where X denotes the time at which a component

would expire if not censored. Suppose that the event that the life of the component be censored is independent of the age X at which the component would expire, but given that the component is censored, the time at which it is censored may depend on X . This might arise, if a component emits warning before expiring; if the warning is seen then the component is taken out, thus censoring its life, otherwise it fails. The random signs model assumes that the probability of seeing the warning is independent of the component's age. This situation is captured in the following definition:

Definition 2.5.1. Let X and Y be life variables with $Y = X - W\delta$, where W , $0 < W < X$, is a random variable and δ is a random variable taking values $\{1, -1\}$, with X and δ independent. The variable $Z \equiv [\min(X, Y), I(X < Y)]$ is called a random sign censoring of X by Y .

Note that

$$\begin{aligned} S_X^*(t) &= Pr\{X > t, \delta = -1\} = Pr\{X > t\}Pr\{\delta = -1\} = \\ &= S_X(t)Pr\{Y > X\} = S_X(t)S_X^*(0). \end{aligned}$$

Note also that $Pr\{Y > X\}$ and $S_X^*(t)$ can be estimated from observing independent copies of Z and that under random signs censoring $S_X(t)$ is equal to the conditional subsurvival function of X .

Cooke (1996 [20]) proved that the random signs model is consistent given subsurvival functions if and only if the conditional subsurvival function of X is greater than the conditional subsurvival function of Y for all $t > 0$. In this case the probability of censoring beyond time t is maximum at the origin. This result suggests that if the random signs model holds then the independent exponential model is difficult to characterize data.

If extra information about the relationship between X and W is known, then more may be said about the form of $\Phi(t)$. For example,

1. Suppose that for sufficiently large X , $W = a$, where a is a positive constant. Then writing $\alpha = S_X^*(0)/S_Y^*(0)$, we have

$$\Phi(t) = \frac{1}{1 + \alpha \frac{S_X(t)}{S_X(t+a)}}$$

for t sufficiently large. If additionally we assume that X is exponential with parameter λ , then

$$\Phi(t) = \frac{1}{1 + \alpha \exp(\lambda a)}$$

for t sufficiently large.

2. If $W = aX$ for a positive constant $a < 1$, then

$$\Phi(t) = \frac{1}{1 + \alpha \frac{S_X(t)}{S_X(t/(1-a))}}$$

For a large class of distributions, including the exponential, $\frac{S_X(t)}{S_X(t/(1-a))} \rightarrow \infty$ as $t \rightarrow \infty$, so that in this case $\Phi(t) \rightarrow 0$ as $t \rightarrow \infty$.

A special case of random signs model will be presented in Chapter 4 as the ‘‘Highly Correlated Model’’.

2.5.2 The Lanseth-Bedford-Lindqvist model

In order to find the joint distribution for $\{Z = \min(X < Y), \delta = 1_{\{X < Y\}}\}$ the following assumptions are done in (Langseth 1999 [52]) and (Lindqvist 2001 [54]):

1. Random signs censoring, that is $\delta = 1_{\{X < Y\}}$ is independent of X
2. $Pr\{Y \leq y | Y < X, X = x\} = \frac{H(y)}{H(x)}, 0 \leq y \leq x$,

where $H(t) = \int_0^t r_X(u) du$. Assumption 2 says that conditional density for Y , given censoring and given the potential failure time, is proportional to the intensity of the underlying failure process. Let $q = Pr\{Y < X\}$ be the probability of catching a critical failure by censoring.

The joint density of Y and X , conditional on $Y < X$ is

$$f(y, x | Y < X) = f(x | Y < X) f(y | Y < X, X = x) = f(x) \frac{r_X(y)}{H(x)} = r_X(x) e^{-H(x)} \frac{r_X(y)}{H(x)}$$

The marginal density of Y conditional on $\{Y < X\}$ is:

$$f(y | Y < X) = \int_y^\infty r_X(x) e^{-H(x)} \frac{r_X(y)}{H(x)} dx = r_X(y) \int_{H(y)}^\infty \frac{1}{u} e^{-u} du = r_X(y) I_e(H(y)),$$

where $I_e(\cdot)$ is the exponential integral. The joint distribution of (Z, δ) can be found from,

$$\begin{aligned}
Pr\{z \leq Z \leq z + \Delta z, \delta = 1\} &= Pr\{z \leq Z \leq z + \Delta z, X < Y\} = \\
&= Pr\{z \leq \min(X, Y) \leq z + \Delta z, X < Y\} = \\
&= Pr\{z \leq X \leq z + \Delta z, X < Y\} = \\
&= Pr\{z \leq X \leq z + \Delta z\}Pr\{X < Y\} = \\
&= r_X(z)e^{-H(z)}(1 - q)\Delta z.
\end{aligned}$$

Similar for $\delta = 0$, we have:

$$\begin{aligned}
Pr\{z \leq Z \leq z + \Delta z, \delta = 0\} &= Pr\{z \leq Z \leq z + \Delta z, Y < X\} = \\
&= Pr\{z \leq Y \leq z + \Delta z, Y < X\} = \\
&= Pr\{z \leq Y \leq z + \Delta z|Y < X\}Pr\{Y < X\} = \\
&= r_X(z)I_e(H(z))q\Delta z.
\end{aligned}$$

The above equations give essentially the joint distribution for (Z, δ) . These two equations give also the likelihood function, by inserting the observed data and taking the product between them. Thus, the parameters of the density function of X , $f_X(x)$ and q can be found.

Note that the marginal distribution function of Z is

$$f_Z(z) = qr_X(z)I_e(H(z)) + (1 - q)r_X(z)e^{-H(z)}.$$

This function is actually a mixture of the two distributions for $\{Y|Y < X\}$ and $\{X|X < Y\}$.

A more general model, called ‘‘Repair Alert Model’’, is proposed in (Støve 2002 [70]), by considering a more general distribution for $\{Y|X = t, Y < X\}$. If $G(t)$ is an arbitrary increasing function with derivative $g(t)$ the assumption 2 becomes:

$$Pr\{Y \leq y|Y < X, X = x\} = \frac{G(y)}{G(x)}, \quad 0 \leq y \leq x.$$

2.5.3 Random Clipping

Perhaps the simplest model which makes the transition from exponential models to dependent models is gotten by assuming that X is always censored by a random

variable $Y = X - W$ (Cooke 1996 [20]). More specifically, we assume that X is exponential distributed and for some positive random variable W independent of X , we observe $X - W$. Of course W may be greater than X , which we interpret as censors at birth.

Let us suppose that censors at birth are simply not recorded. Suppose in other words, that components emitting warnings at birth are simply repaired until the warning disappears, and that the false start is not recorded as an incipient failure at time 0. We call the variable $X - W$ given $X - W > 0$ a random clipping of X .

The following theorem says that, regardless of the distribution of W , the distribution of $X - W$ given $X - W > 0$, is identical to that of X .

Theorem 2.5.1. *Let X be exponential with parameter λ , let $W > 0$ be a random variable independent of X , and $U = X - W$. Then, conditional on $U > 0$, U has the same distribution as X .*

2.6 Colored Poisson representation of competing risk

We consider the process $Z = Z_1, Z_2, \dots$, where Z_i are independent copies of $Z = [\min\{F, M\}, 1_{\{F < M\}}]$, and imagine data by instantly replenishing a component socket as good as new components whenever a component exit service. The components exit service either because of failure (F) or because of a preventive maintenance (M). Think of $\min\{F, M\}$ as the uncolored process and think of the M's as colored magneta and the F's colored fuchsia. The coloring theorem for Poisson processes says:

Theorem 2.6.1. *(coloring theorem for Poisson processes): If the uncolored process is a Poisson process with intensity ν , and if the coloring of a point is determined by the outcome of an independent coin toss: heads for magneta, tails for fuchisa with $Pr\{\text{heads}\} = p$, then the magneta points are a Poisson process with intensity νp , the fuchisa points are a Poisson process with intensity $\nu(1 - p)$, and the magneta and*

fuchisa processes are independent. Conversely, the superposition of two independent Poisson processes is a Poisson process whose intensity is the sum of the intensity of the superposed processes.

A colored Poisson process is equivalent to independent exponential competing risks. A colored Poisson process may be represented as $\gamma = \{M_1, M_2, \dots; \mu, F_1, F_2, \dots; \phi\}$, where $\{M_i\}$ and $\{F_i\}$ are the inter-arrival times of two independent processes starting at $t = 0$, with intensities μ and ϕ ; the uncolored process has the intensity $\nu = \mu + \phi$.

γ may be associated with a subsurvival pair as follows: letting $S_U(t) = \exp(-(\mu + \phi)t)$, $S_M^*(t) = S_U(t)\mu/(\mu + \phi)$; $S_F^*(t) = S_U(t)\phi/(\mu + \phi)$. $\mu/(\mu + \phi)$ is the probability for magna and $\phi/(\mu + \phi)$ is the probability for fuchisa.

Theorem 2.6.2 (Cooke 1996 [20]).

1. *Let $\gamma = \{M_1, M_2, \dots; \mu, F_1, F_2, \dots; \phi\}$ be a colored Poisson process. Then is a unique independent competing risk process $Z = [\min\{F, M\}, 1_{\{F < M\}}]$ associated with γ . Moreover, M and F are exponential distributed with survival function $\exp(-\mu t)$ and $\exp(-\phi t)$*
2. *Let Z_i be independent copies of $[\min\{F, M\}, 1_{\{F < M\}}]$ where M and F are independent and exponential distributed with survival functions $\exp(-\mu t)$ and $\exp(-\phi t)$ and let $Z = Z_1, Z_2, \dots$ be the competing risk renewal process associated with Z . Then Z is a colored Poisson process with intensities μ and ϕ for the M and F processes.*

Remark 2.6.1. Consider the distance between an uncolored point P_i and the previous uncolored point P_{i-1} . This distance follows an exponential distribution with failure rate $\mu + \phi$. Hence, the distance between a maintenance point and its nearest predecessor has the same distribution as the distance between a failure point and its

nearest predecessor. In other words, given that a service sojourn terminates in preventive maintenance, the distribution of length of that sojourn is the same as the distribution for the length of sojourn given termination in failure.

2.7 Uncertainty

Uncertainty bounds convey the restriction on the possible choices of reliability parameters arising from the observable data. It is convenient to distinguish uncertainty due to non-identifiability from uncertainty due to sampling fluctuations. To exclude the effect of sampling fluctuations it is useful to consider how we should proceed if we actually had infinitely many observations of censored life process.

2.7.1 Uncertainty due to non-identifiability: Bounds in the absence of sampling fluctuations

Considering the problem of non-identifiability, Peterson (1976 [63]) presents bounds for the joint distribution of X and Y as well as for its marginals, assuming that $Pr\{X = Y\} = 0$. Peterson further proves that these bounds are sharp. However, this statement and its proofs hold only under the assumption (not stated by Peterson) of continuity of the two subsurvival functions (Bedford and Meilijson 1997 [9]).

Peterson derived lower and upper bounds on the subsurvival function S_X by observable quantities by noting that:

$$Pr\{X \leq t, X \leq Y\} \leq Pr\{X \leq t\} \leq Pr\{X \wedge Y \leq t\}$$

Proof. Let A and B be two sets defined as

$$A = \{X \leq t, X \leq Y\},$$

$$B = \{X \leq t\} = \{X \leq t, X \leq Y\} \cup \{X \leq t, X > Y\}$$

and using the fact that if $A \subset B \Rightarrow Pr\{A\} \leq Pr\{B\}$ we obtain the left hand side inequality

$$Pr\{X \leq t, X \leq Y\} \leq Pr\{X \leq t\}.$$

We can write the set $\{X \leq t\}$ as a union of two sets

$$\{X \leq t, X \leq Y\} \cup \{X \leq t, X > Y\}$$

and using the property of probabilities $Pr\{\cup A_i\} \leq \sum Pr\{A_i\}$ we get

$$Pr\{X \leq t\} \leq Pr\{X \leq t, X \leq Y\} + Pr\{X \leq t, X > Y\}.$$

But $Pr\{X \leq t, X > Y\} \leq Pr\{Y \leq t, X > Y\}$ when $X > Y$, and the right hand side inequality is obtained. \square

If F_Z is the cumulative distribution function of Z , these bounds can also be written as:

$$F_X^*(t) \leq F_X(t) \leq F_X^*(t) + F_Y^*(t) = F_Z(t).$$

Through any point between the functions F_X^* and F_Z there pass a (non-unique) distribution function for X which is consistent with the censored data. This does not say that any distribution function between F_X^* and F_Z is a possible distribution for X . A more general result is given in (Crowder 1994 [28], Bedford and Meijlison 1997 [9]), by the following theorem:

Theorem 2.7.1. *If F is a cumulative distribution function satisfying*

$$F^*(t) \leq F(t) \leq F_{min}(t),$$

then there is a joint distribution for (X, Y) with F as marginal distribution for X if and only if for all t_1, t_2 , with $t_1 < t_2$,

$$F(t_1) - F^*(t_1) \leq F(t_2) - F^*(t_2).$$

In other words, the distance between $F(t)$ and $F^*(t)$ must be increasing in t .

There may be a large number of cumulative distribution function passing through $F(t)$ and $F^*(t)$, but the data can not support all of them. A restriction to the families of cumulative distribution functions can be obtained by considering bounds to time average failure rates. Recall that the failure rate $r_X(t)$ for X is given by

$$r_X(t) = \frac{f_X(t)}{S_X(t)} = -\frac{d(\ln(S_X(t)))}{dt}$$

so that the time average failure rate is

$$-\frac{\ln(S_X(t))}{t} = \frac{\int_0^t r_X(u) du}{t}.$$

Applying this transformation the Peterson bounds become

$$lmin(t) = -\frac{\ln(S_X^*(t) + S_Y^*(0))}{t} \leq \frac{\int_0^t r_X(u) du}{t} \leq -\frac{\ln(S_X^*(t) + S_Y^*(t))}{t} = lmax(t).$$

At each time t , the set of numbers between $lmax(t)$ and $lmin(t)$ corresponds to the time average failure rates at time t which are consistent with the data up to time t . As $t \rightarrow \infty, F_X^*(t) \rightarrow Pr\{X \leq Y\} < 1$, so that $lmin(t) \rightarrow 0$. Hence the lower bound on the admissible values of the time average failure rate decreases as the time becomes larger. Taking sup on the LHS and inf on the RHS of the above equation, we obtain the bounds

$$[\lambda_l, \lambda_u] = \bigcap_{t=1}^{\infty} [lmax(t), lmin(t)].$$

2.7.2 Uncertainty due to sampling fluctuations

The bounds developed in the previous paragraph reflect a lack of knowledge due to non-identifiability of the distribution of X caused by censoring. This uncertainty can not be removed by observations unless the censoring is suspended. In practice we have to deal with another lack of knowledge, namely that caused by a limited observations of realization of Z .

The Peterson bounds may be used to obtain classical confidence bounds depending on time t , which we call “time-wise” bounds. We write the Peterson bounds as:

$$lmin(t) = -\frac{\ln(1 - F_X^*(t))}{t} \leq \frac{\int_0^t r_X(u) du}{t} \leq -\frac{\ln(1 - F_Z(t))}{t} = lmax(t)$$

For each t the probabilities $F_X^*(t)$ and $F_Z^*(t)$ can be estimated from the data. If in $n(t)$ of N independent observations of $(X \wedge Y, 1_{X < Y})$ the event $(X \leq t, X \leq Y)$ is observed, then the quantity

$$\frac{n(t) - NF_X^*(t)}{\sqrt{N}\sigma}$$

is approximately standard normal distributed, where $\sigma^2 = F_X^*(t)(1 - F_X^*(t))$ may be estimated as $(nN - n^2)/N^2$. Hence the classical 5% lower confidence bound for $-\ln(1 - F_X^*(t))/t$ can be written as

$$\lambda_l(t) = -\ln[1 - (n(t)/N - 1.65\sigma(t)/\sqrt{N})]/t.$$

Similarly, if $m(t)$ is the number of observation of the event $(X \wedge Y \leq t)$ in N independent observation of $(X \wedge Y, 1_{X < Y})$ then an upper 95% classical confidence bound for $-\ln(1 - F_Z(t))/t$ is

$$\lambda_u(t) = -\ln[1 - (m(t)/N - 1.65\sigma'(t)/\sqrt{N})]/t,$$

where now σ'^2 is estimated as $(mN - m^2)/N^2$. The curves $\lambda_l(t)$ and $\lambda_u(t)$ have the following interpretation. If we repeatedly draw samples of size N from the distribution of $(X \wedge Y, 1_{X < Y})$, then for each t , in 95% of the N samples the empirical version of $lmin(t)$ is greater than $\lambda_l(t)$ and in 95% of the N samples the empirical version of $lmax(t)$ is less than $\lambda_u(t)$. This does not mean that 95% of the N samples lie above $\lambda_l(t)$ (below $\lambda_u(t)$) for all t .

2.8 Model selection

The probability of censoring after time t seems to have an important role in model selection, via graphical interpretation. Let $c_X(t)$ and $c_Y(t)$ be the failure rates of the conditional subsurvival functions of X and Y respectively. Then, the following statements follow from the theorems presented in the previous sections:

1. If the risks are exponential and independent, then the conditional subsurvival functions are equal and exponential distributed, and $\Phi(t)$ is constant
2. If $\Phi(t)$ is constant then the observed failure rates are proportional
3. If $\Phi(t)$ is constant and the risks are independent, then the naked failure rates are proportional

4. If the risks are independent, r_Y is constant, and r_X is increasing; then $\Phi(0) > \Phi(t)$; if r_X is decreasing then $\Phi(0) < \Phi(t)$; $t > 0$
5. If the risks are independent, then $r_Y(t) = \Phi(t)c_Y(t)$ and $r_X(t) = (1 - \Phi(t))c_X(t)$
6. If the random signs model holds, then $\Phi(0) > \Phi(t), t \geq 0$
7. If the conditional independence model holds with exponential marginals, then the conditional subsurvival functions are equal and $\Phi(t)$ is constant
8. If the mixture of exponentials model holds, then $\Phi(t)$ is increasing for all $t \geq 0$

If $\Phi(t)$ changes its monotonicity over the time of observation, and has no maximum or minimum at the origin, then we have no plausible model for coupling X and Y and we will regard them as independent.

Chapter 3

Competing risk theory-statistical approach

3.1 Introduction

Modern Reliability Data Bases (RDB's) are designed to meet the needs of diverse users, including component designers, reliability analysts and maintenance engineers. To meet these needs RDB's distinguish a variety of ways in which a component's service sojourn may be terminated. Up until quite recently, this data was analyzed from the viewpoint of independent competing risk. Independence is often quite implausible, as eg when degraded failures related to preventive maintenance compete with critical failure. The maintenance crew is trying to prevent critical failures while losing as little useful service time as possible; and hence is creating dependence between these competing risks. We have recently learned how to use simple models for dependent competing risk to identify survival functions and hence to analyze competing risk data. This type of analysis requires new statistical tests, and/or adaptations of existing tests. Competing risk models are described in Chapter 1. In this chapter we present a number of tests to support the analysis of competing risk data.

Competing risk data may be described as a colored point process, where each point event is described by a number of properties, and where a coloring is a grouping of properties into mutually exclusive and exhaustive classes. For example, a maintenance engineer is interested in degraded and incipient failures, as they are associated with preventive maintenance. He is trying also to take the least expensive maintenance action: repair action or adjustment action are favored above replace action. Critical

failures are of primary interest in risk and reliability calculations and a component designer is interested in the particular component function that is lost, in the failure mechanisms and he wishes to prevent the failure of the most expensive components of the system.

In addition to this, two other main operations may be performed on the data: superposition, pooling. Time histories having the same begin and end points may be superposed. The set of event times of the superposition is the union of the times of the superposed processes. In general, superposition is performed in order to obtain a renewal process. If the maintenance team returns components to service as good as new, then all time histories of the components should be superposed. The pooled data are considered as multiple realizations of the same random variable or stochastic process. When time histories are pooled, these are considered as realizations of the same (colored) point process. In general, pooling is performed on identical independent point processes in order to obtain better statistical estimates of the inter-event distribution. To perform these operations on data, a set of assumptions must be made, requiring statistical tests to validate them.

3.2 Model Assumptions

For risk and reliability analysis we ultimately need life distributions and the estimation of the life distribution from time histories calls for renewal processes. However, we are also interested in trends, types of failure, failure modes, failure effects, maintenance operations, etc., and this leads us to competing risk renewal processes.

An analysis scheme is needed in order to build the reliability model. This scheme consists of a structured set of assumption (see Figure 1) directed to the data set, together with statistical sets to validate these assumptions. Typically, each assumption must be validated with “accepted” before proceeding to the next assumption. The assumptions are as follows:

Model assumption 1: The stratum of n sockets is homogeneous

Homogeneity within a group of component can be assumed in the case that the components are similar in design, operating circumstances and maintenance regime.

Nevertheless, this assumption should always be checked with the failure data.

Model assumption 2: The processes are independent

We assume that there are no clusters of failures in the stratum of sockets. In the extreme case all the sockets in the stratum could fail together in a small interval on the calendar time. This would indicate a strong dependency between the sockets and a possible common external cause of the failures in the sockets. This assumption should therefore always be checked.

Model assumption 3: The colored process is stationary

The coloring is stationary if the proportion of, say, "red" and "green" points does not vary significantly with calendar time.

Model assumption 4: The process is "color blind competing risk"

The competing risk processes is describe by a n-tuple of risk or colors, and a countable set of n-tuples of random variables $(X_{11}, ..X_{1n}), (X_{21}, ..X_{2n}), (X_{31}, ..X_{3n})....$ After the $(i - 1) - th$ event, the next event occurs at time $\min\{X_{i1}, ..X_{in}\}$ and is assigned the color of the index realizing the minimum. Hence, we observe $(\min\{X_{i1}, ..X_{in}\}, C(i))$, where $C(i)$ is the color of the $i - th$ event.

The process is colored blind if the distribution of $(\min\{X_{i1}, ..X_{in}\}, C(i))$ is independent of the color of the $(i - 1) - th$ event. Ignoring time dependence of a scale which is small relative to the expected inter-event times, color blindness implies that the processes gotten by splicing together all inter-event times beginning with color $j, j = 1, ...n$, are homogeneous.

Model assumption 5: The process is stationary competing risk

In mathematical terms a stationary series of events is defined by the following requirements:

1. the distribution of the number of events in a fixed interval $(t'_1, t''_1]$ is an invariant under translation, i.e. is the same for $(t'_1 + h, t''_1 + h]$ for all h ;

2. the joint distribution of the numbers of events in fixed intervals $(t'_1, t''_1], (t'_2, t''_2]$ is invariant under translation, i.e. is the same for the pair of intervals $(t'_1 + h, t''_1 + h], (t'_2 + h, t''_2 + h]$ for all h ;
3. generally the same invariance property must hold for the joint distribution of the number of events in a set of k fixed intervals, for all $k = 1, 2, \dots$

Characteristics of a stationary series of events of importance to this work are:

- the distribution of the number of events in an interval of the time window depends only on the length of the interval;
- the expected number of events in an interval of the time window is proportional to the length of the interval;
- there exist no trend in the mean rate of occurrence of failure events throughout the length of the time window.

In our case this means that the process $(\min\{X_{i1}, \dots, X_{in}\}, C(i)), i = 1, 2, \dots$, is stationary. Since the coloring has already been found to be stationary, this is equivalent to asking whether the uncolored process is stationary.

Model assumption 6: The process is renewal competing risk

This assumption implies that a socket is completely or perfectly repaired, similar to replacement to a new one. The plausibility of this assumption can be easily questioned. Yet, we have a major modeling benefit of this assumption by the fact that the series of events is now a renewal process. A renewal process is a process in which the intervals between events are independently and identically distributed.

Model assumption 7: The process is Poisson competing risk

Consider events occurring along the time axis. Let λ be a constant with dimension of reciprocal of time. It will measure the mean rate of occurrence of failure events over the period of time covered in the time window chosen by the user and will be called the probability rate of occurrence. Denoted by $N_{t,t+h}$, the random variable defined as the number of events occurring in $(t, t + h]$. The conditions for a Poisson process of rate λ are that as $h \rightarrow 0$

1. $Pr\{N_{t,t+h} = 0\} = 1 - \lambda h + o(h)$
2. $Pr\{N_{t,t+h} = 1\} = \lambda h + o(h)$
3. And that the random variable $N_{t,t+h}$ is statistically independent of the number and positions of the events in $(0, t]$

Model building and assumptions validation are presented in Figure 1.

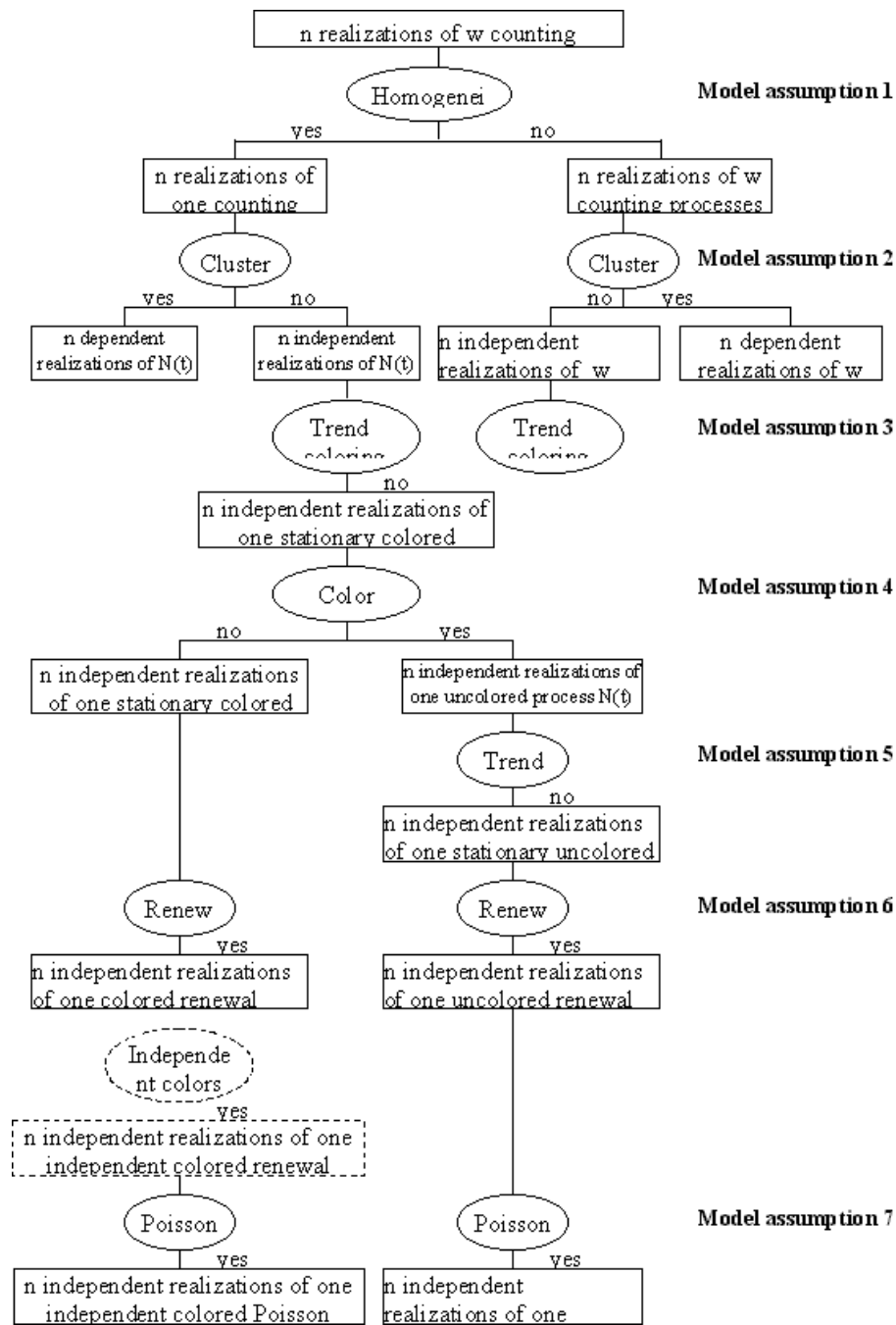


Figure 1. Model assumptions scheme

3.3 Statistical tests

3.3.1 Model assumption 1: The stratum of w sockets (units) is homogeneous

Suppose we have w processes $\{N_i(t)\}, i = 1, \dots, w$, where N_i is the counting process with intensity $I_i(t)$.

We shall test the hypothesis:

$$H_0 : I_1(t) = \dots = I_w(t).$$

Cox and Lewis (1966 [26]) proposed the following statistic when all the processes are defined over the same interval:

$$D(N) = \sum_{i=1}^w \frac{(N_i(t) - N^*(t))^2}{N^*(t)},$$

where

$$N^*(t) = \frac{\sum_{i=1}^w N_i(t)}{w}.$$

We reject H_0 when $D > \chi_{w-1, \alpha}$, where $\chi_{w-1, \alpha}$ is the α quantile of the χ^2 distribution with $w - 1$ degrees of freedom. This test is designed to detect if the w processes have the same intensity $I(t)$ under the assumption of Poisson processes (non-homogeneous). It is no designed to detect if their intensity is constant in time (homogeneous Poisson process). To avoid the assumption that $\{N_i(t)\}$ are Poisson processes, the following approach is recommended by Paulsen et all (1997 [62]): Divide the interval $[0, t]$ in two equal pieces, giving the processes: $N_{i,1}(s), 0 \leq s \leq t/2$ and $N_{i,2}(s), t/2 < s \leq t, i = 1, \dots, w$.

Let $\pi \in w!$ be a permutation of $(1, 2, \dots, w)$ and let

$$N_\pi = \{N_{\pi(i),1} \otimes N_{i,2}\}; \quad i = 1, 2, \dots, w$$

be the π process, where

$$N_{\pi(i),1} \otimes N_{i,2} = \begin{cases} N_{\pi(i),1}(s); & 0 \leq s \leq t/2 \\ N_{\pi(i),1} + N_{i,2}(s) - N_{i,2}(t/2); & t/2 \leq s \leq t \end{cases}$$

We reject H_0 if $D(N_\pi)$ is in the upper or lower 2,5% quantile.

3.3.2 Model assumption 2: The processes are independent

We want a test to detect 'clustering' of failure events from the various processes, that is, the tendency of events in the different processes to occur close in time. Such a clustering means either that there are common peaks in the intensities $I_i(t)$ of the processes or that the processes are stochastically dependent.

$$H_0 : \text{"No clustering across processes"}$$

We choose one of the w processes $N_i(t)$, with a medium or large number of events. For simplicity we choose the process N_1 and let the events of this process occur at times T_1, \dots, T_k . Define intervals $(T_i - \Delta, T_i + \Delta)$ covering these events. There is thus defined a set $S_1(2\Delta)$ on the time axis, namely the union of the intervals of length 2Δ ; let this set have length $T_1(2\Delta)$. If there are $N_1(T)$ events then $T_1(2\Delta) < N_1(T)2\Delta$, with equality if and only if two events do not occur in 2Δ and no events occur in $(T - \Delta, T]$ where T is the end of the period of observation.

Let $n = N(T) = \sum_{k=2}^w N_k(t)$ is the total number of events of the processes and $N^*(T) = \sum_{k=2}^w N_k^*(t)$ is the total number of events falling in any interval of length 2Δ .

For this analysis we consider the approximation in which $N_1(T)2\Delta \ll T$ and $T_1(2\Delta) = N_1(T)2\Delta$. Then under the null hypothesis $N^*(T)$ has a binomial distribution with index n and parameter $T(2\Delta)/T$. If $T(2\Delta)/T = N_1(T)2\Delta/T = p_T(2\Delta)$ is the probability of any observation of the process $N_i(t), i > 2$ falling in one of the intervals with length 2Δ , then the expectation and variance of $N^*(T)$ are:

$$E(N^*(T)) = np_T(2\Delta),$$

$$Var(N^*(T)) = np_T(2\Delta)(1 - p_T(2\Delta)).$$

Using a Normal approximation, we reject H_0 when

$$\frac{N^*(T) - np_T(2\Delta)}{\sqrt{np_T(2\Delta)(1 - p_T(2\Delta))}} > U_\alpha,$$

where U_α is the α quantile of $N(0, 1)$.

If $N^*(T)$ is small (less than 5) we should use the binomial or Poisson distribution rather than $N(0, 1)$. It is suggested that Δ be chosen so that $p_T(\Delta) \approx 0.1$.

3.3.3 Model assumption 3: The colored process is stationary

We construct a test to detect whether the ratio of colored events (R and G) is constant over time. Let $P_R(t)$ be the probability of event occurring at time t is colored R and $P_G(t)$ be the probability of event occurring at time t is colored G .

We test the hypothesis

$$H_0 : P_R(t) = p_R \text{ (independent of time)}$$

We test H_0 by dividing the interval $(0, T)$ in two. Let $N_{R,1}$ be the number of R events in $(0, T/2]$ and $N_{R,2}$ be the number of R events in $(T/2, T)$. Given the total number of events $n_1 = N(T/2)$ and $n_2 = N(T) - N(T/2)$, we have that $N_{R,1}$ and $N_{R,2}$ under H_0 are independent and binomial distributed with probability p_R . Using a normal approximation and the result that if $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ then $X - Y \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$, we reject H_0 when

$$\frac{|N_{R,1}/n_1 - N_{R,2}/n_2|}{\sqrt{(1/n_1 + 1/n_2)(1 - N_R/n)N_R/n}},$$

where $n = n_1 + n_2$, $N_R = N_{R,1} + N_{R,2}$.

3.3.4 Model assumption 4: The process is "color blind competing risk"

Is the colored process 'renewed' to the same degree by R events and G events? If not, the colored process might be a superposition of R and G processes (which could be renewal).

Let, $\{X_{R,k}\}_{k=1,2,\dots}$ be the R process - the sequence of intervals starting with R event and $\{X_{G,k}\}_{k=1,2,\dots}$ be the G process - the sequence of intervals starting with G event.

We test

$$H_0 : \{X_{R,k}\} \text{ and } \{X_{G,k}\} \text{ are realization of identical processes}$$

We reduce H_0 to two 'subhypotheses':

1. $H_{0,1} : I_R(t) = I_G(t)$ (the uncolored R and G processes have the same intensity)

2. $H_{0,2} : P_R(R) = P_G(R)$ (the fraction of R events in the R process and G process are the same)

1) We consider the comparison of k Poisson processes ($k > 2$) and we test for equality the parameters $\lambda_1, \dots, \lambda_k$, no special type of alternative being specified. Suppose that in fixed time periods $t_{0(1)}, \dots, t_{0(k)}$, the number of events observed are n_1, \dots, n_k , where n_i is the observed value of a Poisson variable N_i , of mean $\mu_i = \lambda_i t_{0(i)}$. Under the null hypothesis $\lambda_i = \lambda$, where λ is unknown. Then, following the general result that $n = \sum_1^k n_i$ is a sufficient statistic for λ and the significance test should be based on the conditional distribution

$$\begin{aligned} Pr\{N_i = n_i; i = 1, \dots, k \mid \sum_{i=1}^k N_i = \sum_{i=1}^k n_i = n\} &= \frac{\exp(-\sum_{i=1}^k \lambda_i t_{0(i)}) \prod_{i=1}^k \frac{(\lambda_i t_{0(i)})^{n_i}}{n_i!}}{\exp(-\lambda t_0) \frac{(\lambda t_0)^n}{n!}} = \\ &= n! \prod_{i=1}^k \left\{ \frac{t_{0(i)}}{t_0} \right\}^{n_i} \frac{1}{n_i!}, \end{aligned}$$

where $t_0 = \sum_{i=1}^k t_{0(i)}$ is the total time of observation.

We used the general result that $\sum N_i$ has a Poisson distribution of mean λt_0 . The above equation is equivalent to the statement that the n events are multinomial distributed among k cells with cell probabilities $t_{0(i)}/t_0$. For $k = 2$ we obtain the binomial distribution.

The best known test is based on the index of dispersion, i.e. on the statistic

$$d = \sum_{i=1}^k \frac{(n_i - t_{0(i)}\lambda^*)^2}{t_{0(i)}\lambda^*},$$

where $\lambda^* = n/t_0$. d tends to a chi-square distribution with $k - 1$ degrees of freedom (Cramer 1946, Chapter 30 [27]).

In our case, we reject $H_{0,1}$ if:

$$D = \frac{(N_R(T_R) - T_R\lambda^*)^2}{T_R\lambda^*} + \frac{(N_G(T_G) - T_G\lambda^*)^2}{T_G\lambda^*} > \chi_{1,\alpha}^2,$$

where $\lambda^* = \frac{N_R(T_R) + N_R(T_G)}{T_R + T_G}$ and $N_R(T_R)$ is the number of red events in the red process and $N_R(T_G)$ is the number of red events in the green process.

An alternative test of Lindqvist (1993 [53]) could be applied to test identity within the class of homogeneous Poisson, nonhomogeneous Poisson and Weibull trend renewal process.

This test holds only under the assumption of having renewal process:

$H_{0,1}^* : \{X_{R,k}\}$ and $\{X_{G,k}\}$ have same distribution

Using Kolmogorov-Smirnov statistic, we reject $H_{0,1}^*$ if

$$\text{Sup}|F_R^*(t) - F_G^*(t)| > k(n_R, n_G)$$

2) We test for

$$H_{0,2} : P_R(R) = P_G(R)$$

Is identical with the test used for 'stationary coloring' (see above). Let the R process have n_R events and let N_{RR} of these be R . Let the G process have n_G events and let N_{GR} of these be R .

We reject $H_{0,2}$ if:

$$\frac{|N_{RR}/n_R - N_{GR}/n_G|}{\sqrt{(1/n_R + 1/n_G)(1 - N_R/n)N_R/n}},$$

where $N_R = N_{RR} + N_{GR}$, $n = n_G + n_R$.

3.3.5 Model assumption 5: The process is stationary competing risk

Since the coloring has already been found to be stationary, this is equivalent to asking whether the uncolored process is stationary.

We test for trend in the rate of occurrence. The uncolored process has inter-event times T_1, T_2, \dots, T_n

1) Laplace test (Cox and Lewis 1966 [26])

Suppose first that we consider a trend in the rate of occurrence represented by a smooth change in time

$$\lambda(t) = e^{\alpha + \beta t}.$$

It is sensible to start by using the above relation rather than a linear one $\lambda(t) = \alpha + \beta t$ because for $\alpha > 0$ the latter is non-negative only for restricted values of t and β . Locally, near $\beta = 0$ the exponential relation is equivalent to a linear trend. The probability density that in the interval $(0, T]$ events occur at $T_1 \leq T_2 \leq \dots \leq T_n$ is

$$\lambda(T_1) \exp\left\{-\int_0^{T_1} \lambda(u) du\right\} \lambda(T_2) \exp\left\{-\int_{T_2}^{T_1} \lambda(u) du\right\} \dots \lambda(T_n) \exp\left\{-\int_{T_{n-1}}^{T_n} \lambda(u) du\right\}$$

$$\exp\left\{-\int_{T_n}^T \lambda(u)du\right\} = \left\{\prod_{i=1}^n \lambda(T_i)\right\} \exp\left\{-\int_0^T \lambda(u)du\right\} = \exp\left\{n\alpha + \beta \sum_{i=1}^n T_i - \frac{e^\alpha}{\beta}(e^{\beta T} - 1)\right\}.$$

n and $\sum T_i$ are sufficient statistics. Further for given β , the sufficient statistic for α is the number n of events. Hence the distribution for inference about β , when α is a nuisance parameter, is the conditional distribution of $\sum T_i$ given n . This conditional p.d.f. of the observation, given n events, is

$$\frac{n!\beta^n}{(e^{\beta T} - 1)^n} e^{\beta \sum_{i=1}^n T_i}.$$

The conditional likelihood is

$$L(\beta) = n \ln(\beta) - n \ln(e^{\beta T} - 1) + \beta \sum_{i=1}^n T_i + \ln(n!)$$

so that

$$L'(\beta) = n/\beta - nT/(e^{\beta T} - 1) + \sum_{i=1}^n T_i$$

and the information function is

$$I(\beta) = E\{-L''(\beta)\} = n(1/\beta^2 - T^2 e^{\beta T}/(e^{\beta T} - 1)^2).$$

To test the null hypothesis $\beta = \beta_0$ we can use the statistic

$$\frac{L'(\beta)}{\sqrt{I(\beta)}}.$$

This has mean zero and unit variance and an asymptotic normal distribution.

If we test the null hypothesis $\beta = 0$ the conditional p.d.f is

$$\frac{n!}{T^n}, \quad 0 \leq T_1 \leq T_2 \leq \dots \leq T_n.$$

To interpret this relation consider n random variables U_1, \dots, U_n independently uniformly distributed over $(0, T]$, each having the p.d.f. $1/T$. The joint p.d.f. is $1/T^n$. Now examine the corresponding order statistics $U_{(1)} \leq \dots \leq U_{(n)}$. To calculate their p.d.f. at argument T_1, T_2, \dots, T_n , note that $n!$ different original sequences U_1, \dots, U_n each with the same p.d.f. lead to the same sequence $U_{(1)} \leq \dots \leq U_{(n)}$. That is, conditionally on n , the positions of the events in a Poisson process are independently uniformly

distributed over the period of observation T with mean $T/2$ and variance $T^2/12$. And from central limit theorem we have

$$\frac{\sum_{i=1}^n T_i - nT/2}{T\sqrt{n/12}} \rightarrow N(0, 1) \quad \text{as } n \rightarrow \infty.$$

We reject H_0 when

$$\frac{\sum_{i=1}^n T_i - nT/2}{T\sqrt{n/12}} > U_\alpha.$$

2) The above test depends on taking the Poisson process as null hypothesis. We present now a test in which the intervals X_1, \dots, X_n are independent and identically distributed, not necessarily exponentially distributed (Cox and Lewis 1966 [26]). If a plausible functional form can be chosen for the distribution of the X_i 's, a special test can be constructed by taking the alternative hypothesis a model in which the parameters in the distribution are suitably chosen functions of the z_i 's. We arrange the X_i 's in increasing order $X_{(1)}, X_{(2)}, \dots, X_{(n)}$. Let $r(i)$ be the rank of X_i and define the score for X_i :

$$S_i = 1/n + 1/(n - r(i) + 1).$$

We can use an asymptotic normal distribution for

$$\sum_{i=1}^n S_i(z_i - \bar{z})$$

with mean zero (by symmetry) and variance $\sum_{i=1}^n (z_i - \bar{z})^2 [1 - (1/n + \dots + 1/2)/(n-1)]$. We take $z_i = i$ and reject H_0 when:

$$|V| > U_{\alpha/2}$$

where $V = \sum_{i=1}^n S_i(i - (n+1)/2)/\sigma^2$ and $\sigma^2 = \sum_{i=1}^n (i - (n+1)/2)^2 [1 - (1/n + \dots + 1/2)/(n-1)]$.

3.3.6 Model assumption 6: The process is renewal competing risk

It remains to test for independence of inter-event times X_i . Let $\rho_i = \text{corr}(X_k, X_{k+i})$. Then if (X_k, X_{k+i}) are independent $\rho_i = 0$. But the converse is not always true ¹.

¹If X_k follow a $N(0, 1)$ distribution and $X_{k+i} = X_k^2$ then $\rho_i = \text{corr}(X_k, X_{k+i}) = 0$ because $\text{cov}(X_k, X_{k+i}) = M(X_k^3) - M(X_k)M(X_k^2) = 0$ ($M(X_k) = 0$ and $M(X_k^3) = 0$)

$$H_0 : \{X_i\} \text{ are independent } (\rho_1 = 0)$$

From (Cox and Lewis 1966 [26]) the most useful result is for large sample:

$$\text{var}(\rho_j^*) \approx \frac{1}{n-j} \sum_{-\infty}^{\infty} \rho_i^2 = \frac{1}{n-j} (1 + 2 \sum_{i=1}^{\infty} \rho_i^2).$$

For a renewal process $\text{var}(\rho_j^*) = \frac{1}{n-j}$.

$\rho_1^* \sqrt{n-1}$ will have a unit normal distribution if n is large and $\rho_1 = 0$, so that we reject H_0 when

$$|\rho_1^* / \sqrt{n-1}| > U_{\alpha/2},$$

where $\rho_1^* = \frac{\sum_{i=1}^{n-1} (X_i - X^*)(X_{i+1} - X^*)}{\sum_{i=1}^{n-1} (X_i - X^*)^2}$ and $X^* = \frac{\sum_{i=1}^n X_i}{n}$.

3.3.7 Model assumption 7: The process is Poisson competing risk

Given that the X_i 's are iid, we may test whether these are exponential (thus giving the homogeneous Poisson process). We test for independent exponentially distributed times between events against the alternative of independent times between events with a Gamma distribution.

Recall that the gamma distribution function has the form:

$$f(x) = (k/\mu)^k \frac{x^{k-1} e^{-kx/\mu}}{\Gamma(k)},$$

where $\Gamma(k)$ is gamma function.

We are testing $k = 1$ against $k \neq 1$. This is

$$\begin{aligned} H_0 &: \text{Poisson process} \\ H_1 &: \text{Renewal process} \end{aligned}$$

Cox and Lewis (1966 [26]) suggest two tests:

1) The likelihood function for the n observed times between events is, in this case,

$$L(X; k, \mu) = (k/\mu)^{nk} \left(\prod_{i=1}^n X_i^{k-1} \right) e^{-k \sum_{i=1}^n X_i / \mu} / \{\Gamma(k)\}^n.$$

The maximum likelihood estimators of k and μ are the solution of the equations:

$$\begin{aligned}\frac{\partial \ln L(X;k,\mu)}{\partial k} &= 0 \\ \frac{\partial \ln L(X;k,\mu)}{\partial \mu} &= 0\end{aligned}$$

Hence the maximum likelihood estimator of k is the solution of:

$$\Psi(k) - \ln(k) = -\ln X^* + \frac{1}{n} \sum_{i=1}^n \ln X_i,$$

where $\Psi(k) = \frac{\partial \ln \Gamma(k)}{\partial k}$ is the digamma function. The left-hand side of this equation is a monotonic function of k in the neighborhood of $k = 1$, so that we can write

$$k = \varphi\left(-\ln X^* + \frac{1}{n} \sum_{i=1}^n \ln X_i\right),$$

where φ is a monotonic function of its argument.

A test based on the statistic which appears as the argument of φ will then be an asymptotically most powerful test of our hypothesis. This test was derived by Moran(1951). We reject H_0 if

$$l_n = \frac{2n(\ln X^* - \frac{1}{n} \sum_{i=1}^n \ln X_i)}{1 + \frac{n+1}{6n}} > \chi_{n-1, \alpha}^2.$$

Under the null hypothesis \ln has approximately a chi-squared distribution with $n - 1$ degrees of freedom. The divisor was introduced by Bartlett (1937) to improve this approximation.

2) This test is based on a statistic proposed by Sherman [1950]. This statistic in its original form is a distribution free statistic; the expected values of the order statistics $U_{(i)} = T_i/T_0$ if $N(T_0) = n$ else $U_{(i)} = T_i/T_{n+1}$ are $E(U_{(i)}) = i/(n+1)$, so that $E(U_{(i)} - U_{(i-1)}) = 1/(n+1)$. The Sherman statistic measures the deviations from these expected values as:

$$w_n^* = \frac{1}{2} \sum_{i=1}^{n+1} |U_{(i)} - U_{(i-1)} - 1/(n+1)| = \frac{\sum_{i=1}^{n+1} |X_i - X^*|}{2(n+1)X^*} > \text{constant},$$

where under H_0 , $E(w_n^*) \approx 1/e = 0.368$ and $\text{var}(w_n^*) = 0.05908/n - 0.07145/n$, and the normalized statistic has asymptotically a $N(0, 1)$ distribution.

Chapter 4

Reliability Data Analysis

The problem of competing risk data is considered with an application to Offshore Reliability Data Base (OREDA). Different models for preventive maintenance are discussed which make the failure rate identifiable, as there are more general bounding methods. A statistical test is used for the concordance of the results of the data interpretation and the theoretical models proposed. The results indicate the way to avoid inappropriate models for data analyzing.

4.1 Introduction

Maintenance study requires the use of many modelling assumptions. We focus on issues related to reliability data bases (RDB's) interpretation, and in particular to dependent competing risk. Competing risk models are used to interpret data and a statistical test is used to find the appropriate competing risk model for RDB in question.

Modern RDB's may distinguish ten or more failure modes (ways of ending a service sojourn), often grouped in critical failures, degraded failures and incipient failures. The latter two are usually associated with preventive maintenance, whereas critical failures are of primary interest in risk and reliability calculations. A component exits a service sojourn due to the occurrence of one of its possible failure modes. The failure modes are competing each other to 'kill' the component, hence each failure mode censors the others.

Independent competing risks models have been studied for some time. By observing independent copies of competing risks we can estimate the subsurvival functions. Assuming independence of competing risks we can determine the underlying marginal distributions. In this case we have identifiability. The assumption of independence is questionable when failures are censored by preventive maintenance. The assumption of independence would imply that maintenance engineers take no account of the state of a component when taking the decision to preventively maintain. It is more reasonable to make a dependence assumption between the censoring processes.

In this chapter, we will present the one sided Kolmogorov-Smirnov test for a two sample problem in order to test the exponential independent model against the alternative random signs model. Hence, we are going to test whether the conditional subsurvival function are coming from the same population against the alternative that the conditional subsurvival function of the censoring variable lies entirely below the conditional subsurvival of the censored variable. An algorithm how to calculate the one sided KS statistic is also given.

The performance of the probabilistic model we propose, is illustrated on the Gas Genenrator data used by Langseth (1999 [52]). This is a subset of Phase IV of the Gas Turbine dataset from the Offshore Reliability Database (OREDA 1997 [61]). We have 22 failures in this dataset, out of which 8 are classified as critical and 14 as degraded. The main results coming out from this dataset is that even for a small sample population we can say that the conditionally subsurvival functions are not from the same population and a random signs model is appropriate to interpret this data.

4.2 Gas Turbine Data - OREDA

A number of offshore platforms had been in operating in Europe for a significant length of time, and the Offshore Reliability Data (OREDA) handbook project was established to compile a comprehensive basis of reliability information from failure and repair records already existing in company files and records.

For the purpose of our study, we use Phase IV of the Gas Turbine data set from Offshore Reliability Database (OREDA 1997). Only the Gas Generator subsystem was included in the study. We chose to analyze data from a single offshore installation only to ensure maximum homogeneity of the data sample. The data set consists of

the 23 mechanical units, which are followed over a total of 536.403 operating hours. We have 83 inspections in this data set and 22 failures, out of which 8 are classified as critical and 14 as degraded. Degraded failures can be associated to a preventive maintenance action and critical failure to a corrective maintenance action. The failures are distributed over four different failure mechanisms, namely deformation, leakage, breakage and other mechanical failure.

Inv	Clock	Failure mode	Severity	Censored
1	6125	PM		2
1	12250	PM		2
1	18375	PM		2
1	24500	PM		2
2	6125	PM		2
2	12250	PM		2
2	18375	PM		2
2	24500	PM		2
22	8800	Leakage	N	2
22	8944	Leakage	N	2
22	9114	PM		2
23	240	Leakage	N	2
23	5232	Leakage	C	2
23	5688	Breakage	C	2
23	6216	Leakage	N	2
23	8448	Leakage	N	2
23	12820	PM		2

Figure 1. Phase IV of the Gas Turbine data set from Offshore Reliability Database (OREDA 1997)

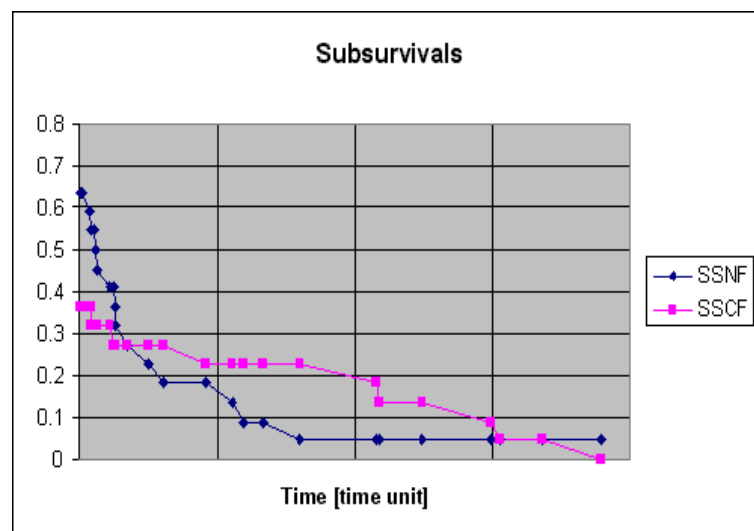


Figure 2. Empirical subsurvival functions

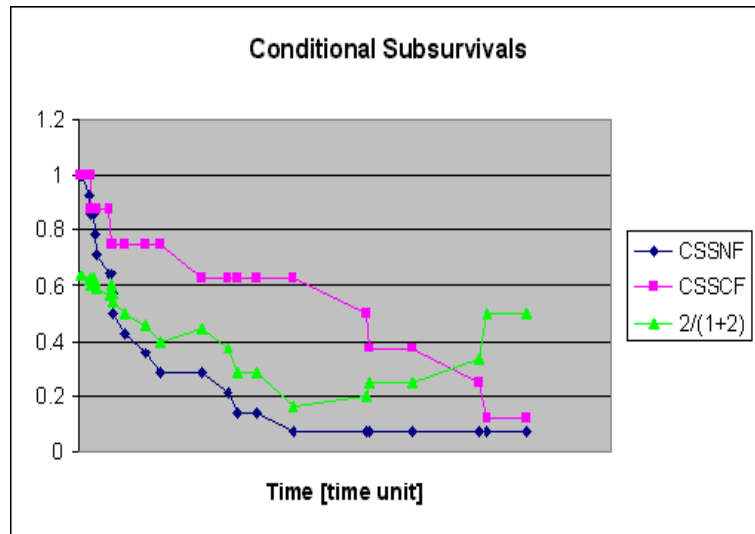


Figure 3. Conditional subsurvival functions and $\Phi(t)$ function

Figure 3 shows that the function $\Phi(t) \left[\frac{2}{1+2} \right]$ is minimum at the origin and the conditional subsurvival function of critical failure dominates the conditional subsurvival function of degraded failure, as predicted by random signs model. Hence, an independent exponential model is not appropriate for this data. (Stove 2002 [70]) used several models in his attempt to model the failure mechanism of Gas Turbine data. The Repair Alert model seems to explain data, but since we have quite few observation of X and Y , the assumption for the conditional subsurvival for Y may be dubious. He proposed as an alternative the Langseth-Bedford-Lindqvist model or the Random Sings model.

Further we will present a statistical test for testing if the independent exponential model fits this data.

4.3 Two-sample Kolmogorov-Smirnov Test

If an independent exponential model holds then the conditional subsurvival functions are equal and the probability of censoring beyond time t is constant. Hence, we want to test if the empirical estimation of the conditional subsurvival functions are from the same population.

Our data consists of two independent random samples drawn independently from each of two population. Let U and V be random variables with survival functions

$S_U(t)$ and $S_V(t)$ equal to the conditional subsurvival functions of X respectively Y . From Gas Turbine Data we have two samples U_1, U_2, \dots, U_8 of size $m = 8$, drawn from the U population and V_1, V_2, \dots, V_{14} of size $n = 14$, drawn from the V population.

The hypothesis of interest in the two-sample problem is that the two-samples are drawn from identical populations,

$$H_0 : S_U(t) = S_V(t) \quad \text{for all } t.$$

The one-sided Kolmogorov-Smirnov two sample test criteria, denoted by $D_{m,n}^+$ is the maximum difference between the empirical functions of $S_U(t)$ and $S_V(t)$:

$$D_{m,n}^+ = \max[S_U^m(t) - S_V^n(t)].$$

Since here the directional differences are considered, $D_{m,n}^+$ is appropriate for a general one-sided alternative:

$$H_1 : S_U(t) \geq S_V(t) \quad \text{for all } t.$$

The null hypothesis H_0 is rejected at the significance level α if

$$D_{m,n}^+ > d_\alpha,$$

where

$$Pr\{D_{m,n}^+ > d_\alpha\} = \alpha.$$

The asymptotic distribution of $\sqrt{\frac{mn}{m+n}}D_{m,n}^+$ is:

$$\lim_{m,n \rightarrow \infty} Pr\{D_{m,n}^+ \leq d_\alpha\} = 1 - e^{-2d_\alpha^2}.$$

If the size of the samples are bigger than 50 then the asymptotic formula can be used to determine the significance level at which the null hypothesis is rejected, otherwise tables should be use. Further we will present an algorithm how to calculate the tail probability for small samples, necessary for programing implementation.

Let $U_{(1)}, U_{(2)}, \dots, U_{(m)}$ and $V_{(1)}, V_{(2)}, \dots, V_{(n)}$ be the order statics of the two samples of size $m = 8$ and $n = 14$ from continuous populations $S_U(t)$ and $S_V(t)$. To compute $Pr\{D_{m,n}^+ \leq d_\alpha\}$, where $D_{m,n}^+ = \max[S_U^m(t) - S_V^n(t)]$, we first arrange the combined sample of $m + n$ observation in increasing order of magnitude (Table 1).

Sample	v	v	v	v	u
v	v	v	u	u	u
v	v	v	v	u	v
u	u	v	v	u	

Table 1. Failure times of Gas Turbine data set

The arrangement can be plotted on a Cartesian coordinate system by a path which starts at the origin and moves one step up for a u observation and one step to the right for a v observation, ending at (n, m) . The observed values of $mS_V^m(t)$ and $nS_V^n(t)$ are the coordinates of all points (i, j) on the path, where i and j are integers. The number d_α is the largest of the difference

$$\frac{i}{m} - \frac{j}{n} = \frac{ni - mj}{mn}.$$

The vertical distance from any point (i, j) on the path to the line $nu - mv = 0$ and situated below it is $\max[j - \frac{ni}{m}]$. Hence, nd_α for the observed sample is the distance from the diagonal to that point on the path which is farthest from the diagonal line and is situated below it.

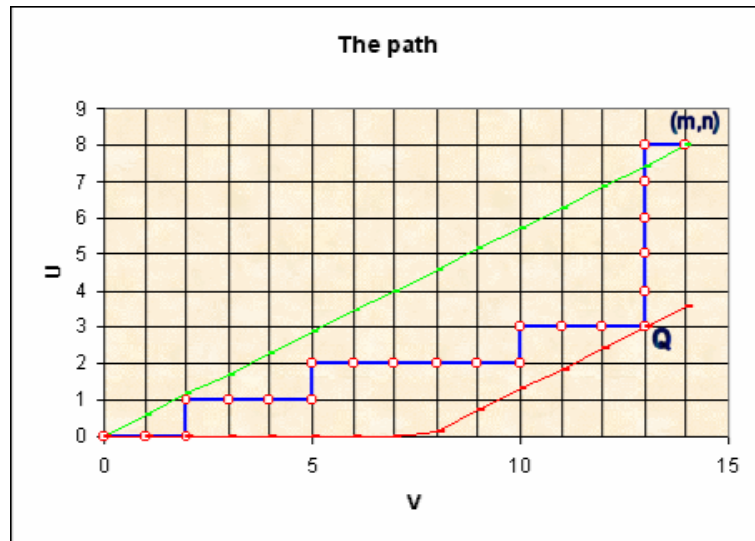


Figure 4. Cartesian representation of the combine sample

For our case the farthest point is Q and $d_\alpha = 0.554$. The total number of arrangements of mU and nV r.v. is C_{m+n}^m , and under H_0 each of the corresponding paths is

equally likely. The probability of an observable value of $D_{m,n}$ not less than d_α then is the number of paths which have points at a below distance from the diagonal not less than nd_α , divided by C_{m+n}^m . We mark off a line at vertical distance nd_α from the diagonal, below it, as in Figure 4. Denote by $A(m, n)$ the number of paths from $(0, 0)$ to (m, n) which lie entirely above this line. Then

$$Pr\{D_{m,n}^+ \leq d_\alpha\} = \frac{A(m, n)}{C_{m+n}^m}.$$

$A(i, j)$ at any intersection (i, j) satisfies the recursion relation:

$$A(i, j) = A(i - 1, j) + A(i, j - 1),$$

with boundary conditions $A(0, j) = A(i, 0) = 1$. Thus $A(i, j)$ is the sum of the numbers at the intersection where the previous point on the path could have been while it still was within the boundaries. Since $A(8, 14) = 310, 751$, we have:

$$Pr\{D_{8,14}^+ \leq d_\alpha\} = 0.0282$$

Hence we reject the null hypothesis that the conditional subsurvival functions are coming from the same population at the significant level $\alpha = 0.0282$.

4.4 Conclusions

Figure 3 shows that the conditional subsurvival function for critical failures dominates the conditional subsurvival function for degraded failures, as predicted by random signs model.

The statistical test rejected the null hypothesis that the exponential independent model is appropriate for this data at the significance level $\alpha = 0.0282$, indicating that the random signs model might be indicated to interpret data. The algorithm of calculating the tail probability for small samples can be implemented also to other sets of data with a small number of events. We mention also the example of the pressure relief valves data from a Swedish nuclear station. The Kolmogorov-Smirnov-test can be applied to test if the exponential independent model describes data. The null hypothesis (exponential independent model) is rejected at the significance level $\alpha = 0.0209$. It is worth mentioning that this test can yield important conclusions even if one or both of the competing risks are scarce. The test is powerful in spite

of having only 4 “alarm” and “unintended discovery” events, because there are then a large number of events of competing risk. Thus, although the estimate of the conditional subsurvival function of “alarm” and “unintended discovery” is very noisy, the estimate of the competing conditional subsurvival function is not.

Given the Glivenko-Cantelli theorem that the estimated distribution function converges with probability one to the real distribution function when the number of observation increase, this mean that the quality of estimating the parameters of the distribution is increasing but the population from which the sample is drawn remains the same. Hence, we can conclude that the conditional subsurvival functions are not coming from the same population and even for a small number of samples, the exponential independent model is not appropriate for this data.

4.5 Competing risk perspective on reliability data bases

This section considers competing risk models suitable for analysis of modern reliability databases. Commonly used models are reviewed and a new simple and attractive model is developed. This model grew out of a study of real failure data from an ammonia plant. The use of graphical methods as an aid in model choice is advocated in this section.

4.5.1 Introduction

Since Daniel Bernoulli’s attempt in the 18th century to separate the risk of dying due to smallpox from other causes, the competing risk theory has spread through various fields of science such as statistics, medicine and reliability analysis.

The competing risk approach in reliability is closely related to the development of modern reliability databases (RDB’s) in the second half of the last century. RDB’s consist of several data fields containing information on failure modes, failure causes, maintenance/repair actions, severity of failure, component characteristics and operating circumstances. For every specific field we can often distinguish ten or more competing risks, which compete to terminate a service sojourn of the component.

The general theory of competing risk is advanced here as a proper mathematical language for modeling reliability data.

To check the performance of different probabilistic models, we discuss a dataset coming from two identical compressor units at an ammonia plant of Norsk Hydro, covering an observation period from 1968 to 1989 (Erlingsen[34]). The competing risk models available in the reliability literature are especially developed for the nuclear sector, where strict regulations are imposed. Consequently these models are often not appropriate for the various fields of the compressor unit data. Due to the fact that the compressor unit consists of several heterogeneous sub-components, we therefore introduce another competing risk model, called the “mixture of exponentials model”, to interpret the competing risks between different failure modes.

4.5.2 Analysis of Norsk Hydro Data Sets

The data set proposed to discuss different competing risk models comes from one Norsk Hydro ammonia plant operating two identical compressor units, for the period of observation 2-10-68 up to 25-6-89. This yields 21 years of observation and more than 370 events. As every modern reliability data base, this data base has the following compressor unit history:

- Time of component failure
- Failure mode: leakage, no start, unwanted start, vibration, warming, overhaul, little gas stream, great gas stream, others
- Degree of failure: critical, non-critical
- Down time of the component
- Failure at the compressor unit: 1 - first unit failed, 2 - second unit failed, 3 - both units failed
- System and Sub-System Failure
- Action taken: immediate reparation, immediate replacement, adjustment, planned overhaul, modification, others
- Revision periods: 18 revision periods with a duration from 4 to 84 days

Date	Calendar time	Operation time	Revision time	Failure mode	Degree of failure	Failure time	Degree time	Sys. Time	Sys. Degree time	Down time (h)	Unit	System	Subsystem	Subnature mode	Action taken
10/2/68	0	0	0			0	0	0	0	0					
10/3/68	1	1	1	UST	C	1	1	1	1	1.25	1	SMO	PUM	UST	AKR
10/6/68	4	4	4	UST	C	3	3	4	4	135	2	SPE	ANN	LIG	AKR
10/6/68	4.5	4.5	4.5	UST	C	0.5	0.5	0.5	0.5	0.08	1	SPE	ANN	LIG	AKR
11/7/68	36	36	36	VIB	N	32	36	35	36	0	3	SMO	PUM	VIB	AKR
12/8/68	67	67	67	LEK	N	31	31	63	67	0	2	SPE	TAN	LEK	AKU
1/2/69	92	92	92	UST	C	25	88	25	88	5.33	2	SPE	FOR	ANN	JUS
1/22/69	112	R	E	V	I	S	I	O	N	1272	3	1/6/69			
4/23/69	203	150	38.2	LEK	N	58	83	150	150	0	1	KOM	ROR	LEK	AKR
4/23/69	203.25	150.25	38.4	LEK	N	0.25	0.25	0.25	0.25	0	2	KOM	ROR	LEK	AKR
4/23/69	203.5	150.5	38.6	LEK	N	0.25	0.25	150	150	0	2	INS	TRY	LEK	AKR
4/23/69	203.75	150.75	38.8	LEK	N	0.25	0.25	58	83	0	2	SPE	VEN	LEK	AKR
7/18/69	289	236	124	LEK	N	86	86	86	86	0	2	SPE	VEN	LEK	JUS
8/3/69	305	252	140	UST	C	16	160	252	252	154	1	ELM	ANN	UST	AKR
8/18/69	320	267	155	OVH	N	15	31	31	31	0	3	SPE	FIL	OVH	OVH
8/20/69	322	269	157	LEK	N	2	2	233	233	0	3	SMO	SEP	LEK	AKR
8/28/69	330	277	165	UST	C	8	25	10	185	0.5	1	SPE	VEN	LEK	AKR
8/28/69	330.5	277.5	165	UST	C	0.5	0.5	127	277	1.25	1	INS	NIV	ISI	AKU
9/4/69	337	284	172	UST	C	7	7	15	283	2.5	3	SMO	PUM	UST	JUS
10/4/69	367	R	E	V	I	S	I	O	N	720	11	3/69			
12/1/69	425	342	28	LEK	N	58	73	65	75	0	2	SPE	ROR	LEK	AKR
12/28/69	453	370	56	VIB	N	28	28	28	28	0	2	SPE	PUM	VIB	AKU
1/2/70	457	374	60	LEK	C	4	90	224	427	15	2	KOM	ROR	LEK	AKR

Failure mode	
LEK	leakage
IST	no start
UST	unwanted start
LIG	little gas stream
STG	great gas stream
VIB	vibration
VAR	warming
OVH	overhaul
ANN	others
Degree of failure	
N	non-critical
C	critical
Action taken	
AKR	immediate repair
AKU	immediate replacement
JUS	adjustment
OVH	planned overhaul
MOD	modification
ANN	others
UNIT	
1	compressor 1 failed
2	compressor 2 failed
3	both compressor failed
System	
SMO	

Figure 5. Compressor units data

The analysis performed on data is both statistical and probabilistic. Nevertheless, the most time consuming operation is "cleaning data". Experience showed that 2/3 of the time is spent on extracting the most relevant information from the huge amount of information gathered in a reliability data base. The main operation on data and the statistical analysis is presented in (Bunea 2002 [15]). Further the probabilistic analysis is studied.

A software tool was developed in the higher order programming languages Visual Basic and Excel. The operator can choose two exhaustive classes of competing risks (corresponding to the X and Y in Section 2) for the four fields of interest: failure mode, degree of failure, action taken, system. The analysis may be performed for the complete observation period or for a certain time window by specifying the limiting dates of interest.

The user interface consists of the following sections:

- Time window:** Start: 22 march 1968, Stop: 25 june 1989.
- Failure mode:**
 - Risk 1: LEK, IST, UST, LIG, STG, VIB, VAR, OVH, ANN
 - Risk 2: LEK, IST, UST, LIG, STG, VIB, VAR, OVH, ANN
- Degree of failure:** Risk 1: Critical, Risk 2: Noncrit
- Action taken:**
 - Risk 1: AKR, AKU, JUS, OVH, MOD, ANN
 - Risk 2: AKR, AKU, JUS, OVH, MOD, ANN
- System:**
 - Risk 1: SMO, SPE, KOM, INS, ELM, GEA
 - Risk 2: SMO, SPE, KOM, INS, ELM, GEA
- SubSystem:**
 - Risk 1: PUM, TAN, FOR, ROR, TRY, VEN, FIL, SEP, NIV, HUS, LAG, VIB, KJL, TEM, PAK, LEV, VIK, POS, OLJ, ANN
 - Risk 2: PUM, TAN, FOR, ROR, TRY, VEN, FIL, SEP, NIV, HUS, LAG, VIB, KJL, TEM, PAK, LEV, VIK, POS, OLJ, ANN

Figure 6. User interface

Figures 7-10 illustrate the different field analyses performed on the data. The upper graphs show the empirical subsurvival functions, $\widehat{S}_X^*(t)$, $\widehat{S}_Y^*(t)$, empirical conditional subsurvival functions, $\widehat{C}\widehat{S}_X^*(t)$, $\widehat{C}\widehat{S}_Y^*(t)$, and the estimated probability of censoring beyond time t , $\widehat{\Phi}(t)$. The bottom graphs show the estimated Peterson bounds for the average failure rate, for both classes of risk, as given in Section 2. Cooke and Bedford[21] explained how to use such graphical displays to choose an appropriate competing risk model. In Bunea et al.[14] we proposed a statistical test to check whether an independent exponential model is appropriate for the data, against the alternative of the random signs model or the mixture of exponentials model. The one sided Kolmogorov-Smirnov (KS) test was used to test the null hypothesis that the two empirical conditional subsurvival functions are drawn from two identical populations, against the alternative that one conditional subsurvival function is larger than the other. If the null hypothesis is not rejected at the chosen significance level, then the empirical conditional subsurvival functions may come from two identical populations, and the independent exponential model or the conditional independence model may be appropriate.

Failure Mode

The risk engineer tries to avoid a functional failure of the system. Discussions with maintenance personnel indicate that the most dangerous failure modes among the listed failure mechanism are "leakage", "no start", "unwanted start" and "vibration".

We let this be the risk class corresponding to X . These risks are censored by the other failure modes, which together correspond to Y . The total number of failures is 359 out of which 166 are censoring events and 193 are events that we want to prevent. The large number of unwanted events can be explained by a poor maintenance policy. Figure 7 shows a slightly increasing probability of censoring after time t , which is not consistent with previously used models. The bottom graphs indicate that a constant time average failure rate is not reasonable for the distribution of the first competing risk class. Hence, the independent exponential model is not appropriate in this case. In fact, choosing the significance level $\alpha = 0.05$, gives the critical value of the KS statistic $d_\alpha = 1.2230$. The empirical KS statistic calculated from data is 1.24783, and hence the null hypothesis is rejected at the level $\alpha = 0.05$. The graphs seem to indicate that the mixture of exponentials model may be appropriate in this case.

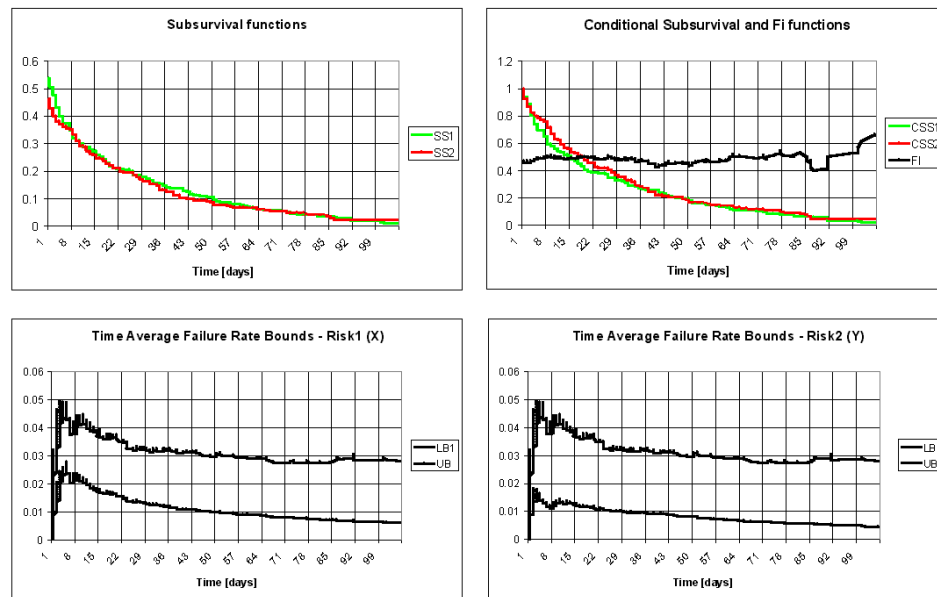


Figure 7. Analysis of “failure effect” field

Action Taken

The maintenance engineer is mainly interested in this field. His goal is to avoid the most expensive maintenance operation, immediate replacement, which corresponds to X in the model. He will prefer every other maintenance operation, together corresponding to Y , to this one. 85 replacements and 274 other maintenance actions are found. The empirical conditional subsurvival functions are crossing once and the estimated probability of censoring after time t has an inflexion point (Figure 8). Thus, the graphical interpretation of data might suggest that the independent exponential model does not hold. However, the Kolmogorov-Smirnov test does not reject the hypothesis that the empirical conditional subsurvival functions are coming from the same population. Thus, the independent exponential model might be applied. Given the uncertainty involved in the graphical visualization and the fact that no model is available in the literature for this case, we regard these risks as independent.

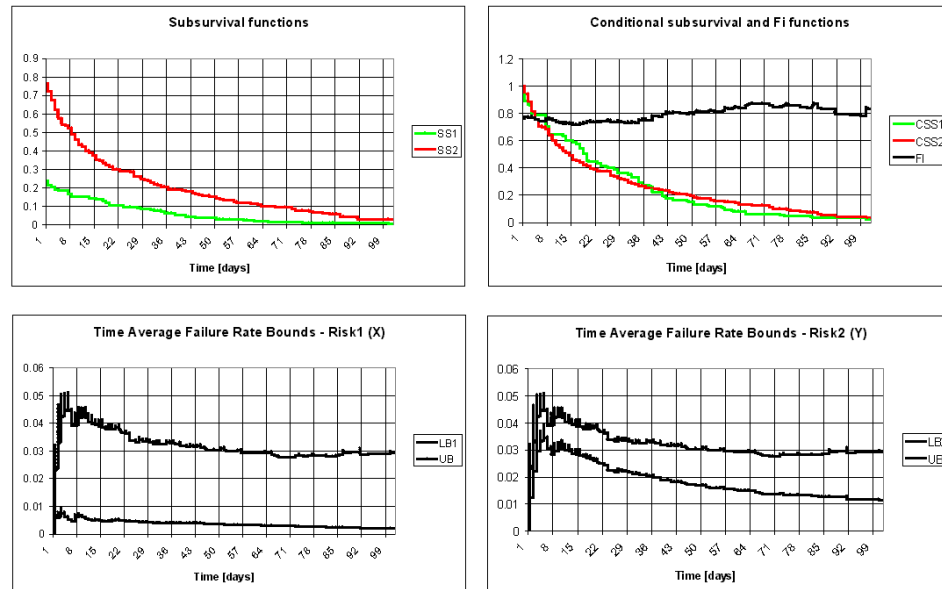


Figure 8. Analysis of “action taken” field

Degree of Failure

The risk engineer and the maintenance engineer are both interested in this field. Obviously, they want to prevent a “critical” failure. From the risk point of view this is the event with major consequences on the state of the system. On the other hand a

critical failure is usually associated with a corrective maintenance and a non-critical with a preventive maintenance. To keep the maintenance costs lower a corrective maintenance should be avoided. 92 critical failures (X) and 267 non-critical failures (Y) are recorded in the database. The empirical conditional subsurvival functions are more or less equal and the estimated probability of censoring after time t is roughly constant (Figure 9). The exponential independence model seems to be appropriate to fit the data, but the estimated Peterson bounds seem to be inconsistent with an exponential distribution for the critical failure. The KS-test does not reject the hypothesis that the empirical conditional subsurvival functions are equal. Recalling that the conditional independence model provides equal conditional subsurvival functions, this model seems most compliant with the graphical and statistical analyses.

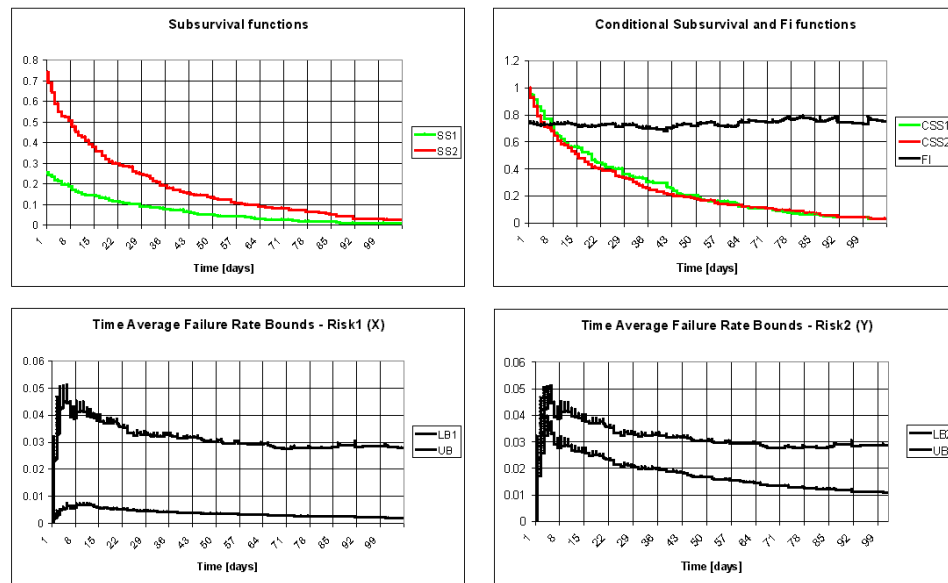


Figure 9. Analysis of “degree of failure” field

System

A third type of engineer is interested in this field: the designer. He is concerned with avoiding the failure of an expensive component. Since the electrical components are more expensive than mechanical components, their failure should be considered as the X , while the mechanical components correspond to Y . 208 failures of electrical components and 151 failures of mechanical components are detected. The empirical conditional subsurvival of the censoring variable dominates the one of the unwanted

event and the probability of censoring after time t is increasing (Figure 10). This is the opposite case of the random signs model, which models the behaviour of a very competent maintenance team. For the X -related risk, the exponential distribution is not compatible with the Peterson bounds in Figure 10. The KS-test also rejects the exponential independent model at the significance level $\alpha = 0.05343$.

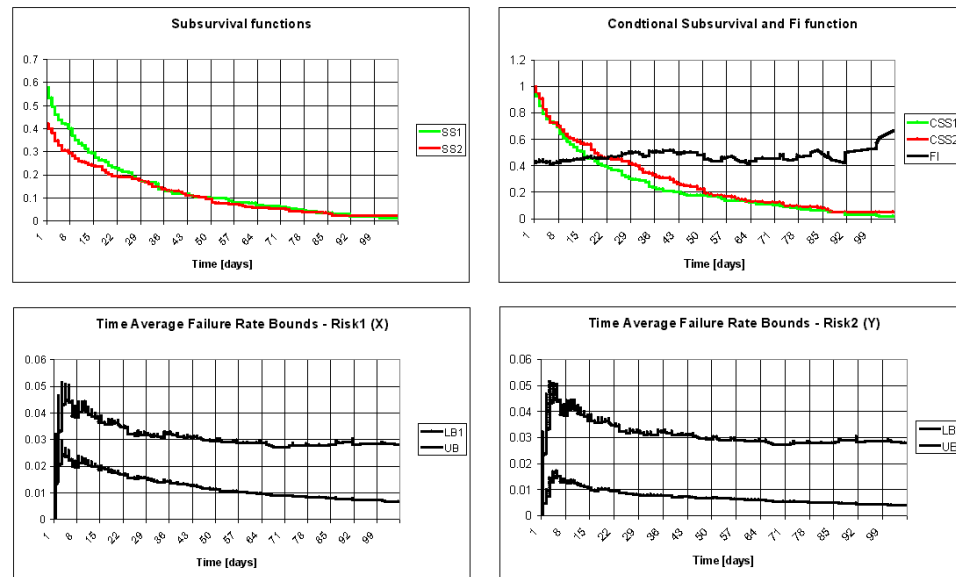


Figure 10. Analysis of “system” field

Based on this we used the mixture of exponentials model of Section 1.3.3. The following estimates were obtained by the moment method: failure rate of the censoring variable, $\lambda_y = 0.0015$; failure rates of the mixture, $\lambda_1 = 0.03757$, $\lambda_2 = 0.00936$, mixing coefficient, $p = 0.59$. Estimated parametric (continuous line) and nonparametric (dotted line) curves are presented in Figure 11. The figure indicates very good model fit for these data.

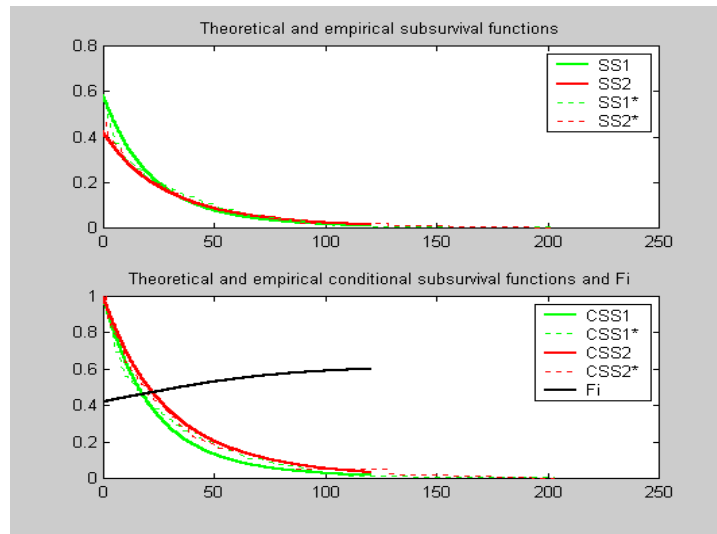


Figure 11. Model validation

The good fit of the mixture of exponentials model can be explained by heterogeneity. The compressor unit is vast and the component histories constituting the data come from pumps, valves, electro-motors, etc., which are expected to have very different failure characteristics. The Peterson bounds suggest a mixture of a few components with very high average failure rates and other components with low average failure rates. A similar behavior of data has been found in the pressure relief data from Swedish nuclear facilities (Cooke et al.[22]).

4.5.3 Conclusions

A new simple and attractive model has been developed for the case when the conditional subsurvival functions of the censoring variable dominates the conditional subsurvival functions of the other risks. The model agrees with empirical findings for the “Failure Mode” and “System” fields, where we observe a decreasing time average failure rate for X and a roughly constant one for Y . Such simple data analyses should be performed routinely by engineers. A suitable model may then be fit after this primary analysis of the data.

For future work, it remains to find a model appropriate for the “action taken” analysis case, when the conditional subsurvival functions are crossing eachothers (see Figure 8).

4.5.4 Statistical analysis of Norsk Hydro Data Base

In order to increase the available observations for the theoretical processes, hence to reduce the uncertainty in model estimation, one may pool the data obtained from unit 1 and 2. This operation has to take into account the assumptions 1 and 2. The assumption of homogeneity within the 2 units data is accepted at the level $\alpha = 0.1931$ under the assumption of non-homogeneous Poisson processes and it is rejected at the level $\alpha = 0.0328$ using the approach proposed by Paulsen et al (1996 [62]). The assumption of independence is also accepted at level of significance $\alpha = 0.95$.

	Assum. 3	Assum. 4-1 $I_R(t) = I_G(t)$	Assum. 4-2 $P_R(R) = P_G(R)$
Failure Mode Class I: LEK,IST,UST,VIB Class II: LIG,STG,VAR,OVH,ANN	0.6312	rejected	0.7889
Action taken Class I: AKU Class II: AKR,JUS,OVH,MOD,ANN	0.8969	rejected	0.5616
Degree of failure Class I: Critical Class II: Non-critical	0.9813	rejected	0.9977
System Class I: SPE,ELM,GEA Class II: SMO,KOM,INS	0.9195	rejected	0.9427
Sub-system Class I: PUM,TAN,FOR Class II: ROR,TRY,,ANN	0.0783	rejected	0.7152

Table 2. Significance levels of rejection the null hypothesis for model assumptions 3 (the colored process is stationary) and model assumption 4 (the process is color blind competing risk)

Having the assumptions of homogeneity and independence validated, the next step can be approach: the statistical analysis of competing risk concept (assumptions 3

and 4). Following the choice of competing risk classes proposed in (Bunea et al 2002 [15]) for different fields of interest, one can obtain the significance levels for rejection the null hypothesis of assumptions 3 and 4 (Table 2).

Since the “color blind” assumption has been rejected, it is not necessary to check whether the uncolored process is stationary competing risk (Figure 12). However for the state of the art, one can ask whether the uncolored process is stationary. Due to the fact that the homogeneity was accepted under the assumption of Poisson processes, Laplace’s test is used. The null hypothesis is not rejected (significance level $\alpha = 0.3957$).

The most important assumption of this analysis, the process is renewal competing risk, is not rejected at the level of significance $\alpha = 0.5067$.

At this point we have to find the competing risk model, which is appropriate to interpret data. In literature one can find two methods: via graphical interpretation (Bunea et al [15], Cooke and Bedford 2002 [21]) and via statistical testing (Bunea et al [14], Dewan et al [31]). We propose here to test if the renewal competing risk process is Poisson. The assumption is rejected at the significance level $\alpha = 0.00128$.

4.5.5 Conclusions

Figure 12 shows the main assumptions that have been made through this present work. The validation/invalidation of the assumptions is marked on the graph. The analysis is not complete due to the fact that no tests are available in the literature to test the independence of the colors. One attempt of testing independent exponential distributions, can be found in (Bunea et al 2002 [14]). At this point of color “independence” a probabilistic interpretation of data has been developed in a new type of analysis (Cooke 1996 [20]). One can see that if the independence assumption holds then a constant hazard rate is characteristic for our process, hence we have to deal with a Poisson process. Nevertheless, the probabilistic analysis performed on this data (Bunea et al 2002 [15]), indicates that independent model might hold but exponential distribution not.

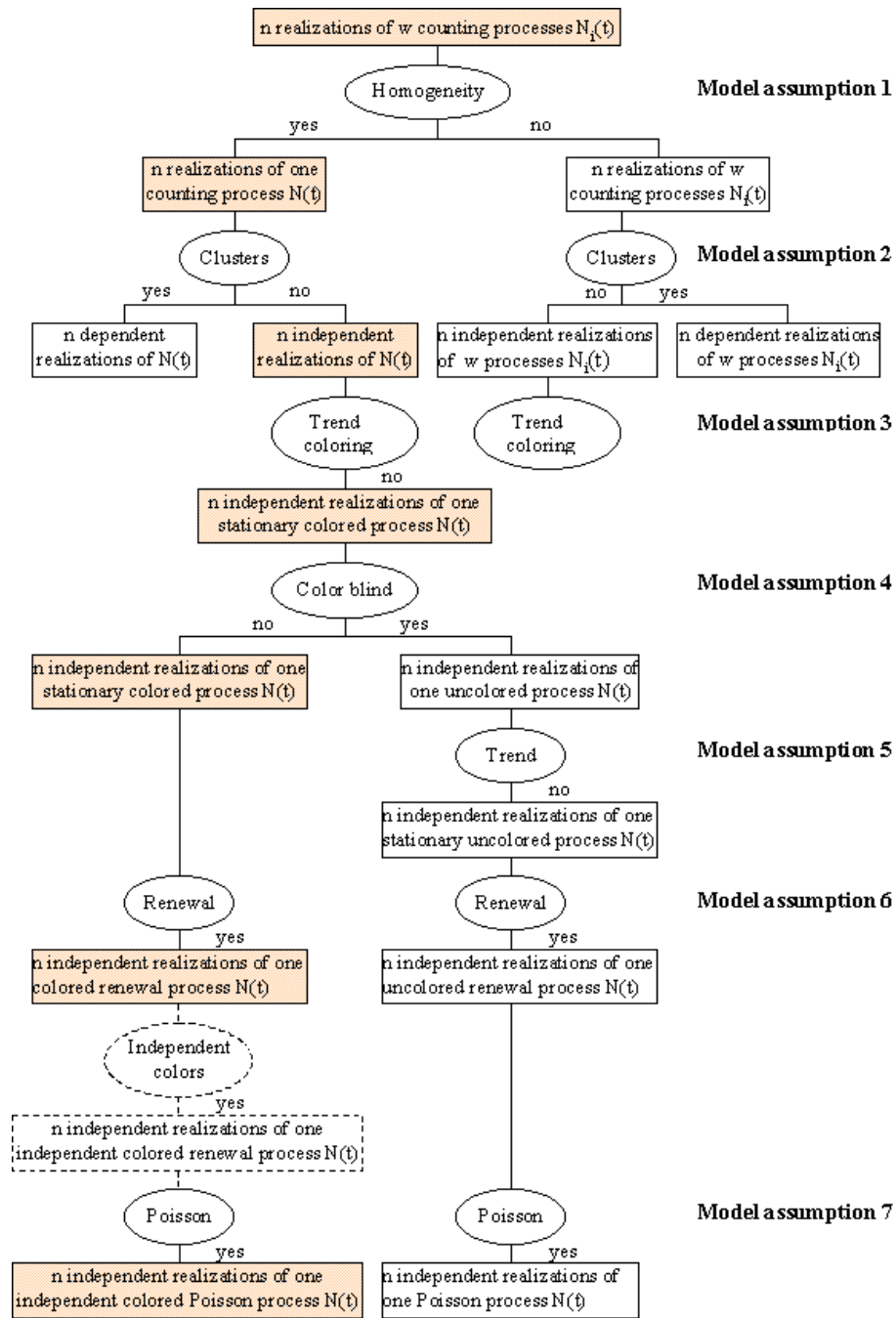


Figure 12. Model assumptions scheme

Chapter 5

The effect of model uncertainty on maintenance optimization

Much operational reliability data available, for example in the nuclear industry, is heavily right censored by preventive maintenance. The standard methods for dealing with right censored data (Total Time on Test statistic, Kaplan–Meier estimator, adjusted rank methods) assume the independent competing risk model for the underlying failure process and the censoring process, even though there are many dependent competing risk models that can also interpret the data. It is not possible to identify the “correct” competing risk model from censored data. A natural question is whether this model uncertainty is of practical importance. In this paper we consider the impact of this model uncertainty on maintenance optimization and show that it can be substantial. We present three competing risk model classes which can be used to model the data, and determine an optimal maintenance policy. Given these models, we consider the error that is made when optimizing costs using the wrong model. It is shown that model uncertainty can be expressed in terms of the dependence between competing risks, which can be quantified by expert judgement. This enables us to reformulate the maintenance optimization problem to take into account model uncertainty.

5.1 Introduction

Acronyms

PM preventive maintenance

RC replacement cost

RT replacement time

The standard methods, assuming independent censoring, used to treat right censored data are non-conservative, in the sense that other dependent censoring models estimate the underlying failure process more pessimistically (see [8]). Without making non-testable assumptions (such as independence of the failure and censoring processes), the true distribution function is not identifiable from the data. Hence, in addition to the usual uncertainty caused by sampling fluctuation we have the extra problem of model uncertainty.

In this chapter we test the effect of model uncertainty on the problem of optimizing maintenance. We assume that data is available which contains censors from an existing PM program, and use this data to estimate an optimal age replacement PM program.

In Section 3, we take three model classes of competing risk. The independent model is used as the most extreme pessimistic model of existing PM. The other extreme model is used for the most optimistic model of existing PM. The dependent competing risk model is used for the general case and the dependence between competing risks is given by a copula. The minimally informative copula with respect to the uniform distribution and Archimedean copula are studied - the later will be use to approximate the first one, due to numerical difficulties in working with the minimally informative copula for strong dependence between risks. We present a method by which expert judgement may be used to quantify model uncertainty. In Section 4 we recall the theory of optimal age replacement policies and in Section 5 we will present three numerical examples to determine the error that is made when optimizing costs using the wrong model. The last section shows that model uncertainty does lead to substantial uncertainty in the estimation of optimal maintenance intervals and to excessive costs. This chapter extends and develops results given in (Bedford and Mesina 2000 [10]), in particular by showing how expert judgement may be used to quantify model uncertainty.

5.2 Three Models for Competing Risk

In this section we present three competing risk models in which the marginal distribution functions are identifiable. Two of them are the “extreme” cases - independent model and high correlated censoring model and the third one assumes that the dependence between competing risks is given by a copula.

5.2.1 Model 1: Independent competing risks

If F_X has a density function $f_X(t)$, then the failure rate $r_X(t)$ of X is

$$r_X(t) = f_X(t)/S_X(t) = -(dS_X(t)/dt)/S_X(t).$$

Since

$$d[\log(S_X)] = dS_X/S_X,$$

we have

$$S_X(t) = \exp\left\{-\int_0^t r_X(s)ds\right\}.$$

But from competing risk data we observe a different rate of failure for X . The observed failure rate for X is defined as

$$obr_X(t) = \lim_{\delta \rightarrow 0} Pr\{X > t, X < Y, X \in (t, t + \delta) | Z > t\} / \delta = -\frac{dS_X^*(t)/dt}{S_X^*(t) + S_Y^*(t)}.$$

For the most frequently made assumption in the literature, that of probabilistic independence between X and Y , we have

$$S_X^*(t) + S_Y^*(t) = Pr\{X > t, Y > t\} = Pr\{X > t\}Pr\{Y > t\} = S_X(t)S_Y(t).$$

Using the above results Cooke [20] showed that if the competing risks X and Y are independent with differentiable survival functions, then the failure rate is equal with the observed failure rate

$$r_X(s) = obr_X(s).$$

Now, the underlying marginal distributions of X and Y can be identified in terms of the observable sub-survivor functions,

$$S_X(t) = \exp\left(\int_0^t \frac{dS_X^*(s)}{S_X^*(s) + S_Y^*(s)}\right). \quad (5.2.1)$$

5.2.2 Model 2: Highly Correlated Censoring

Clearly, independent censoring does not capture the notion that preventive maintenance is carried out when the equipment has given some sign of future failure.

The most extreme case is described as follows: Preventive maintenance aims to prevent the failure of the component at a time immediately before failure. If that aim is not achieved then the PM action is applied immediately after failure. The PM is unsuccessful with probability p and successful with probability $1 - p$, independently of the time at which the failure occurs. We model this by taking $Y = X + \delta\varepsilon$, where $\varepsilon > 0$ is very small but depends on X , and $\delta = \{1, -1\}$ with probability p respectively $1 - p$ is independent of X . For very small ε Model 2 gives the following relationships:

$$S_X^*(t) = Pr\{X > t, X < Y\} = Pr\{X > t, \delta = 1\} = Pr\{\delta = 1\}Pr\{X > t\} = pS_X(t),$$

and

$$S_Y^*(t) = Pr\{Y > t, Y < X\} = Pr\{\delta = -1\}Pr\{Y > t\} \approx (1-p)Pr\{X > t\} = (1-p)S_X(t).$$

Hence the *normalized* subsurvivor functions (normalized so that they take that value 1 at $t = 0$) are approximately equal,

$$\frac{S_X^*(t)}{p} \approx \frac{S_Y^*(t)}{1-p}, \quad (5.2.2)$$

and both are equal to $S_X(t)$. Now, this condition can be checked from the data. If it does not hold then Model 2 is not correct. An example is shown in Figure 1 where we have taken

$$S_X(t) = \exp(-t^{0.5}).$$

We took a sample 1000 times for the model above with $p = 1/3$ and then we plot the empirical functions $\frac{\hat{S}_X^*(t)}{p}$, $\frac{\hat{S}_Y^*(t)}{1-p}$ and the theoretical $S_X(t)$.

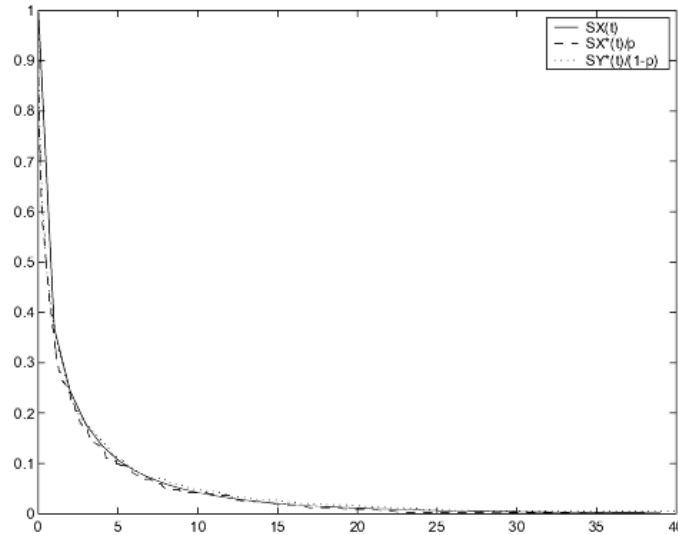


Figure 1. Highly correlated censoring

If Equation 5.2.2 does hold then the model might be correct, but the independent model might also hold with the same observable data. Assuming Model 1 (independence) when Model 2 holds would lead to an incorrect assessment of the marginals. The following proposition is obtained by using Equation 5.2.1 [10]:

Proposition 5.2.1. *Suppose X and Y have a joint distribution described by Model 2. Let \tilde{X} and \tilde{Y} be independent with*

$$S_X^*(t) = S_{\tilde{X}}^*(t) \quad \text{and} \quad S_Y^*(t) = S_{\tilde{Y}}^*(t).$$

Then

$$S_{\tilde{X}}(t) = [S_X(t)]^p \quad \text{and} \quad S_{\tilde{Y}}(t) = [S_X(t)]^{1-p}.$$

Model 2 is a special case of the random signs model of Cooke [18]. This model can be used when the subsurvivor functions satisfy the inequality

$$\frac{S_X^*(t)}{p} \geq \frac{S_Y^*(t)}{1-p}. \quad (5.2.3)$$

5.2.3 Model 3: Dependent Competing risks

In this model we assume that the dependence structure between X and Y is given by a copula. As defined by Schweizer and Wolff [69] the copula of two random variables X and Y is the distribution C on the unit square $[0, 1]^2$ of the pair $(F_X(X), F_Y(Y))$ (recall that for a continuous random variable X with pdf F_X , the random variable $F_X(X)$ is always uniformly distributed on $[0, 1]$). The functional form of $C : [0, 1]^2 \rightarrow \mathbf{R}$ is

$$C(u, v) \equiv H(F_X^{-1}(u), F_Y^{-1}(v)),$$

where H is the joint distribution function of (X, Y) and F_X^{-1} and F_Y^{-1} are the right-continuous inverses of F_X and F_Y . Under independence of X and Y the copula is $C(u, v) = uv \equiv \Pi$, and any copula must fall between $M(u, v) \equiv \min(u, v)$ and $W(u, v) \equiv \max(u + v - 1, 0)$, the copulas of the upper and lower Fréchet bounds [59]. As we saw in the first model, under the assumption of independence of X and Y , the marginal distribution functions of X and Y are uniquely determined by the sub-survival functions of X and Y . Zheng and Klein [75] showed the more general result that, if the copula of (X, Y) is known, then the marginal distributions functions of X and Y are uniquely determined by the competing risk data. This result is captured in the following theorem:

Theorem 5.2.2. *Suppose the marginal distribution functions of (X, Y) are continuous and strictly increasing in $(0, \infty)$. Suppose the copula C is known and the corresponding probability measure for any open set of the unit square is positive. Then F_X and F_Y , the marginal distribution functions of X and Y , are uniquely determined by the subdistribution functions.*

In the Appendix 4.6.1 we show briefly why the marginals are identifiable when the densities and subdensities exist.

We now discuss the problem of choosing a copula. There are many measures of association for the pair (X, Y) , which are symmetric in X and Y . The best known measures of association are Kendall's tau and Spearman's rho (we will use the more modern term "measure of association" instead of the term "correlation coefficient" for a measure of dependence between random variables).

Kendall's tau for a vector (X, Y) of continuous random variables with joint distribution function H is defined as follows: Let (X_1, Y_1) and (X_2, Y_2) be i.i.d. random vectors, each with joint distribution H , then Kendall's tau is defined as the probability of concordance minus the probability of discordance:

$$\tau(X, Y) = Pr\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - Pr\{(X_1 - X_2)(Y_1 - Y_2) < 0\}$$

or

$$\tau(X, Y) = Pr\{sgn(X_1 - X_2) = sgn(Y_1 - Y_2)\} - Pr\{sgn(X_1 - X_2) \neq sgn(Y_1 - Y_2)\}.$$

The other measure of association (Spearman's rho) is defined as follows: Let X and Y be continuous random variables then the Spearman's rho is defined as the product moment correlation of $F_X(X)$ and $F_Y(Y)$:

$$\rho_r(x, y) = \rho(F_X(X), F_Y(Y)) = \frac{Cov\{F_X(X), F_Y(Y)\}}{\sqrt{Var\{F_X(X)\}Var\{F_Y(Y)\}}}.$$

Simple formulae relating the measures of association to copula density are given in the Appendix 4.6.2.

Since the measure of association is to be treated as a primary parameter, it is necessary to choose a family of copulae which are as "smooth" as possible and which model all possible measures of association in a simple way. Meeuwissen and Bedford (1997 [57]) proposed using the unique copula with the given Spearman's rho that has minimum information with respect to the independent distribution, and also he gave a method to calculate numerical this copula. Now, due to the difficulty of the interpretation of Spearman's rho by a non-specialist and due to the difficulty of quantifying it, we will use as a primary parameter Kendall's tau. Kendall's tau has the advantage of a definition which can be explained to a non-specialist, but the value can not be estimated using only the competing risk data, because of "identifiability problem". Thus we need to use some prior knowledge or subjective information to obtain information about the value of tau. To model the uncertainty over tau we will use expert judgement. This will be discuss later in this paper, but for now it remains to clarify the way that we obtain the copula.

Work of Zheng and Klein [75] suggests that the important factor for an estimate of the marginal survival function is a reasonable guess at the strength of the association between competing risks and not the functional form of the copula. For this reason we will choose a class of copula with which it is easy to work from the mathematical

point of view. A such class is Archimedean copula. First, we recall some definitions about the Archimedean copula and some properties of Kendall's tau for a certain Archimedean family of copula.

Let X and Y be continuous random variables with joint distribution H and marginal distribution F_X and F_Y . When X and Y are independent, we have $H(x, y) = F_X(x)F_Y(y)$, and this is the only case when the joint distribution is write into a product of F_X and F_Y . But, there are some families of distributions in which we have $\lambda(H(x, y)) = \lambda(F_X(x))\lambda(F_Y(y))$, see [59]. Using the function $\varphi(t) = -\log \lambda(t)$ (λ must be positive on the interval $(0,1)$), we can also write H as a sum of the marginals F_X and F_Y , $\varphi(H(x, y)) = \varphi(F_X(x)) + \varphi(F_Y(y))$, or in terms of copula $\varphi(C(u, v)) = \varphi(u) + \varphi(v)$. Copulas of this form are called *Archimedean copulas*. The function φ is called an *additive generator* of the copula. If $\varphi(0) = \infty$, φ is a strict generator and $C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v))$ is a strict Archimedean copula. For our goal we choose an one-parameter family of copulae which has a strict generator. The Gumbel family is defined as follows:

$$C_\alpha(u, v) \equiv \exp(-[(-\log u)^\alpha + (-\log v)^\alpha]^{1/\alpha})$$

for

$$\alpha \in [1, \infty).$$

The generator is the function $\varphi_\alpha(t) = (-\log t)^\alpha$.

As shown in the Appendix 4.6.2, we can directly write α as a function of Kendall's tau,

$$\alpha_\tau = 1/(1 - \tau).$$

It remains now to quantify the uncertainty in Kendall's tau using expert opinion. Experts can not be directly asked to quantify their uncertainty over tau, instead they are asked to give uncertainties over physically realizable quantities [7]. Consider two sockets with failure times X_1 and X_2 and the PM times Y_1 and Y_2 . The expert can be asked for the probability that an attempt to preventively maintain socket one would occur before the PM for socket two, given that the failure of socket one occurs before the failure of socket two. Let this probability be q . By symmetry we have the same probability for the occurrence of the PM for socket two before the PM for socket one if the failure time of socket one is greater than the failure time of socket two. Also

we get that the probability of occurrence of the PM for socket two before the PM for socket one if the failure time of socket one is smaller than the failure time of socket two is equal to $1 - q$.

If the experts can give a distribution over $q \equiv Pr\{Y_1 > Y_2 | X_1 > X_2\}$, then we can this translate to a distribution over Kendall's tau. Indeed, after a little simple algebra we have:

$$\begin{aligned} & Pr\{(X_1 - X_2)(Y_1 - Y_2) > 0\} = \\ &= Pr\{(X_1 > X_2) \cap (Y_1 > Y_2)\} + Pr\{(X_1 < X_2) \cap (Y_1 < Y_2)\} = \\ &= Pr\{X_1 > X_2\}Pr\{Y_1 > Y_2 | X_1 > X_2\} + Pr\{X_1 < X_2\}Pr\{Y_1 < Y_2 | X_1 < X_2\} = q. \end{aligned}$$

Similarly we find:

$$Pr\{(X_1 - X_2)(Y_1 - Y_2) < 0\} = 1 - q$$

and so

$$\tau = 2q - 1.$$

Note that q can be considered an observable quantity because q is the approximate average rate for which $\{Y_1^{(n)} > Y_2^{(n)} | X_1^{(n)} > X_2^{(n)}\}$ holds when a large sample of pairs $(X_1^{(n)}, Y_1^{(n)})$, $(X_2^{(n)}, Y_2^{(n)})$ is observed.

Now for each replacement time θ and measure of association τ we can calculate the long term specific cost and furthermore we can optimize this replacement cost finding the minimal one. This is discussed in the next section.

5.3 Maintenance Optimization

We consider the effect of uncertainty about the underlying lifetime distribution on the selection of the maintenance policy. To keep things simple we just consider the *age replacement policies*. Recall that an age replacement policy is one for which replacement occurs at failure or at age θ , whichever occurs first. Unless otherwise specified, θ is taken to be a constant.

In the finite time span replacement model we will try to minimize expected cost $C(t)$ experienced during $[0, t]$, where cost may be computed in money units, time, or some appropriate combination. For an infinite time span, an appropriate objective function is expected cost per unit of time, expressed as

$$\gamma(\theta) \equiv \lim_{t \rightarrow \infty} \frac{C(t)}{t}.$$

Letting $N_1(t)$ denote the number of failures during $[0, t]$ and $N_2(t)$ denote the number of planned preventive maintenance during $[0, t]$, we may express the expected cost during $[0, t]$ as

$$C(t) \equiv c_1 E\{N_1(t)\} + c_2 E\{N_2(t)\},$$

where c_1 is the cost of critical failure and c_2 is the cost for planned replacement. We only consider non-random age replacement in seeking the policy minimizing the specific cost $\gamma(\theta)$ for an infinite time span.

Starting from the definition of the specific cost

$$\gamma(\theta) \equiv \lim_{t \rightarrow \infty} \left[c_1 \frac{E\{N_1(t)\}}{t} + c_2 \frac{E\{N_2(t)\}}{t} \right]$$

Barlow and Proschan [2] showed that

$$\gamma(\theta) \equiv \frac{c_1 F(\theta) + c_2 S(\theta)}{\int_0^\theta S(t) dt},$$

where F and S are the lifetime distribution function respectively the lifetime survival function.

Then $\gamma(0) = \infty$ and $\gamma(\infty) = c_1 / \int_0^\theta S(t) dt$. Differentiating γ to find the optimum, $\frac{d\gamma(\theta)}{d\theta} = 0$, we obtain the equation

$$r(\theta) \int_0^\theta S(t) dt - F(\theta) = \frac{c_2}{c_1 - c_2}.$$

When $F_X(x)$ has an increasing failure rate, the optimal replacement time θ_0 is the unique solution of the above equation. For a r.v. with constant failure rate or decreasing failure rate the specific cost has not an optimum ($\text{sign}(\frac{d\gamma(\theta)}{d\theta})$ is constant), thus this type of maintenance policy is not appropriate for a such r.v.

When we have as primary parameter Kendall's tau and the information over τ is given by a distribution function $F_\tau(\tau)$ with density $f_\tau(\tau)$, the specific cost is dependent on τ and θ :

$$\gamma(\tau, \theta) \equiv \frac{c_1 F(\tau, \theta) + c_2 S(\tau, \theta)}{\int_0^\theta S(\tau, t) dt}.$$

So the long term specific cost given θ is

$$\gamma(\theta) = \int_0^1 \gamma(\tau, \theta) f_\tau(\tau) d\tau$$

and the optimal replacement time θ_0 is obtaining minimizing $\gamma(\theta)$.

5.4 Numerical Examples

We now give the results of three sets of numerical experiments to show the effect of using Model 1 when Model 2 actually holds, to show the dependence of replacement cost with the measure of association (Kendall's tau) and finally to find the optimal replacement time of the average specific cost.

For the first part of numerical computations, we consider two underlying distributions for X . The first, Distribution 1, is that X has failure rate $r_X(t) = t^{3/2}$. The second, Distribution 2, assumes a failure rate of $r_X(t) = t^2$, while Distribution 3 assumes a failure rate $r_X(t) = t^3$. Both failure rates are continuous and increasing, and correspond to Weibull distributions.

Since the costs of critical failure can be much higher than those of planned maintenance (because of other consequences to the system beyond the need simply to replace the failed unit), we assume that c_1 is much larger than c_2 . Since actual plant data shows a considerable number of preventive maintenance actions we assume that p is small. Specifically we take $c_1/c_2 = 10$ and $c_1/c_2 = 20$, and also $p = 0.3$ and $p = 0.1$, thus giving us 4 different cases on which the two models are compared. Both replacement times (RT) and replacement costs (RC) are given in Table 5.1. The replacement times are the optimal replacement times calculated under the assumption that the model under consideration is correct. For Model 2 the replacement costs are equal to the optimal replacement costs. For Model 1 they are equal to the replacement costs of Model 2 (which is actually the correct model), evaluated with the optimal replacement time calculated for Model 1. Hence the costs given for Model 1 are always higher than those of the true Model 2. Table 5.2 gives the ratio of the two model outcomes (Model 1 divided by Model 2) for the time and costs of each of the distributions.

For the second part, we consider three sub-survival functions for X which for the extreme cases (independence and high correlation) take the same failure rates for X as in the first part, and for every sub-survival function of X , we take other three sub-survival functions Y in such a way that Inequality 5.2.3 is satisfied (for Weibull distributions with the same shape parameter of S_X^* and S_Y^* , the scale parameter of S_Y^* , a_Y , must be greater than the scale parameter of S_X^* , a_X). Specifically we take $a_X/a_Y = 1/2$, $a_X/a_Y = 1/4$, $a_X/a_Y = 1/8$ and for p and c_1/c_2 we take the same values as in the first numerical example.

Figure 2, 3 and 4 show the way in which the RC (normalized by RC for the

c_1/c_2	10				20			
p	0.3		0.1		0.3		0.1	
	RT	RC	RT	RC	RT	RC	RT	RC
Dist 1, Mod 1	0.8280	10.922	1.2849	9.8756	0.6127	18.885	0.9508	21.699
Dist 1, Mod 2	0.3860	7.5921	0.3860	7.5921	0.2574	9.9433	0.2574	9.9433
Dist 2, Mod 1	0.8687	4.9674	1.5047	6.5283	0.5950	7.2682	1.0305	10.102
Dist 2, Mod 2	0.4758	4.2823	0.4758	4.2823	0.3259	6.1916	0.3259	6.1916
Dist 3, Mod 1	0.3393	3.3011	0.3353	3.3280	0.2322	4.6522	0.2303	4.6821
Dist 3, Mod 2	0.3556	3.2006	0.3556	3.2006	0.2393	4.5459	0.2393	4.5459

Table 5.1: Optimal maintenance times and costs

c_1/c_2	10				20			
p	0.3		0.1		0.3		0.1	
	RT	RC	RT	RC	RT	RC	RT	RC
Dist 1	2.15	1.44	3.33	1.3	2.38	1.9	3.69	2.18
Dist 2	1.83	1.16	3.16	1.52	1.83	1.17	3.16	1.63
Dist 3	0.95	1.03	0.94	1.04	0.97	1.02	0.96	1.03

Table 5.2: Ratio's of maintenance times and costs

independent case) depends on Kendall's tau.

To obtain a distribution for Kendall's tau we ask an expert to give quantiles for the probability q defined in the previous section. If the expert gives 5% and 95% quantiles then we can fit a beta distribution. Specifically if $Pr\{q \leq 0.7\} = 0.05$ and $Pr\{q \leq 0.95\} = 0.95$, then the 5% and 95% quantiles for τ are $Pr\{\tau \leq 0.4\} = 0.05$ and $P\{\tau \leq 0.9\} = 0.95$. Taking the beta distribution as appropriate for τ , we easily obtain the parameters of this distribution a and b given the above quantiles by Newton's method as: $a = 5.6705$ and $b = 2.7322$.

The specific costs for different values of Kendall's tau are shown in Figure 5 and the average specific cost with optimal replacement time are shown in Figure 6.

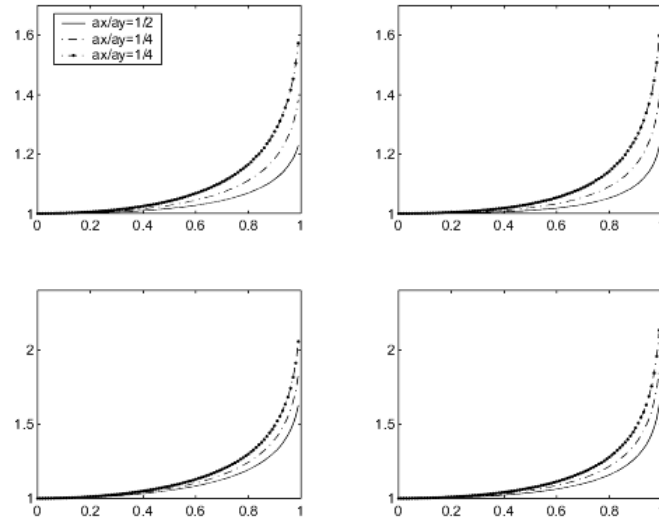


Figure 2. Dependence between RC and the measure of association for the pairs of the sub-survivals for X and Y given by the first sub-survival for X and the other three for Y ; a) $p = 0.3$ and $c_2/c_1 = 0.1$; b) $p = 0.3$ and $c_2/c_1 = 0.05$; c) $p = 0.1$ and $c_2/c_1 = 0.1$; d) $p = 0.1$ and $c_2/c_1 = 0.05$;

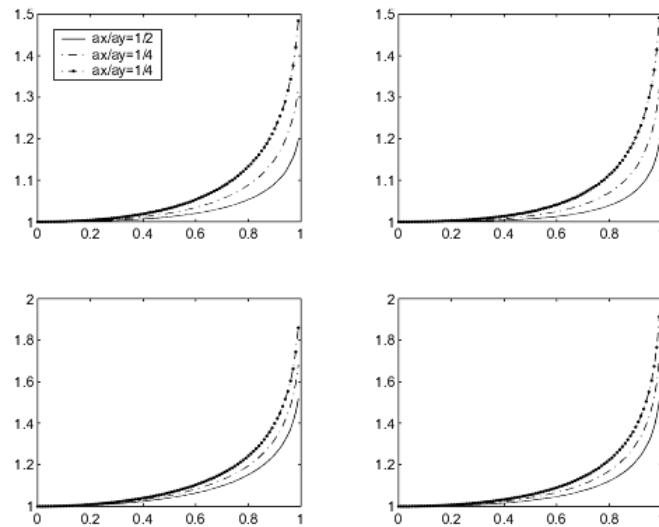


Figure 3. Dependence between RC and the measure of association for the pairs of the sub-survivals for X and Y given by the second sub-survival for X and the other three for Y ; a) $p = 0.3$ and $c_2/c_1 = 0.1$; b) $p = 0.3$ and $c_2/c_1 = 0.05$; c) $p = 0.1$ and $c_2/c_1 = 0.1$; d) $p = 0.1$ and $c_2/c_1 = 0.05$;

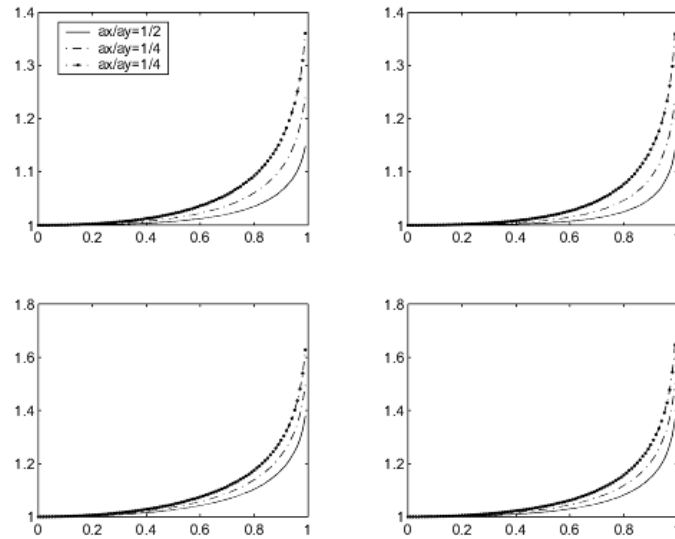


Figure 4. Dependence between RC and the measure of association for the pairs of the sub-survivals for X and Y given by the third sub-survival for X and the other three for Y ; a) $p = 0.3$ and $c_2/c_1 = 0.1$; b) $p = 0.3$ and $c_2/c_1 = 0.05$; c) $p = 0.1$ and $c_2/c_1 = 0.1$; d) $p = 0.1$ and $c_2/c_1 = 0.05$;

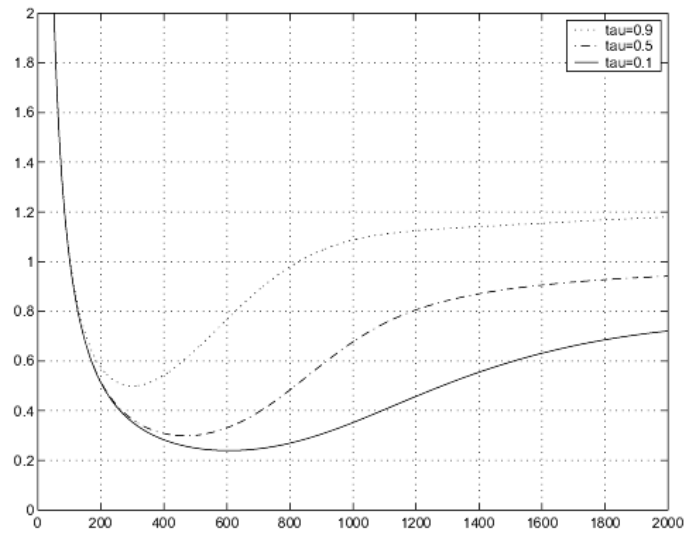


Figure 5. Specific cost for three values of Kendall's tau: 0.1, 0.5 and 0.9

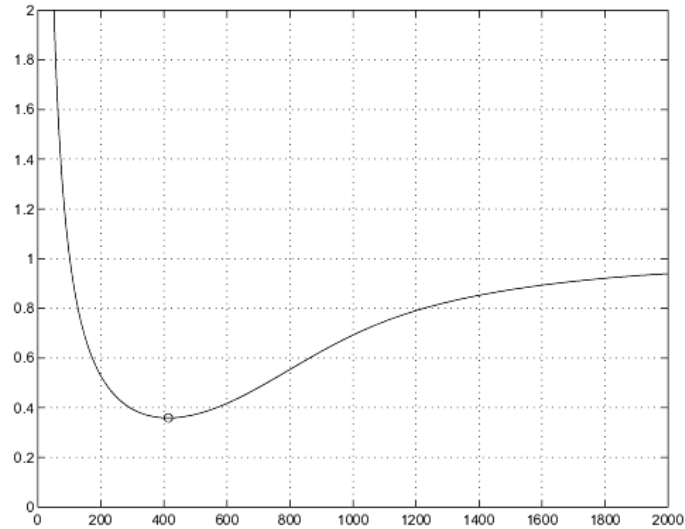


Figure 6. Average specific cost and optimal replacement time

5.5 Robustness of optimal replacement time with respect to choice of the copula

Work of Zheng and Klein [75] suggests that the important factor for an estimate of the marginal survival function is a reasonable guess at the strength of the association between competing risks and not the functional form of the copula. For this reason we chose [13] Archimedean copula with which it is easy to work from the mathematical point of view and we calculated for one family of Archimedean copula the average specific cost with optimal replacement time. Now we introduce other two classes of copula and another family of Archimedean copula.

5.5.1 Three families of copulae

Minimally informative copula

This copula was originally introduced to knowledge dependence in uncertainty analysis [19]. Minimally informative copula has minimal information (taken with

respect to the uniform distribution) with the given correlation among all candidate distributions. The relative information function of a continuous distribution function with density $f_X(x)$ with respect to the continuous distribution function with density $f_Y(y)$ is:

$$I(f_X|f_Y) = \int \log\left(\frac{f_X(t)}{f_Y(t)}\right) f_X(t) dt.$$

Then the information function of minimally informative copula taken with respect to the uniform distribution is:

$$I(c|u) = \int \int_{I^2} c(x, y) \log\{c(x, y)\} dx dy,$$

where $c(x, y)$ is the copula density. Meeuwissen and Bedford [57] showed how the minimally informative copula with given correlation coefficient could be determined. This copula has density of the form:

$$c(x, y) = k(x, \theta)k(y, \theta) \exp^{\theta(x-0.5)(y-0.5)},$$

where $\theta = \theta(\rho)$ is a certain monotone increasing function of the correlation coefficient ρ , and the function $k(\cdot, \theta)$ is determined as the solution to an integral equation:

$$k(x, \theta) = \left[\int_0^1 k(y, \theta) \exp^{\theta(x-0.5)(y-0.5)} dy \right]^{-1}.$$

Meeuwissen and Bedford proposed in 1997 [57] a discrete approximation to the density of minimally informative copula. This method is based on a so-called DAD algorithm. This algorithm works because we know the general form taken by the copula, but relies on the fact that the correlation is determined by the mean of the symmetric function $F_X(t)F_Y(t)$. In order to have asymmetric specifications, Bedford (2002 [6]) developed a D_1AD_2 algorithm, to show how non-symmetric functions can be used to determine the unique joint distribution (copula) with given uniform marginals and rank correlation.

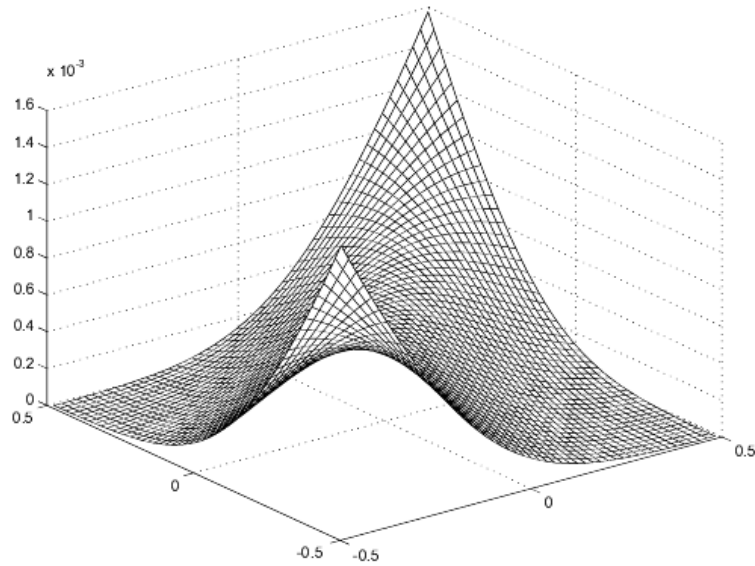


Figure 7. The density of minimally informative copula with correlation 0.7

Diagonal band distribution

Definition 5.5.1. For positive measures of association the density of the diagonal band distribution, $b_\alpha(x, y)$ has a mass in a band around the diagonal $y = x$. The bandwidth β equals $1 - \alpha$; i.e.

$$P\{X - \beta < Y < X + \beta\} = 1.$$

The density $b_\alpha(x, y)$ takes values 0, $1/2\beta$ and $1/\beta$ on the five regions it divides the unit square into,

$$b_\alpha(x, y) = \frac{1/2}{1 - \alpha} (1_{\{\alpha - 1 \leq x - y \leq 1 - \alpha\}} + 1_{\{1 - x - y \geq \alpha\}} + 1_{\{1 - x - y \leq -\alpha\}}),$$

with $0 \leq \alpha \leq 1$ and $0 \leq x, y \leq 1$.

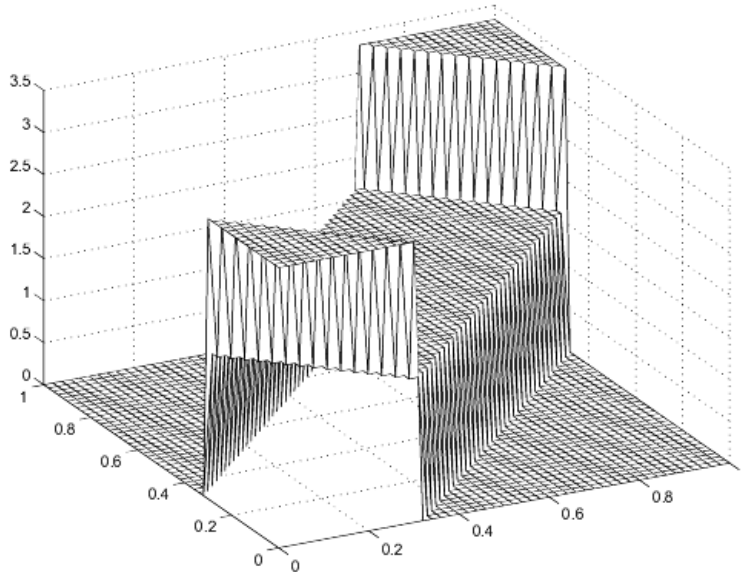


Figure 8. The density of diagonal band distribution with correlation 0.7

Zheng and Klein assumed a copula with positive probability measure on any open set in order to get identifiability, so just using diagonal band distribution is not possible. Hence, we will consider mixtures of diagonal band distribution with relative information with respect to the uniform density given a correlation.

Definition 5.5.2. Let $0 \leq p \leq 1$. A probability distribution $M(\alpha)$, $M : [-1, 1] \rightarrow [0, 1]$ is called a mixing function if its probability density function consists of an absolutely continuous part $m(\alpha) \geq 0$ with $\int_{-1}^1 m(\alpha) d\alpha = p$ and a discrete part with atoms $p_j > 0$ at α_j , $-1 \leq \alpha_{j-1} \leq \alpha_j \leq 1$, $\cup_j \{\alpha_j\} = \mathcal{A}$ and $\sum_j p_j = 1 - p$. \mathcal{A} may be the empty set.

Definition 5.5.3. Let $M(\alpha)$ be a mixing function. A mixture $f_M(x, y)$ of diagonal band densities $b_\alpha(x, y)$ is defined as

$$f_M(x, y) = \int_{-1}^1 b_\alpha(x, y) dM(\alpha).$$

Meeuwissen [56] proved that mixtures f_M with correlation ρ have less relative information with respect to the uniform distribution $I(f_M|u)$ than the diagonal band

density with the same correlation. This is another reason to consider mixture of diagonal band distributions to model an incompletely specified joint probability distribution. Using the results of Meeuwissen we can approximate the mixing densities that give mixture with minimal relative information with respect to the uniform density very well with a mixture of the uniform density and a beta density. Thus we can determine directly that mixing measure in the class of beta densities that gives a mixture of diagonal band densities with minimal relative information with respect to the uniform density given correlation.

Archimedean copula

We considered in Section 4.2.3 only the Gumbel family of copulae:

$$C_\alpha(u, v) = \exp(-[(\ln \frac{1}{u})^\alpha + (\ln \frac{1}{v})^\alpha]^\frac{1}{\alpha})$$

$$\varphi_\alpha(t) = (-\ln t)^\alpha \quad \text{with } \alpha \in [1, \infty) \quad \text{and } C_1 = \Pi, C_\infty = M.$$

For this study we will take into account another family of Archimedean copula.

$$C_\alpha(u, v) = \exp(1 - [(1 - \ln u)^\alpha + (1 - \ln v)^\alpha - 1]^\frac{1}{\alpha})$$

$$\varphi_\alpha(t) = (1 - \ln t)^\alpha - 1 \quad \text{with } \alpha \in [0, \infty) \quad \text{and } C_1 = \Pi, C_\infty = M.$$

Using Theorem 4.6.3, we obtain the following relation between Kendall's tau and the parameter α :

$$\tau(\alpha) = \exp(2) * (\text{gamma}(2 - \alpha, 2) * 2^\alpha + \alpha - 3) / \alpha.$$

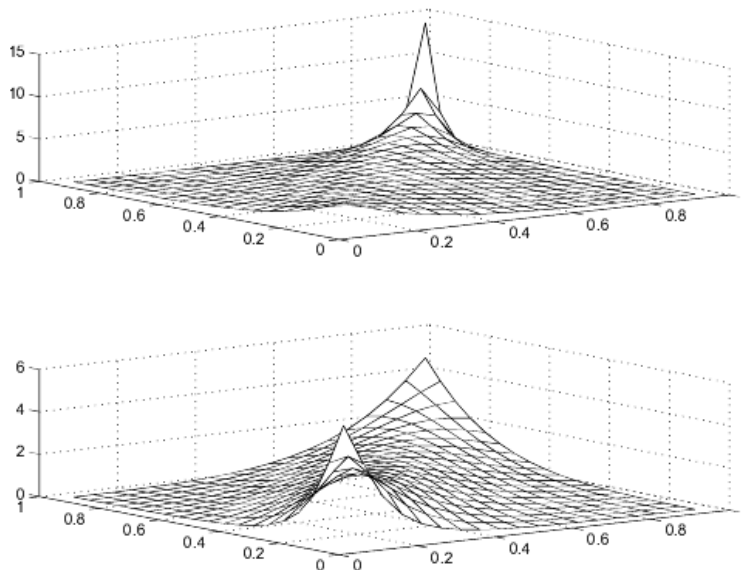


Figure 9. The densities of two Archimedean families of copula with correlation 0.8

Using the same simulation study presented in section 4.4, one can obtain the average specific cost and optimal replacement time for different families of copula. (Figure 10)

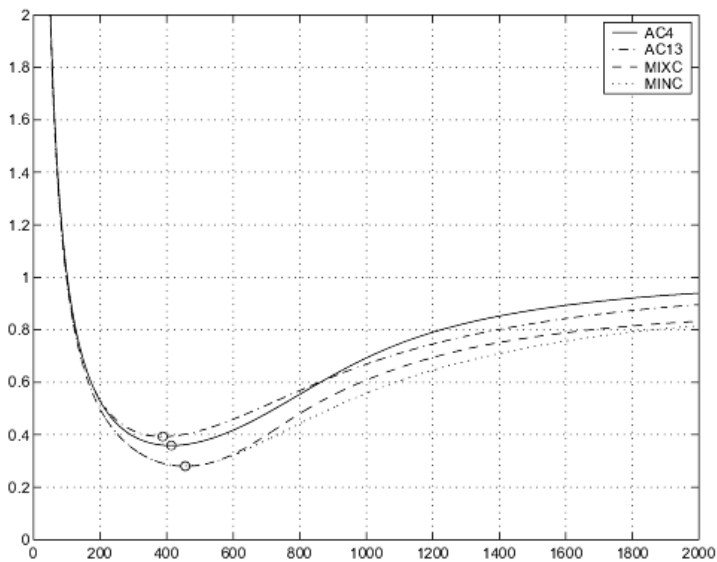


Figure 10. The average specific cost with optimal replacement time

5.6 Discussion

The results presented in Table 1 and 2 show that the “optimal replacement interval” and “optimal replacement costs” can be dramatically non-optimal when the wrong model is used to estimate the underlying failure distribution from censor data. The difference is least when the failure rate increases quickly. when the failure rate increase more slowly, the difference is larger. For one case calculated here the specific costs obtained by using the independent model would be more than twice the best possible specific costs using the correct model. In the second part we consider the effect of model uncertainty due to impossibility of identifying the “correct” competing risk model from censored data. Using the expert judgement to quantify the dependence between competing risks , we have shown that the replacement cost is highly sensitive to the measure of association Kendall’s tau. Figure 1, 2 and 3 show that sensitivity is higher for the first model and for a certain case RC can be twice than RC for independent case. Figure 4 shows also that the difference between optimal replacement costs and optimal replacement time can be more than twice and Figure 5 presents the long term specific cost and the optimal replacement time.

The work carried out here demonstrates the importance of using good expert judgement from experts with insight into the maintenance process. If the experts are able to select the correct correlation level then this will aid model selection considerably.

The results show little difference among the four different families of copula, regarding the optimal replacement time. Figure 10 suggests that the important factor for determine the optimal replacement time is the measure of association between risks X and Y and not the functional form of the copula. This results confirm Zheng and Klein study over the robustness of copula-graphic estimator.

The results also show that the optimal replacement times for all families of copula studied can be found in a small interval. This is most important since it suggests that we can robustly determine the optimal replacement time even without being certain about the optimal costs.

5.7 Appendix

5.7.1 Part 1

We show briefly why the marginals are identifiable in the case that densities and subdensities exist. By definition we have that the subdistribution function of X is

$$F_X^*(t) \equiv P\{X \leq t, X < Y\}.$$

After a straightforward calculation we get:

$$\begin{aligned} F_X^*(t) &= \int_0^t \int_x^\infty h(x, y) dx dy = \int_0^t H_X(x, \infty) - H_X(x, x) dx = \\ &= F_X(t) - \int_0^t H_X(x, x) dx = F_X(t) - \int_0^t C_u(F_X(x), F_Y(x)) f_X(x) dx, \end{aligned}$$

where $h(x, y)$ is the joint density function of X and Y and $H_X(x, \infty)$ respectively $H_X(x, x)$ denote the first order partial derivative $\frac{\delta}{\delta x} H(x, y)$ calculated in (x, ∞) respectively in (x, x) . We obtain an analogous formula for F_Y^* . From this formula it follows that the marginal distributions functions F_X and F_Y are solutions of the following system of ordinary differential equations:

$$\begin{cases} \{1 - C_u(F_X(t), F_Y(t))\} F_X'(t) = F_X^{*'}(t) \\ \{1 - C_v(F_X(t), F_Y(t))\} F_Y'(t) = F_Y^{*'}(t) \end{cases}$$

with initial conditions $F_X(0) = F_Y(0) = 0$, where $C_u(F_X(t), F_Y(t))$ and $C_v(F_X(t), F_Y(t))$ denote the first order partial derivatives $\frac{\delta}{\delta u} C(u, v)$ and $\frac{\delta}{\delta v} C(u, v)$ calculated in $(F_X(t), F_Y(t))$.

5.7.2 Part 2

To see which are the relations between the measures of association and copula we will recall three theorems (see [59]).

Theorem 5.7.1. *Let X and Y be continuous random variables whose copula is C . Then Kendall's tau for X and Y (which we will denote by either $\tau(X, Y)$ or τ_C) is given by*

$$\tau(X, Y) = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1.$$

Theorem 5.7.2. *Let X and Y be continuous random variables with copula C . Then Spearman's rho for X and Y (which we will denote by either $\rho(X, Y)$ or ρ_C) is given by*

$$\rho(X, Y) = 12 \int_0^1 \int_0^1 uv dC(u, v) - 3,$$

$$\rho(X, Y) = 12 \int_0^1 \int_0^1 C(u, v) dudv - 3.$$

Recall also from [59] the following theorem which enables us to determine the parameter α (and implicitly the copula) when we know Kendall's tau.

Theorem 5.7.3. *Let X and Y be random variables with an Archimedean copula C generated by $\varphi \in \Omega$. Kendall's tau for X and Y is given by*

$$\tau_C = 1 + 4 \int_0^1 \frac{\varphi(t)}{\varphi'(t)} dt.$$

If C_α is a member of the Gumbel family, then for $\alpha \geq 1$,

$$\frac{\varphi(t)}{\varphi'(t)} dt = \frac{t \log t}{\alpha},$$

so that $\tau(\alpha) = 1 - 1/\alpha$. Now it is easy to see that

$$\alpha_\tau = 1/(1 - \tau).$$

Chapter 6

Two-Stage Bayesian Models - Application to ZEDB project

6.1 Introduction

ZEDB is the major German effort to collect data from nuclear facilities. The goal of the project is to create a reliability data base which contains all major plant events: failure events, operational experience, maintenance actions. In this respect ZEDB is the German equivalent for the Nordic t-book project. As mathematical tool to analyse ZEDB data, a 2-stage Bayesian model was chosen. The first evaluation of ZEDB, in order to obtain estimates of failure rates or failure probabilities per demand for groups of components (as generators and pumps), was performed in 1998.

The standard two stage model, applied to the problem of assimilating failure data from other plants was developed in [Kaplan, 1983, Iman and Hora, 1987,1989, 1990, Prn 1990]. The SKI data bank [1987] uses a two stage model developed by Pörn [1990]. This model was reviewed in [Cooke et al, 1995], and further discussed in [Meyer and Hennings, 1999].

First we identify the standard conditional independence assumptions and derive the general form of the posterior distribution for failure rate λ_0 at plant of interest 0, given failures and observation times at plants 0, 1, ... n . Any departure for the derived mathematical form necessarily entails a departure from the conditional independence assumptions. Vaurio's one stage empirical Bayes model is discussed as an alternative to the two stage model [Vaurio, 1987]. Hofer [Hofer et al 1997, Hofer and Peschke,

1999, Hofer, 1999] has criticized the standard two-stage model and proposed an alternative, which is also discussed. Finally, the methods of Pörn and Jeffrey for choosing a non-informative prior distribution are discussed.

The ZEDB software verification is accomplished by independently coding the model in a higher order programming language and comparing results. This effort is based on the published information over the models and their implementation. No appeal is made to unpublished information regarding the details of the numerical implementation. Hence the differences which arise may be taken to represent those which any independent calculation based on public information might produce.

The following conclusions are drawn:

1. Two stage models provide a valid method for assimilating data from other plants. The conditional independence assumptions, as also used in ZEDB, are reasonable and yield a tractable and mathematically valid form for the failure rate a plant of interest, given failures and operational times at other plants in the population.
2. Choice of hyperprior must be defensible. Non-informativeness is not a good defense if it leads to improper distributions. The influence of the hyperpriors does not decay as observation times get longer, if the number of plants in the population remains fixed.
3. Improper hyperpriors do not always become proper after observations. Improper hyperpriors should be avoided if propriety after observations cannot be demonstrated.
4. The present implementation produced for verification of the implementation of the ZEDB results yields agreement with the gamma model of ZEDB which is consistent with previously noted variations due to truncation of the hyperparameters.
5. This present implementation produces good agreement with the lognormal model of ZEDB.
6. The lognormal model, as used in standard ZEDB evaluations, enjoys a significant advantage over the gamma model in that, as observation time increases, a natural truncation of the hyperparameters m, s is possible.

7. In the context of a literature survey, Vaurio's one-stage empirical Bayes model has been investigated. It is elegant and simple. It will not work with zero observed failures or with a population of two plants.
8. In the context of a literature survey, Hofer's criticism for the normal two-stage model and his own variant have been investigated. The latter appears to rest on shifting viewpoints involving conflicting assumptions. Consistent application of the standard conditional assumptions collapses his model into the same form (equation 4) which he criticizes as a 'wrong chance model'. Further discussion should wait until the conditional independence assumptions and mathematical derivation are clarified.

6.2 Bayesian Two Stage Hierarchical Models

Bayesian two stage or hierarchical models are widely employed in a number of areas. The common theme of these applications is the assimilation of data from different sources, as illustrated in Figure 1. Agents producing failure data in this case are component types at nuclear power plants, but it could also be hospitals / treatments producing deceased patients [Mashall and Spiegelhalter 2000], factories producing pollution, or units of any kind producing malfunctions, etc [Richardson and Green, 1997, Clayton et al, 2000] . The data from agent i is characterized by an exposure T_i and a number of events X_i . The exposure T_i is not considered stochastic, as it can usually be observed with certainty. The number of events for a given exposure follows a fixed distribution type, in this case Poisson. The parameter(s) of this fixed distribution type are uncertain, and are drawn from a prior distribution. The prior distribution is also of a fixed type, yet with uncertain parameters. In other words, the prior distribution itself is uncertain. This uncertainty is characterized by a hyperprior distribution over the parameter(s) of the prior.

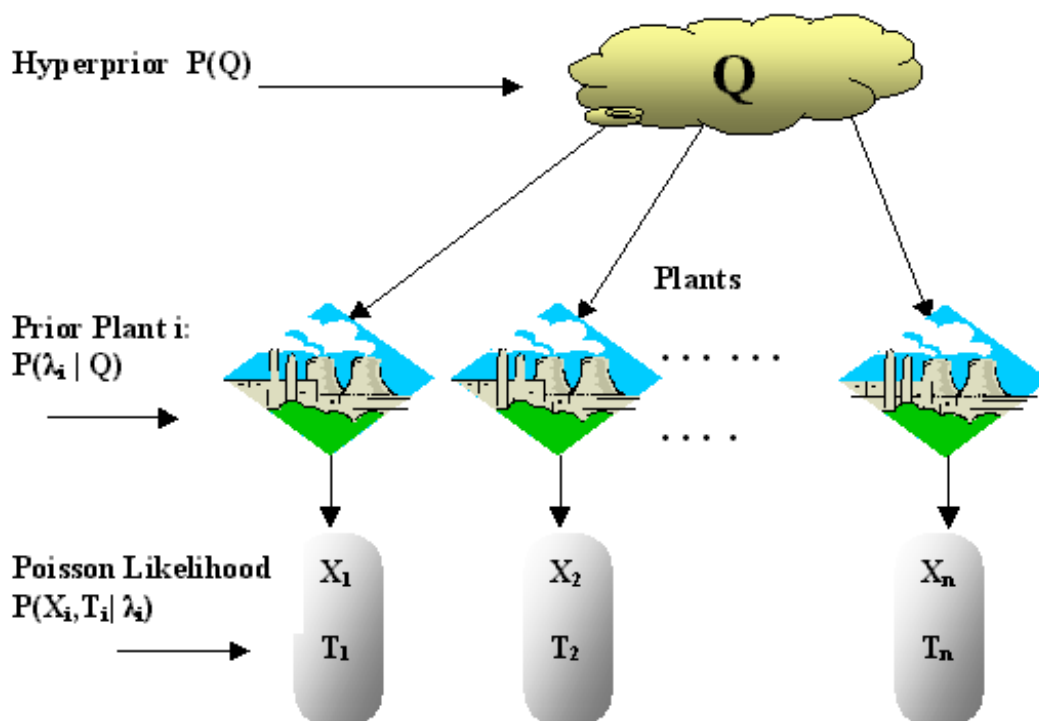


Figure 1. Bayesian Two Stage Hierarchical model

In Figure 1, the hyperprior is a distribution $P(Q)$ over the parameters Q of the prior distribution from which the Poisson intensities $\lambda_1, \dots, \lambda_n$ are drawn. In sum, our model is characterized by a joint distribution:

$$P(X_1, \dots, X_n, \lambda_1, \dots, \lambda_n, Q) \quad (1)$$

A two-stage model is really nothing more than a joint distribution (1). To be useful, however, we must derive conditional distributions. Typically we want to use data from "other plants" to make predictions about a given plant. This is very attractive in cases where the data from the given plant is sparse.

To yield tractable models, such models must make two types of assumptions. First, conditional independence assumptions [Pörn, Iman and Hora] are made to factor (1). Second, assumptions must be made regarding the fixed distribution types and the hyperprior distribution $P(Q)$. The conditional independence assumptions may be read from Figure 1, by treating this figure as a "belief net". In particular, this figure says:

CI.1 Given Q, λ_i is independent of $\{X_j, \lambda_j\}_{j \neq i}$,

CI.2 Given λ_i, X_i is independent of $\{Q, \lambda_j, X_j\}_{j \neq i}$.

The expression " X_i is independent of $\{Q, \lambda_j, X_j\}_{j \neq i}$ " entails that X_i is independent of Q , and X_i is independent of λ_j , which we sometimes write as $X_i \perp \lambda_j$.

Virtually all 2-stage Bayesian models make the above conditional independence assumptions, with one possible exception (see below). Their plausibility in the present context is discussed in [Cooke et al. 1995]. With these assumptions we can derive the conditional probability $P(\lambda_0|X_0, \dots, X_n)$ for the failure rate at plant 0, given X_i failures observed at plant i , $i = 0, \dots, n$. This is sometimes called the posterior probability for λ_0 .

6.2.1 Derivation of Posterior Probability for λ_0

Certain features are common to all two stage Bayesian models, at least those subscribing to CI.1 and CI.2. We derive these features characterizing the posterior probability of λ_0 in this section.

We assume throughout that the plant of interest is plant 0. Since the observation times T_0, \dots, T_n are not stochastic, we suppress them in the notation. We seek an expression for

$$P(\lambda_0|X_0, \dots, X_n) \quad (2)$$

We step through this derivation, giving the justification for each step. A more detailed exposition is found in [Cooke et al 1995]. " \propto " denotes proportionality, "CI.i" means that conditional independence assumption i is invoked, $i = 1, 2$; and "TP" denotes the law of total probability; and "FB" denotes Fubini theorem. The Fubini theorem authorizes switching the order of integration if the integrals are finite ¹ [Royden 1968, p269],

$$P(\lambda_0|X_0, \dots, X_n) \propto \quad (\text{Bayes theorem}) \quad P(X_0|\lambda_0, X_1, \dots, X_n)P(\lambda_0|X_1, \dots, X_n)$$

¹The Fubini theorem says that if $f(x, y)$ is (Lebesgue) integrable with respect to the product measure, $dx \times dy$, over some domain $X \times Y$, then $\int \int f(x, y) dx \times dy = \int [\int f(x, y) dx] dy = \int [\int f(x, y) dy] dx$. A function f over $X \times Y$ is integrable if $\sup_{\alpha_i < f(A_i)} \{\sum_{i=1..m} \alpha_i \mu(A_i)\} < \infty$, where the sup is taken over all finite partitions $\{A_i\}$ of $X \times Y$ and $\alpha_i \in \mathbb{R}$; and where $\mu(A_i)$ is the Lebesgue measure of A_i . There are examples in which $\int \int f(x, y) dx \times dy \neq \int [\int f(x, y) dx] dy = \int [\int f(x, y) dy] dx$, but then of course the product integral does not exist [De Barra, 1974, p. 202].

$$\propto \quad (CI.2) \quad P(X_0|\lambda_0)P(\lambda_0|X_1, \dots, X_n)$$

$$\propto \quad (TP, Bayes \text{ theorem})$$

$$P(X_0|\lambda_0) \int_{\lambda_1 \dots \lambda_n} \int_q P(\lambda_0|\lambda_1, \dots, \lambda_n, q, X_1, \dots, X_n) P(q, \lambda_1, \dots, \lambda_n|X_1, \dots, X_n) dq d\lambda_1 \dots d\lambda_n$$

$$\propto \quad (CI.1) \quad P(X_0|\lambda_0) \int_{\lambda_1 \dots \lambda_n} \int_q P(\lambda_0|q) P(q, \lambda_1, \dots, \lambda_n|X_1, \dots, X_n) dq d\lambda_1 \dots d\lambda_n$$

$$\propto \quad (Bayes \text{ theorem})$$

$$\propto P(X_0|\lambda_0) \int_{\lambda_1 \dots \lambda_n} \int_q P(\lambda_0|q) P(X_1, \dots, X_n|q, \lambda_1, \dots, \lambda_n) P(q, \lambda_1, \dots, \lambda_n) dq d\lambda_1 \dots d\lambda_n \quad (3)$$

$$\propto \quad (CI.1, 2; Fubini \text{ theorem})$$

$$P(X_0|\lambda_0) \int_q P(\lambda_0|q) \int_{\lambda_1 \dots \lambda_n} [\prod_{i=1..n} P(X_i|\lambda_i) P(\lambda_i|q) d\lambda_1 \dots d\lambda_n] P(q) dq$$

$$\propto P(X_0|\lambda_0) \int_q P(\lambda_0|q) \prod_{i=1..n} \int [P(X_i|\lambda_i) P(\lambda_i|q) d\lambda_i] P(q) dq \quad (4)$$

Expression (4) is normalized by integrating over all λ_0 .

6.2.2 Summary of Significant Features

If conditional independence assumptions CI.1, CI.2 hold, then necessarily the posterior probability of λ_0 given (X_0, \dots, X_n) has the form (4). Equivalently, if the posterior probability does not have the form (4), then necessarily CI.1, CI.2 do not both hold.

1. If $Q = q_0$ is known with certainty, then there is no influence from X_1, \dots, X_n on λ_0 . Indeed, in this case the posterior density in λ_0 is simply proportional to $P(X_0|\lambda_0)P(\lambda_0|q)$.

2. As the numbers $X_i, T_i, i = 1 \dots n$ get large, $X_i/T_i \rightarrow \lambda_i$, then the Poisson likelihood $P(X_i|\lambda_i)$ converges to a Dirac measure concentrating mass at the point $X_i = T_i\lambda_i$. In the limit the "hyperposterior"

$$\prod_{i=1..n} \left[\int P(X_i|\lambda_i)P(\lambda_i|q)d\lambda_i \right] P(q) \quad (5)$$

becomes

$$\left[\prod_{i=1..n} P(\lambda_i = X_i/T_i|q) \right] P(q) \quad (6)$$

(6) corresponds to the situation where $P(q)$ is updated with observations $\lambda_1, \dots, \lambda_n$. Note that as the observation time increases, the number n does not change. If n is only modest (say in the order 10) then the effect of the hyperprior will never be dominated by the effect of observations². We say that the hyperprior persists in the posterior distribution $P(\lambda_0|X_0, \dots, X_n)$.

3. The persistence of the hyperprior is a very significant feature of the two stage Bayesian models which practitioners have not always fully appreciated. For one thing, it means that the choice of hyperprior must be very defensible, as it will never be overruled by observational data.
4. It is shown in [Cooke et al 1995], [Hennings and Meyer, 1999] that improper hyperpriors $P(q)$ do not always become proper when multiplied by $\left[\prod_{i=1..n} P(\lambda_i = X_i/T_i|q) \right] P(q)$. In other words, the hyperposterior may well remain improper. This is very serious, as posterior expectations may then be infinite, and numerical results will essentially depend on the method of truncation. Hence, "non-informativeness" does not yield defensible hyperpriors, unless the propriety after observations can be demonstrated.

²The following thumbnail calculation illustrates this point. Suppose λ is drawn from a *Gamma*(α, β) distribution with α fixed and with β following a prior *Gamma*(h, s). The prior variance is hs^{-2} . After updating the prior with observations $\lambda_1 \dots \lambda_n$, the variance of $G(h, s|\lambda_1 \dots \lambda_n)$ is $(n\alpha + h)(\sum \lambda_i + s)^{-2}$. Assuming that α is order h and λ_i is order s , we see that the observations reduce the variance with a factor $(n + 1)$. The Bayesian confidence bounds are roughly linear in the standard deviation, and hence are shrinking by a factor $(n + 1)^{1/2}$. Hence, for $n = 10$, the 5%, 95% confidence band in β is shrunk by a factor 3.3.

The conclusion from the above general discussion is that the Bayesian two stage hierarchical model is sensitive to the choice of hyperprior, and that improper hyperpriors should be avoided unless propriety of the posterior can be proved.

6.2.3 Selected Literature Review

We restrict to two stage models implemented in the present context of assimilating data from other plants. Two stage Bayesian models have been implemented by various authors. Perhaps one of the first to propose these models in the context of assimilating data from different plants was [Kaplan, 1983]. Kaplan used a log normal prior with a Poisson likelihood, which of course is not a natural conjugate. This method has been implemented by ZEDB. [Iman and Hora, 1989, 1990] and [Hora and Iman, 1987] proposed a natural conjugate gamma prior. [Vaurio, 1987] proposed a one-stage empirical Bayes approach, using other plants to determine the prior. [Frhner, 1985] proposed a method of choosing a hyperprior for two stage models. The SKI data bank [1987] uses a two stage model developed by Pörn [1990]. This model was reviewed in [Cooke et al, 1995], and further discussed in [Meyer and Hennings, 1999]. Recently [Hofer et al 1997], [Hofer and Peschke, 1999] and [Hofer, 1999] have suggested that an incorrect chance mechanism underlies the two-stage models, and have proposed their own model. In this section we briefly review these developments.

In the two stage Bayesian models considered here (Figure 1) use a Poisson likelihood. The prior is usually gamma or log normal. The second stage places a hyperprior distribution over the parameters of the prior gamma or log normal distribution. We briefly recall the definitions and elementary facts of the Poission, Gamma, and log normal distributions in Table 1 below:

Name	Density	Expectation	Variance
Poisson	$P(X T, \lambda) = \frac{(\lambda T)^X}{X!} e^{-\lambda T}, \lambda > 0, T > 0$	λT	λT
Gamma	$f(\lambda \alpha, \beta) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \beta^\alpha e^{-\beta\lambda}, \alpha > 0$	α/β	α/β^2
Lognormal	$f(\lambda \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma\lambda}} e^{-\frac{(\ln \lambda - \mu)^2}{2\sigma^2}}, \sigma > 0$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$

Table 1

Using a gamma prior with parameters as above, the term in (4):

$$\prod_{i=1..n} \int P(X_i|\lambda_i)P(\lambda_i|q)d\lambda_i$$

becomes after carrying out the integration:

$$\prod_{i=1}^n \frac{\Gamma(X_i + \alpha)}{\Gamma(X_i + 1)\Gamma(\alpha)} \left(\frac{\beta}{\beta + T_i}\right)^\alpha \left(\frac{T_i}{\beta + T_i}\right)^{X_i} = \prod_{i=1}^n \binom{X_i + \alpha - 1}{X_i} \left(\frac{\beta}{\beta + T_i}\right)^\alpha \left(\frac{T_i}{\beta + T_i}\right)^{X_i} \quad (7)$$

where

$$\binom{X_i + \alpha - 1}{X_i} = \frac{(X_i + \alpha - 1)!}{X_i!(\alpha - 1)!}$$

Further calculation to solve equation (4) must be performed numerically. It is shown in [Cooke et al. 1995] that improper hyperpriors may remain improper after assimilating observations. The asymptotic behavior of the "hyperposterior"

$$P(\alpha, \beta | X_1 \dots X_n, T_1 \dots T_n) \propto P(X_1 \dots X_n, T_1 \dots T_n | \alpha, \beta) P(\alpha, \beta)$$

will essentially be determined by the maximum of $P(X_1 \dots X_n, T_1 \dots T_n | \alpha, \beta)$. The significant fact is that $P(X_1 \dots X_n, T_1 \dots T_n | \alpha, \beta)$ has no maximum; it is asymptotically maximal along a ridge, see (Figure 2).

Vaurio

[Vaurio, 1987] proposed an analytic empirical Bayes approach to the problem of assimilating data from other plants. A simple one-stage Bayesian model for one plant would use a Poisson likelihood with intensity λ , and a *Gamma*($\lambda | \alpha, \beta$) prior. Updating the prior with X_i failures in time T_i yields a *Gamma*($\lambda | \alpha + X_i, \beta + T_i$) posterior. Vaurio proposes to use data from the population of plants to choose the *Gamma*($\lambda | \alpha, \beta$) prior by moment fitting. Any other two moment prior could be used as well. Data from other plants are not used in updating, hence, this is a one-stage model.

We sketch Vaurio's model in the simple case that the observation times at all $n+1$ plants are equal to T . The population mean and (unbiased) variance are estimated as:

$$m = \left(\sum_{i=0..n} X_i / T \right) / (n + 1) \quad (8)$$

$$v = \left(\sum_{i=0..n} (X_i/T - m)^2 \right) / n. \quad (9)$$

A shifted variance estimate, which is positive when at least one of the $X_i > 0, i = 0, ..n$; is defined as:

$$V = v + m/nT. \quad (10)$$

V and m are used to solve for the shape α and scale β of a gamma prior $G(\lambda_i|\alpha, \beta)$:

$$\alpha = m^2/V \quad (11)$$

$$\beta = m/V. \quad (12)$$

Using the familiar gamma-Poisson one stage model, the posterior mean and variance for λ_i after observing X_i failures in time T , are:

$$E(\lambda_i|X_i, T) = (\alpha + X_i)/(\beta + T), \quad (13)$$

$$Var(\lambda_i|X_i, T) = (\alpha + X_i)/(\beta + T)^2. \quad (14)$$

The model is consistent, in the sense that as $X_i, T_i \rightarrow \infty$, with $X_i/T_i \rightarrow \lambda_i$, his model does entail that $E(\lambda_i|X_i, T_i) \rightarrow \lambda_i$. Elegance and simplicity are its main advantages. Disadvantages are that it cannot be applied if all $X_i = 0$, or if the population consists of only 2 plants. Further, numerical results in section 4 indicate that the model is non-conservative when the empirical failure rate at plant 0 is low and the empirical failure rates at other plants are high. A final criticism, which applies to most empirical Bayes models is that the data for the plant of interest is used twice, once to estimate the prior and once again in the Poisson likelihood. Thus, X_i occurs in (13) twice, once as X_i , and again in the estimate of α and β . This may contribute to the non-conservatism noted in section 4.

Fröhner

[Fröhner, 1985] proposed a two stage model in which the hyperprior is a mixture of the gamma posteriors at each plant. Fröhner explicitly states that this model is useful

when the gamma posteriors are sufficiently smooth, a point that was emphasized in a discussion with [Kaplan, 1985]. This condition will not hold as the observation times at each plant get large, as the gamma posteriors converge to Dirac distributions. Because of this limitation, the model will not be treated further in this report.

Hofer

Hofer has published a number of articles [Hofer et al, 1997][Hofer, 1999] and [Hofer and Peschke, 1999] in which the two-stage models are faulted for using a ‘wrong chance mechanism’, and a new model is proposed. He does not explicitly formulate conditional independence assumptions, and does not derive the posterior by conditionalizing the joint as done above. Rather, the model is developed by shifting between the point of view of ‘observing λ_i ’ and ‘observing (X_i, T_i) ’. [Hofer 1997] criticizes [Hora and Iman 1990] for using the wrong order of integration of improper integrals³. In later publications, a “deeper” reason is found to reside in the use of a “wrong chance mechanism”. Hofer’s model appears⁴ to result in a posterior of the following form:

$$P(\lambda_0|X_0..X_n) \propto P(X_0|\lambda_0) \int_{(q)} P(\lambda_0|q) \int \int_{\forall \lambda_{1..n}} \frac{[\prod_{i=1..n} P(\lambda_i|q)]P(q)}{\int_{(q)} [\prod_{i=1..n} P(\lambda_i|q')]P(q')dq'} \\ [\prod_{i=1..n} P(\lambda_i|q)]P(\lambda_1... \lambda_n)d\lambda_1...d\lambda_n dq \quad (15)$$

Notice that this does not appear to have the form of (4).

Although Hofer does not explicitly formulate his conditional independence assumptions, he does use them. E.g. he uses CI.1 to derive the expression in the

³An improper integral is an integral with one or more infinite bounds, or an integral with finite bounds where the integrand is unbounded at one or more points between (\leq, \geq) the integration limits. As noted in the footnote in section 5.2.1, the order of integration over $X \times Y$ can only make a difference if the function in question is not integrable with respect to the product measure. Even stronger is the following statement (De Barra 1974, p 202): if a function of two variables is non negative, then if one of the iterated integrals is finite, so is the other and the order of integration makes no difference. Even so, the integral with respect to the product measure may not exist. Further, whenever the variables’ ranges are truncated to a domain of finite measure in which the integrand is bounded, then the function is integrable with respect to the product measure and the order of integration is immaterial.

⁴This represents our, admittedly uncertain, conjecture as to the intended posterior distribution.

denominator (see equation (4) of [Becker and Hofer, 2001]). If CI.1 holds, then necessarily

$$\int_{(q)} [\prod_{i=1..n} P(\lambda_i|q')]P(q')dq' = \int_{(q)} P(\lambda_1\dots\lambda_n|q)P(q)dq = P(\lambda_1\dots\lambda_n) \quad (16)$$

and (15) reduces to (4). If CI.1 does not hold, then the origin of the product $\prod P(\lambda_i|q)$ is unclear. Hofer says that $P(\lambda_1\dots\lambda_n) = \prod r(\lambda_i)$, where $r(\lambda_i)$ is a noninformative prior, which he takes to be constant. This entails that the λ_i are unconditionally independent. It is not difficult to show that if λ_i are unconditionally independent, and independent given q , that then λ_i is independent of q ⁵. This would make the entire two stage model quite senseless. If $P(\lambda_1\dots\lambda_n) = \prod r(\lambda_i) = \text{constant}$ in the numerator of (15), but not in the denominator, then (15) is not equivalent to (4), but rests on conflicting assumptions.

In any event, if (15) does not reduce to (4) then the assumptions CI.1, CI.2 do not both hold. Hofer does not say which assumptions are used to derive (15), in fact (15) is not derived mathematically, but is “woven together” from shifting points of view. The danger of such an approach is that conflicting assumptions may be inadvertently introduced. This appears to be the case, as the λ_i are at one point assumed to be independent, and at another point are assumed to be conditionally independent given q . A consistent application of Hofer’s (implicit) conditional independence assumptions would reduce (15) to (4), which of course is the model Hofer faults for using a “wrong chance mechanism”.

A standard mathematical derivation of this model would easily remove all uncertainty. This involves:

1. Clearly stating the modeling assumptions
2. Writing down the joint distribution, and
3. Deriving the posterior distribution via conditionalization.

⁵Independence implies that for all λ_i , $P(\lambda_1, \dots, \lambda_n) = \prod P(\lambda_i) = \prod \int P(\lambda_i|q)P(q)dq$. By conditional independence; $P(\lambda_1, \dots, \lambda_n) = \int P(\lambda_1, \dots, \lambda_n|q)P(q)dq = \int \prod P(\lambda_i|q)P(q)dq$. The λ_i are identically distributed given q ; take $\lambda_i = \lambda; i = 1, \dots, n$. These two statements imply $[\int P(\lambda|q)^n P(q)dq]^{1/n} = \int P(\lambda|q)P(q)dq$. Since the integrand is non-negative, this implies that $P(\lambda|q) = \text{constant} = P(\lambda)$ (Hardy Littlewood and Polya 1983, p 143).

The gravity of the of "wrong chance mechanism" charge warrants such a derivation. This would also clarify the order of integration issue: if the order of integration is not material, it would be helpful to acknowledge this directly. Further discussion of this model should be deferred, pending a mathematical derivation.

Pörn

Pörn (1990) introduces a two stage model with a gamma prior for λ , similar to (Hora and Iman 1990). He provides an argument for choosing the following non-informative (improper) densities for the parameters ν, μ' :

$$g(\nu) = \frac{1}{\nu}, \nu > 0, k(\mu') = \frac{1}{\sqrt{\mu'(1 + \mu')}}, \mu' > 0 \quad (17)$$

where $\nu = \frac{1}{\alpha}$ is the coefficient of variation and $\mu' = \frac{T\alpha}{\beta}$ is the expected number of failures at time T , given α and β . Assuming independence between these parameters and transforming back to the hyperparameters α and β , $\alpha = \frac{1}{\nu^2}\beta = \frac{T}{\nu^2\mu}$, a joint (improper) hyperprior density for α and β is obtained proportional to:

$$\frac{1}{\beta\sqrt{\alpha(\alpha + \beta/T)}} \quad (18)$$

Jeffrey's rule

Another frequently used principle, called Jeffrey's rule, is to choose the non-informative prior $P(q)$ for a set of parameters q proportional to the square root of the determinant of the information matrix $\Phi_n(q)$:

$$P(q) \propto \sqrt{|\Phi_n(q)|}, \quad \text{where} \quad \Phi_n(q) = E\left\{\frac{\partial^2 L}{\partial q_i \partial q_j}\right\} \quad (19)$$

and L is the log-likelihood function for the set of the parameters.

Hora and Iman (1990) apply this rule to the two-dimensional parameter vector (α, β) for the Gamma distribution of the failure rate λ . They get the (approximate) improper hyperprior:

$$\frac{1}{\alpha^{1/2}\beta}, \alpha, \beta > 0.$$

Heuristic interpretation of hyperparameters

In view of the importance attaching to the method of truncation, it is important to have a heuristic interpretation of the hyperparameters. Pörn suggests a heuristic for μ' : $\mu' = \frac{\sum_{i=1}^n T_i}{n}$ (the number of expected failures in time T).

No heuristic is indicated for ν , beyond saying that it is the coefficient of variation of prior distribution for λ . Hence, the heuristic for μ' suggests that μ' is of the same order as the x_i , which is of order $\lambda_i T$. The coefficient of variation is the standard deviation of λ_i divided by the expectation of λ_i . Further, the order of ν , α and β is given by:

$$\begin{aligned}\nu &\sim \frac{1}{\sqrt{\mu'}} \sim \frac{1}{\sqrt{x_i}}; \\ \alpha &= \frac{1}{\nu^2} \sim x_i; \\ \beta &= \frac{\alpha T}{\mu'} \sim \frac{x_i T}{\lambda_i T} \sim T.\end{aligned}$$

Note also that the order relation $\nu \sim 1/\sqrt{\mu'}$ would suggest a strong negative correlation between the uncertainty distribution over ν and μ' .

6.2.4 Conclusions

1. Two stage models provide a valid method for assimilating data from other plants. The conditional assumptions are reasonable and yield a tractable and mathematically valid form for the failure rate a plant of interest, given failures and operational times at other plants in the population.
2. Choice of hyperprior must be defensible. Non-informativeness is not a good defense if it leads to improper distributions. The influence of the hyperpriors does not decay as observation times get longer, if the number of plants in the population remains fixed.
3. Improper hyperpriors do not always become proper after observations. Improper hyperpriors should be avoided if propriety after observations cannot be demonstrated.

4. Vaurio's one-stage empirical Bayes model is elegant and simple. It will not work with zero observed failures or with a population of two plants.
5. Hofer's two-stage model appears to rest on shifting viewpoints involving conflicting assumptions. Consistent application of the standard conditional assumptions collapses his model into the form (4), which he criticizes as a 'wrong chance model'. Further discussion should wait until the conditional independence assumptions and mathematical derivation are clarified.
6. Pörn and Jeffrey give rules for choosing distributions over hyperparameters. These typically yield improper distributions are subject to the reservations mentioned above.

6.3 ZEDB Software Verification

To perform the verification tasks associated with the theoretical models presented above, a software implementation in the higher order programming language MATLAB is presented. We have not attempted to replicate the ZEDB results exactly, as that would require using the same step sizes and truncation bounds. Rather, we have used such information as is available in the published literature, using our own judgment regarding numerical implementation. Differences between our results and those of ZEDB reflect differences that may arise from an independent implementation based on public information. Although ZEDB recommends the lognormal model, both the lognormal and gamma models are supported, and both are benchmarked here ⁶.

Three data sets are used to check the concordance with the results from [Becker and Hofer, 2001] (2001). These are:

Data Set 1 (4 in [Becker and Hofer, 2001]):

Number of failures	7	1	3	<u>2</u>	1	2	0
Time of observation	24000	24000	24000	<u>24000</u>	24000	24000	24000

# failures cont'd	0	2	0	0	0
Time of observation	24000	24000	24000	24000	24000

⁶The ZEDB results for the lognormal model are published in [Becker and Hofer, 2001, without details on numerical procedures; the results for the gamma model were not made available beforehand.

Data Set 2 (2 in [Becker and Hofer, 2001]):

Number of failures	1	0	0	0	1	<u>2</u>
Time of observation	20000	2000	4000	6000	10000	<u>12000</u>

Data Set 3 (3 in [Becker and Hofer, 2001]):

Number of failures	0	0	<u>1</u>
Time of observation	12000	2000	<u>3000</u>

Remark: The underlined field is the plant of interest.

The ZEDB results are presented in Table 2. Also shown are the results of Vaurio's model as computed by a spreadsheet provided by ZEDB.

	Dataset 1	Dataset 2	Dataset 3
	Lognormal	Model	
Mean	6.7518E-5	9.1360E-05	1.0456E-4
Std.dev	3.9866E-5	5.9964E-05	1.2892E-4
5%	2.0140E-5	2.5387E-05	1.1200E-5
50%	5.9078E-5	7.7103E-05	6.2277E-5
95%	1.4403E-4	2.0576E-04	3.3980E-4
	Gamma	Model	
Mean	7.4138E-05	2.0364E-04	3.2123E-04
Std.dev	3.0598E-05	7.4684E-05	2.1575E-04
5%	3.2518E-05	1.2220E-04	1.2141E-04
50%	6.9926E-05	1.7247E-04	2.5766E-04
95%	1.3044E-04	3.4473E-04	7.4076E-04
	Vaurio's	Model	
Mean	7.84E-5	1.29E-4	2.84E-4
Std.dev	4.91E-5	7.64E-5	2.43E-4
5%	1.83E-5	3.34E-5	2.84E-5
50%	6.84E-5	1.14E-4	2.19E-4
95%	1.73E-4	2.74E-4	7.64E-4

Table 2 ZEDB results with Lognormal and Gamma models.

We first discuss the Gamma model, then the lognormal model.

6.3.1 Gamma model

The computation may be broken into three steps:

1. Truncate the range of (α, β) to a finite rectangle.
2. Identify a range for λ_0 which contains all the "plausible" values.
3. For every "plausible" value of λ_0 , evaluate numerically the integrals over α and β , and interpolate to find the 5%, 50% and 95% quantiles.

Running the code for 100 values of λ_0 with a high degree of numerical accuracy may take 1 hour, but rough results (40 values) may be obtained in 20 minutes.

The likelihood $P(X_1, \dots, X_n, T_1, \dots, T_n | \alpha, \beta)$ as a function of α and β (2.7) is presented in Figure 2. Values for $(X_1, \dots, X_n, T_1, \dots, T_n)$ are taken from data set 1. For uniform hyperpriors, this likelihood is proportional to the hyperposterior distribution $P(\alpha, \beta | X, T)$. Note that $P(\alpha, \beta | X, T)$ does not peak but "ridges". This means that a "natural" truncation for α and β cannot be defined; that is, we cannot define a finite rectangle for α and β which contains most of the hyperposterior mass. In our simulations, these ranges were chosen in a manner similar to [Cooke et al 1995], using Pörn's heuristic. The inability to localize the hyperposterior mass for (α, β) means that we cannot localize the posterior mass

$$\begin{aligned}
 P(\lambda_0 | X_0, \dots, X_n) &\propto \\
 &\propto P(X_0 | \lambda_0) \int \int_{\alpha, \beta} P(\lambda_0 | \alpha, \beta) \prod_{i=1..n} \left[\int P(X_i | \lambda_i) P(\lambda_i | \alpha, \beta) d\lambda_i \right] P(\alpha, \beta) d\alpha d\beta
 \end{aligned}$$

For each finite rectangle for α, β , the mass in λ_0 will be localized, but other choices for α, β could significantly shift the region in which λ_0 is localized. This means, of course, that the method of truncation in step 1 will influence the plausible values in step 2, and can have a significant effect on the results.

Figure 3 and 4 represent the hyperposterior distribution for Pörn's approach and Jeffrey's hyperpriors; also with $(X_1, \dots, X_n, T_1, \dots, T_n)$ from data set 1 below. The posterior density and cumulative distribution of λ_0 are presented in Figure 5 and 6.

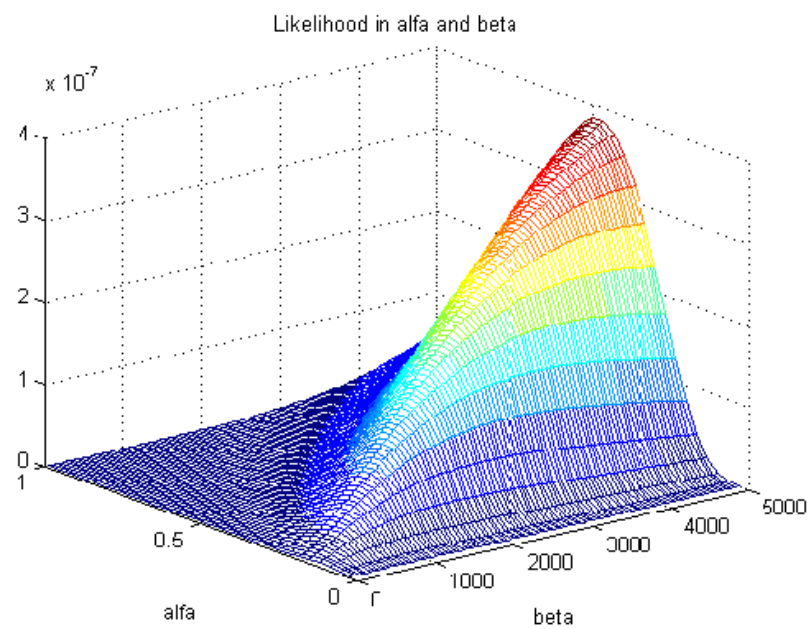


Figure 2

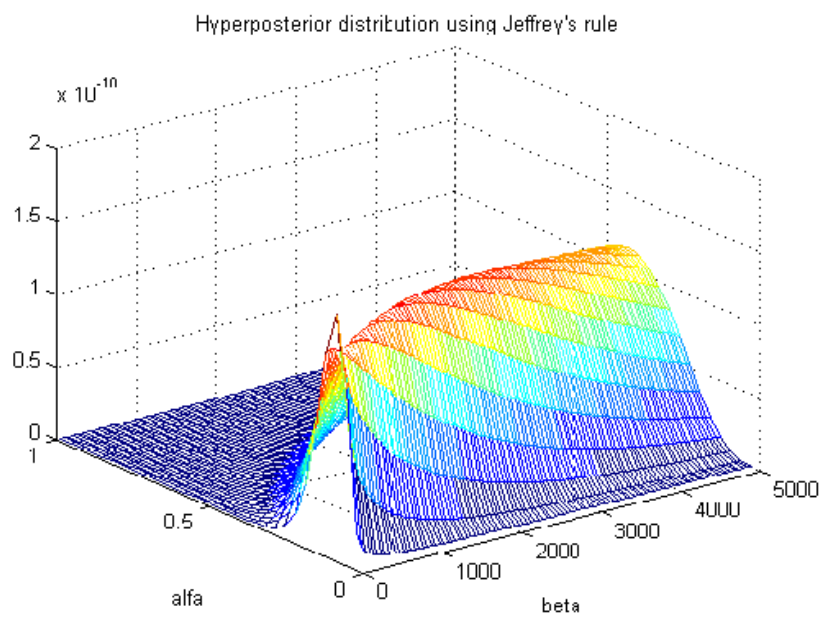


Figure 3

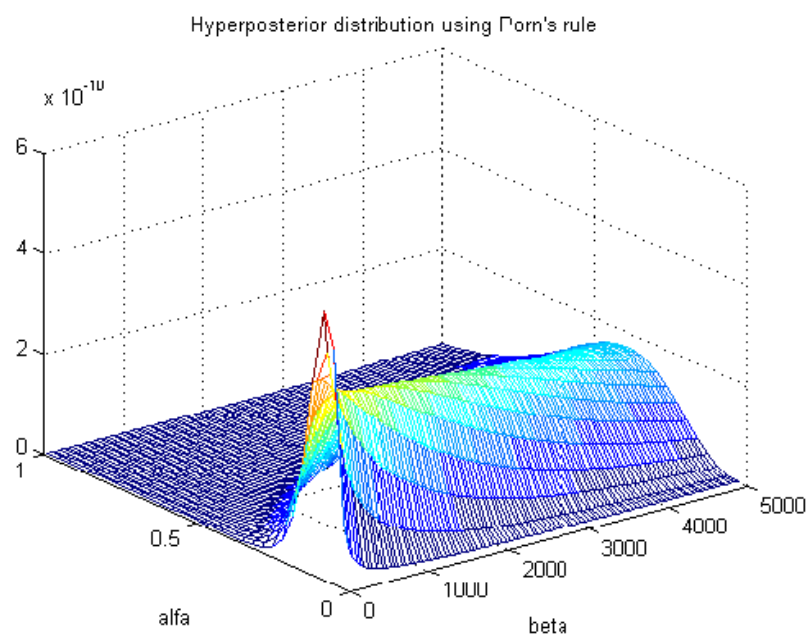


Figure 4

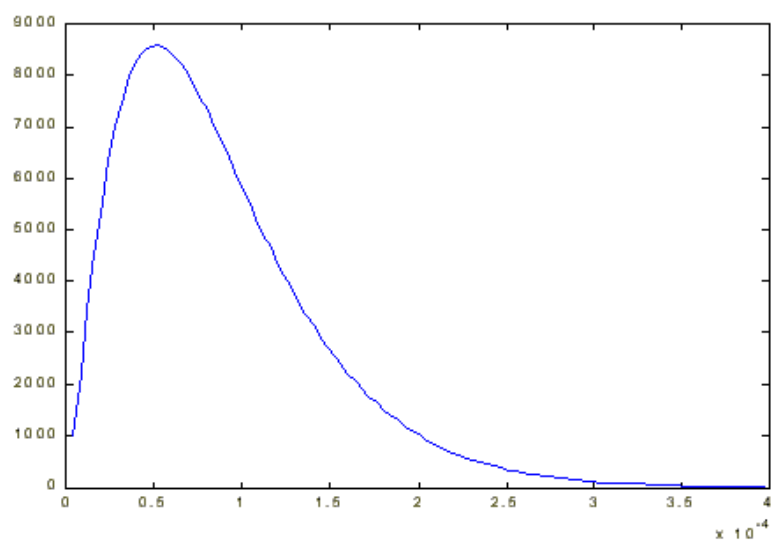


Figure 5

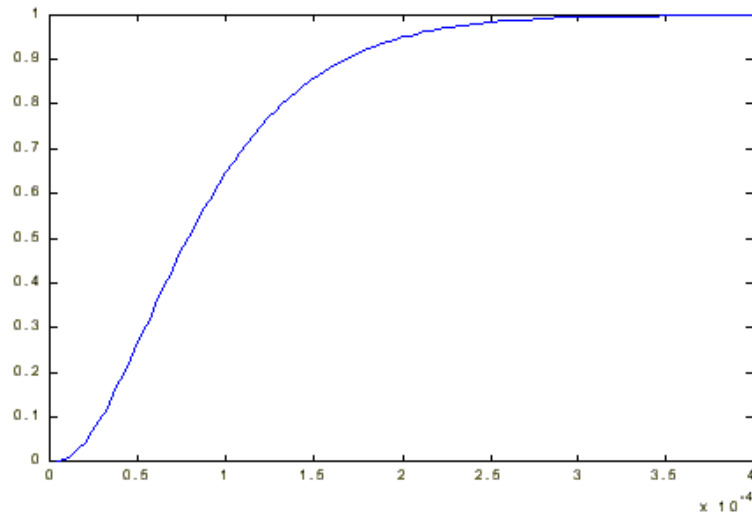


Figure 6

Tables 3 - 5 compare our results for the uniform, Pörn and Jeffrey prior, and give the integration ranges for α and β for our computation and for the ZEDB results. Table 6 compares the TUD and ZEDB results. Note that the 5% quantile for dataset 3 is a more than a factor three lower in the TUD results. In dataset 1 the agreement is better, as there are more plants, more operational hours and more failures.

These differences are consistent with the results reported in [Cooke et al 1995], where 'stress-testing' the gamma model by exploring the range of plausible choices for α, β resulted in differences up to a factor 5.

	Uniform	Pörn	Jeffrey	Ranges α, β	
5%	2.3971 E-5	2.8429 E-5	2.8665 E-5	TUD	ZEDB
50%	8.0511 E-5	8.4990 E-5	8.5678 E-5	α : 0.033..1	α : 0.002..351
95%	2.0012 E-4	2.0598 E-4	2.0670 E-4	β : 50..5000	β : 582..232,219

Table 3. The 5%, 50% and 95% quantiles of the posterior distribution of λ_0 for data set 1

	Uniform	Pörn	Jeffrey	Ranges α, β	
5%	4.2371 E-5	3.9306 E-5	4.3845 E-5	TUD	ZEDB
50%	1.4603 E-4	1.4278 E-4	1.4908 E-4	α : 0.03..1.2	α : 0.01..1926
95%	2.0012 E-4	2.0598 E-4	2.0670 E-4	β : 50..5000	β : 789..350,098

Table 4. The 5%, 50% and 95% quantiles of the posterior distribution of λ_0 for data set 2

	Uniform	Pörn	Jeffrey	Ranges α, β	
5%	3.3976 E-5	3.6503 E-5	3.6394 E-5	TUD	ZEDB
50%	1.7514 E-4	2.1247 E-4	2.0711 E-4	α : 0.0154..0.3846	α : 0.01..503
95%	5.8914 E-4	6.6524 E-4	6.5640 E-4	β : 50..5000	β : 126..91385

Table 5. The 5%, 50% and 95% quantiles of the posterior distribution of λ_0 for data set 3

	Gamma	Model TUD	(Pörn)	Gamma	Model	ZEDB
	Dataset 1	Dataset 2	Dataset 3	Dataset 1	Dataset 2	Dataset 3
5%	2.8429 E-5	3.9306 E-5	3.6503 E-5	3.2518E-05	1.2220E-04	1.2141E-04
50%	8.4990 E-5	1.4278 E-4	2.1247 E-4	6.9926E-05	1.7247E-04	2.5766E-04
95%	2.0598 E-4	3.3217 E-4	6.6524 E-4	1.3044E-04	3.4473E-04	7.4076E-04

Table 6 Comparison TUD (Pörn) and ZEDB for gamma model.

6.3.2 Lognormal model

ZEDB adopted the lognormal distribution as a prior, based on the maximum entropy principle invoked by [Jaynes, 1968].

The lognormal density is:

$$f(\lambda|\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma\lambda}} e^{-\frac{(\ln\lambda-\mu)^2}{2\sigma^2}}, \quad \text{with } -\infty < \mu < +\infty \quad \text{and } \sigma > 0 \quad (20)$$

The uncertainty over parameters μ and σ is expressed by hyperpriors. [Becker 2001] takes into account four types of hyperprior distribution based on Jeffrey's rule. [Becker 2001] proposes four different implementations of Jeffrey's rule. We caution against the multivariate implementation and version of the Jeffrey's rule when parameters of different kind e.g. location and scale parameters, are considered. In this case, as [Box and Tiao, 1974] suggested, it is wiser to choose parameters, which can

be assumed independent and then apply the one parameter version of the rule. This is done only in the first and the fourth case below. The higher order language MAPLE was used for the verification of the codes and the results obtained in all cases; case 1 is used by ZEDB.

1st case

Jeffrey's rule is applied to the parameters μ and σ : In this case the hyperprior has the well-known form $f(\mu, \sigma) \propto 1/\sigma^2$. The same result is obtained, if μ and σ are assumed to be independent, and Jeffrey's rule is applied twice. This corresponds to the model implemented by ZEDB.

2nd case

Jeffrey's rule is applied to the parameters $\alpha = E(X)$ and $CF = \frac{\sqrt{VAR(X)}}{E(X)}$ (coefficient of variation). We recall that $\alpha = e^{(\mu+\sigma^2/2)}$ and $CF = \sqrt{e^{\sigma^2}}$. Solving for the parameters μ and σ we obtain:

$$f(x|\alpha, v) = \frac{1}{\sqrt{2\pi x} \sqrt{\ln(v^2 + 1)}} e^{-\frac{\ln \alpha + 1/2 \ln(1+v^2) + \ln x}{2 \ln(1+v^2)}}.$$

The resulting hyperprior in terms of μ and σ has the form

$$f(\mu, \sigma) \propto \sqrt{\frac{2e^{-2\mu-3\sigma^2}(e^{\sigma^2} - 1)}{\sigma^6}}.$$

We note that in [Becker and Hofer, 2001], the resulting hyperprior in the second case was used as being proportional to the expression obtained above without the square root. Of course this approach is false since the Jeffrey's rule is not correctly applied.

3rd case

Jeffrey's rule is applied to the parameters $\alpha = E(X)$ and $b = \sqrt{VAR(X)}$. We recall that $VAR(X) = CF^2\alpha^2$. The distribution of X in terms of α and b now has the form

$$f(x|\alpha, b) = \frac{1}{\sqrt{2\pi x} \sqrt{\ln(1 + \frac{b^2}{\alpha^2})}} e^{-\frac{\ln \alpha + 1/2 \ln(1 + \frac{b^2}{\alpha^2}) + \ln x}{2 \ln(1 + \frac{b^2}{\alpha^2})}}.$$

Just as above, the resulting hyperprior in terms of μ and σ has the form

$$f(\mu, \sigma) \propto \sqrt{\frac{2e^{-4(\mu+\sigma^2)}(e^{\sigma^2} - 1)}{\sigma^6}}.$$

As in case 2, the square root has been omitted in [Becker and Hofer, 2001].

4th case

Jeffrey's rule is applied to the parameters $\alpha = E(X)$ and σ , assuming independence between them. In this case has the following form

$$f(x|\alpha, \sigma) = \frac{1}{\sqrt{2\pi x} \sqrt{\ln(e^{\sigma^2})}} e^{-\frac{\ln \alpha + 1/2 \ln(e^{\sigma^2}) + \ln x}{2 \ln(e^{\sigma^2})}}.$$

Applying Jeffrey's rule twice, one time for each parameter we arrive at the hyperprior

$$f(\mu, \sigma) = \sqrt{\frac{e^{-2\mu - \sigma^2}(2 + 4\sigma^2)}{\sigma^4}}.$$

For each case the steps in the calculation are similar to those for the gamma model, except that in step 1, truncation is applied to the parameters of the lognormal density.

The likelihood $P(X_1, \dots, X_n, T_1, \dots, T_n | \mu, \sigma)$ as a function of μ and σ (7) is presented in Figure 7, with values for $(X_1, \dots, X_n, T_1, \dots, T_n)$ taken from case 1. For uniform hyperpriors, this likelihood is proportional to the hyperposterior distribution $P(\mu, \sigma | X, T)$. Note that, in contrast to Figure 2, $P(\mu, \sigma | X, T)$ does peak. This means that a "natural" truncation for μ and σ can be defined as any rectangle containing the peak. The choice which such rectangle will have negligible influence on the results.

Figure 8 shows the hyperposterior distribution. Again, the contrast with Figures 3 and 4 is striking. The mass is captured within the μ, σ rectangle containing the peak in Figure 7. Figures 9 and 10 show the posterior density and posterior cumulative distribution function.

The results are presented in Table 7, Table 8 and Table 9 for each hyperprior distribution discussed. For case 1 we include the effect of omitting the square root in the Jeffrey prior. The corresponding ZEDB results are shown in each table. The differences are smaller than with the gamma model the differences noted above.

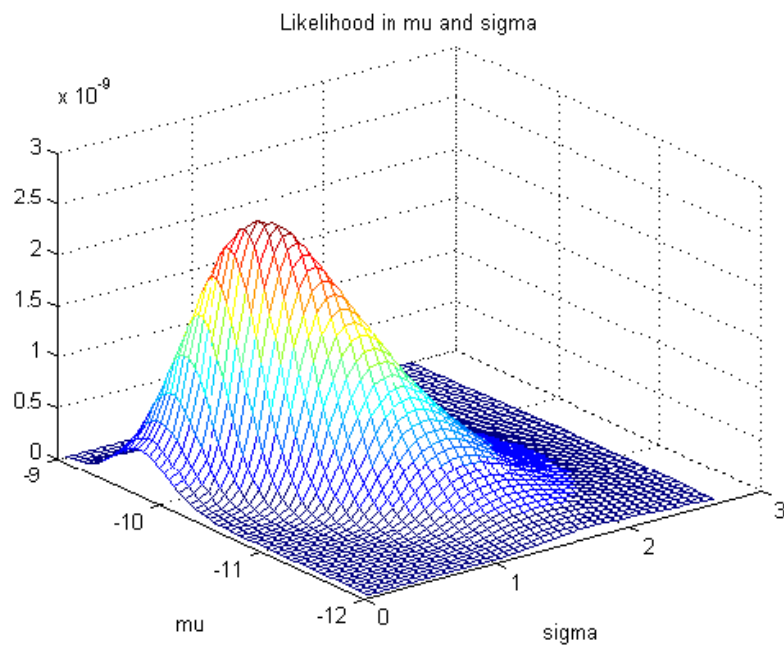


Figure 7

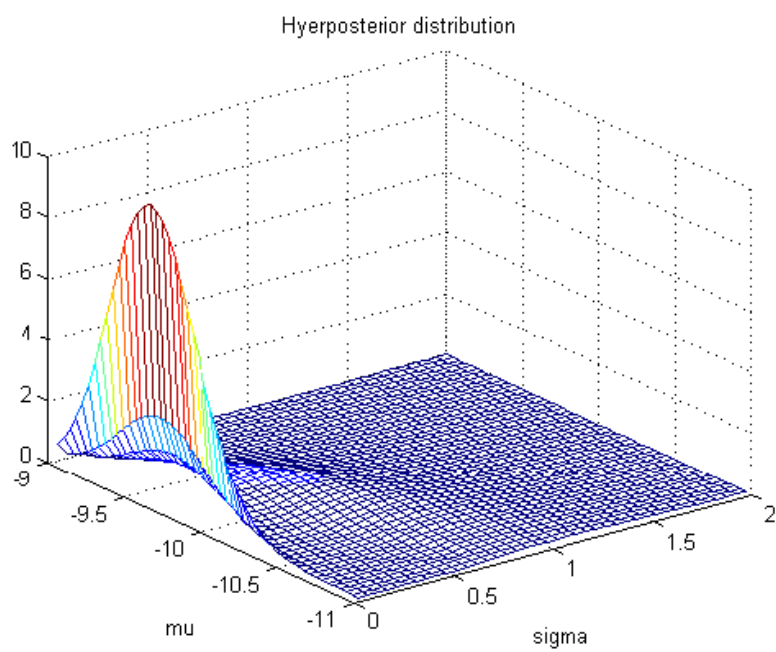


Figure 8

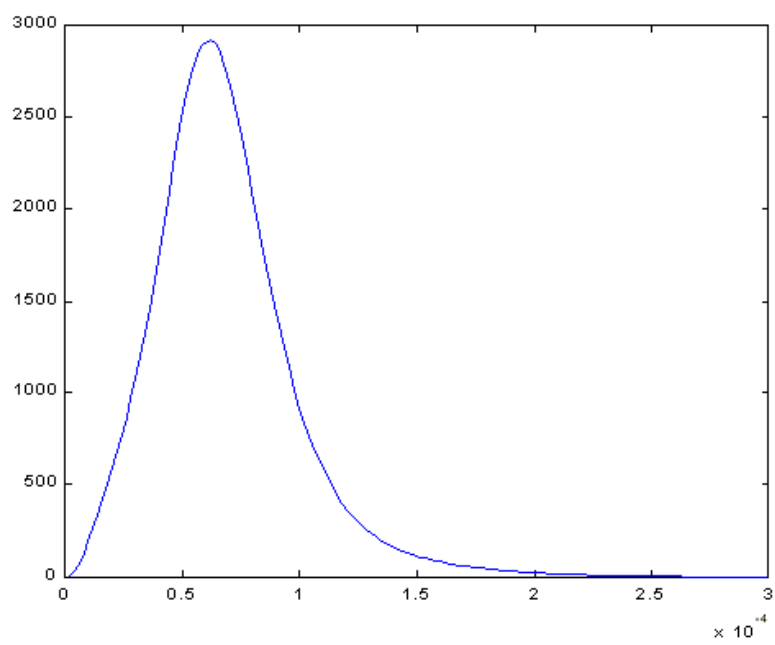


Figure 9

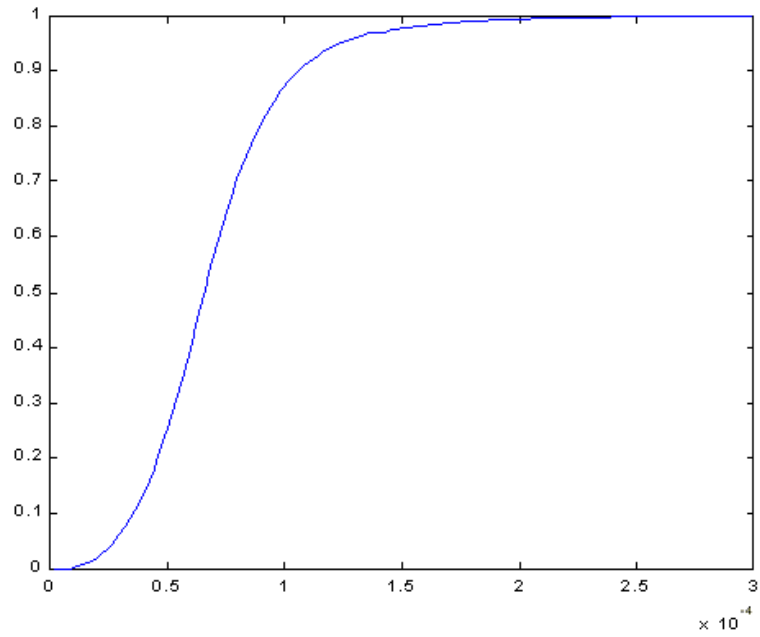


Figure 10

Hyperprior Quantiles	5%	50%	95%
1st case	2.3671 E-5	6.1333 E-5	1.2171 E-4
2nd case w.o.sqrt	3.5021 E-5	5.5355 E-5	8.3108 E-5
2nd case w. sqrt.	2.8031 E-5	5.8052 E-5	1.0671 E-4
3rd case w.o. sqrt	2.9692 E-5	4.8409 E-5	7.4464 E-5
3rd case w. sqrt.	2.5477 E-5	5.4498 E-5	1.0220 E-4
4th case	2.2738 E-5	5.7508 E-5	1.2220 E-4
ZEDB (1rst case)	2.01 E-5	5.91 E-5	1.44 E-4

Table 7. The 5%, 50% and 95% quantiles of the posterior distribution of λ_0 for data set 1.

Hyperprior Quantiles	5%	50%	95%
1st case	2.6037 E-5	7.7139 E-5	1.3811 E-4
2nd case w. sqrt.	7.838 E-6	4.8169 E-5	1.2464 E-4
3rd case w. sqrt.	3.0166 E-6	1.9669 E-5	8.2809 E-5
4th case	8.5411 E-6	4.8516 E-5	1.2635 E-4
ZEDB (1rst case)	2.5387 E-5	7.7103 E-5	2.0576 E-4

Table 8. The 5%, 50% and 95% quantiles of the posterior distribution of λ_0 for data set 2.

Hyperprior Quantiles	5%	50%	95%
1st case	2.4142 E-5	6.9354 E-5	2.5011 E-4
2nd case w. sqrt.	2.1794 E-5	3.7944 E-5	1.3568 E-4
3rd case w. sqrt.	2.1261 E-5	3.2612 E-5	7.8947 E-5
4th case	2.186 E-5	3.8602 E-5	1.4675 E-4
ZEDB (1rst case)	1.12 E-5	6.2277 E-5	3.398 E-4

Table 9. The 5%, 50% and 95% quantiles of the posterior distribution of λ_0 for data set 3.

6.4 Truncation

Using a gamma prior, the method of truncation seems to have a large influence on the posterior distribution of λ . It has been shown in section 5.2.3 that the likelihood in α and β has no maximum, but it is asymptotically maximal along a ridge. This behavior can cause a persistence of impropriety of hyperpriors after observing data. [Cooke et al 1995] showed that different choices of truncation ranges can affect the median and the 95% quantile by a factor 5. In (4), the term $\prod_{i=1..n} \int [P(X_i|\lambda_i)P(\lambda_i|q)d\lambda_i]$ cannot be calculated analytically when we have a lognormal distribution as prior for λ . Hence, we cannot study the asymptotic behavior of the "hyperposterior"

$$P(\mu, \sigma | X_1 \dots X_n, T_1 \dots T_n) \propto P(X_1 \dots X_n, T_1 \dots T_n | \mu, \sigma) P(\mu, \sigma)$$

analytically. Performing numerical integration (Figure 10), one can see that a maximum occurs in the likelihood in μ and σ . Hence, if the parameters of the lognormal distribution μ and σ are truncated in a way that includes the bulk of mass around the maximum, then how they are truncated will not make a significant difference. To save time in the computation process, truncation is performed around the significant values of likelihood in μ and σ (Figure 7). Figure 11 shows this same likelihood, but a larger integration rectangle for μ and σ ; integration over the larger rectangle produces effectively the same result. The intervals for integration over λ , were determined as in [Niemann, 1996]. One remark can be made: if the domains of integration are not large enough the posterior cumulative distribution of λ will not go to one. Using an iterative loop in software implementation, the natural interval of integration can be found.

The possibility of truncating the domains of integration so as to include the bulk of mass around the maximum of the prior is a significant argument in favor of the lognormal prior over the gamma prior. This possibility must be verified in each given dataset. Indeed, problems can arise if the number of failures is very small. The Poisson term will concentrate mass at $\lambda = 0$ if no failures are observed in long observation times. This will drive the likelihood, $\prod P(X_i|\lambda_i)P(\lambda_i|q)$ as a function of q , to values which concentrate mass of λ near zero. From Table 1 we see that if the variance of a gamma distribution is proportional to the mean, hence they go to zero at the same rate. For the lognormal the variance is proportional to the square of the mean, and hence the variance goes to zero faster than the mean. We can anticipate concentration of mass near $\sigma = 0$ in such cases. In any event, with sufficient observation times failures will be observed and the lognormal prior will peak away from zero and hence admit a good truncation heuristic ⁷. For the gamma this is not the case. Figure 12 shows the likelihood of μ, σ for dataset 3. Here a peak is not visible at all, but mass seems to concentrate at $\sigma = 0$. These figures also give the integration ranges for μ, σ , and λ .

⁷The ZEDB code uses a method based on the mean and standard deviation of the sample of raw data in order to achieve this [Software documentation Bayes20].

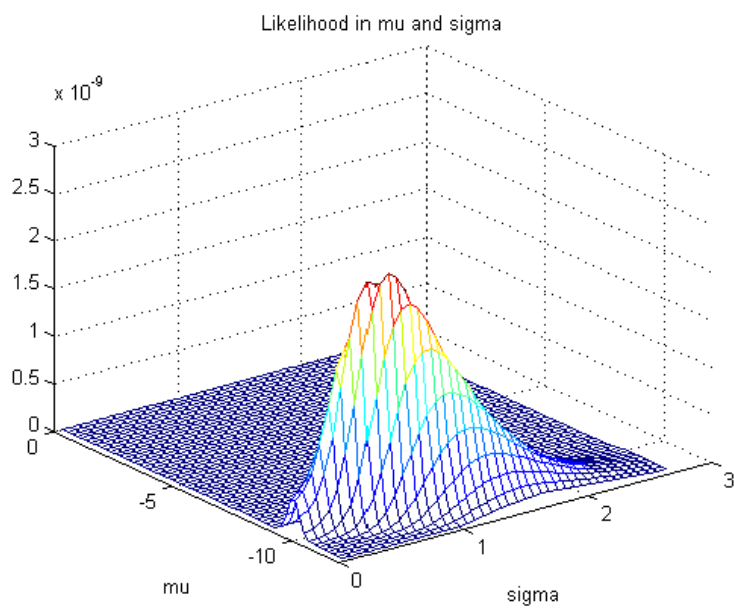


Figure 11 Likelihood μ, σ dataset 1; $\mu = -17.5 \dots -3, \sigma = 0.1 \dots 4, \lambda = 2 * 10^{-6} \dots 3 * 10^{-4}$.

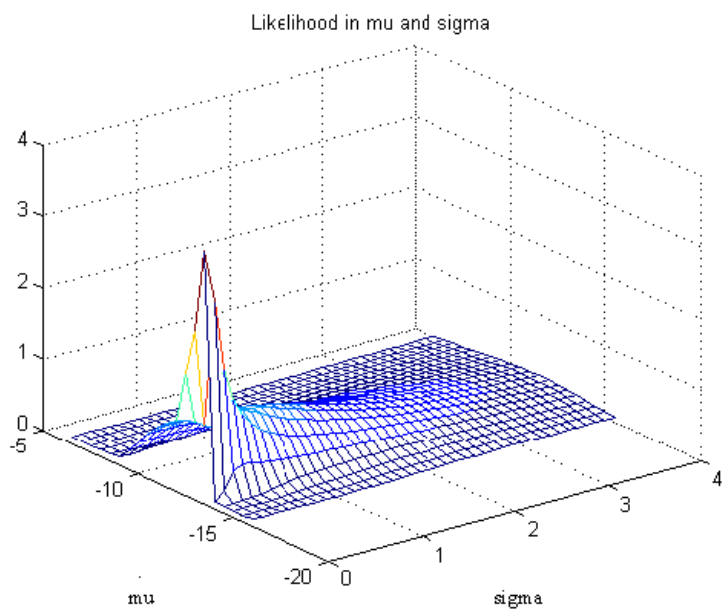


Figure 12 Likelihood μ, σ dataset 3: $\mu = -15.5 \dots -6, \sigma = -0.1 \dots 4, \lambda = 2 * 10^{-5} \dots 3 * 10^{-3}$.

6.5 Conclusions

1. Two stage models provide a valid method for assimilating data from other plants. The conditional independence assumptions, as also used in ZEDB, are reasonable and yield a tractable and mathematically valid form for the failure rate a plant of interest, given failures and operational times at other plants in the population.
2. Choice of hyperprior must be defensible. Non-informativeness is not a good defense if it leads to improper distributions. The influence of the hyperpriors does not decay as observation times get longer, if the number of plants in the population remains fixed.
3. Improper hyperpriors do not always become proper after observations. Improper hyperpriors should be avoided if propriety after observations cannot be demonstrated.
4. The present implementation produced for verification of the implementation of the ZEDB results in agreement with the gamma model of ZEDB which is consistent with previously noted variations due to truncation of the hyperparameters.
5. This present implementation produces good agreement with the lognormal model of ZEDB.
6. The lognormal model, as used in standard ZEDB evaluations, enjoys a significant advantage over the gamma model in that, as observation time increases, a natural truncation of the hyperparameters μ, σ is possible.
7. In the context of a literature survey, Vaurio's one-stage empirical Bayes model has been investigated. It is elegant and simple. It will not work with zero observed failures or with a population of two plants.
8. In the context of a literature survey, Hofer's criticism for the normal two-stage model and his own variant have been investigated. The latter appears to rest on shifting viewpoints involving conflicting assumptions. Consistent application of the standard conditional assumptions collapses his model into the form (4), which he criticizes as a 'wrong chance model'. Further discussion should wait until the conditional independence assumptions and mathematical derivation are clarified.

Appendix

1	2	3.01	5	8.02	11	17	22	31	40	54.56	81
1	2	3.01	5	8.02	12	17	22.42	31	40	55	83.03
1	2	3.04	5.08	8.02	12	17	22.42	31	41	55	84.69
1	2	3.17	5.08	8.63	12	17	23	31	41	55.02	85
1	2	3.40	5.63	8.63	12	17	23	32	41	56	85
1	2	3.48	6	9	12	17	25	32	41.01	58	86
1	2	3.96	6	9	12	17	25	32	42	59	86
1	2	4	7	9	12	18	26	32	42.25	59.4	86
1	2	4	7	9	13	18	26	33	43	60	86
1	2	4	7	9	13	18	26	33.67	43	61.29	87
1	2	4	7	9	13	18	26	34	43	62	91
1	2	4	7	9	13	18	26	34	44	62	91
1	2	4	7	9	13	18	26	34	44	63	91
1	2.63	4	7	9	13.25	18	26	34	44	63	91
1.01	3	4	7	9.08	13.35	19	26.38	34	45.71	63	100
1.01	3	4	7	9.42	14	19	27	34	46	65	100
1.01	3	4.01	7	10	14	19	27	35	48	65	101
1.04	3	4.02	7.83	10	14.01	19	27	35	48	67	105
1.08	3	4.07	8	10	15	19	28	35.04	48.35	70	110
1.17	3	4.19	8	10	15	19.21	28	36	49	72	127
1.21	3	4.25	8	10	15	19.48	28	36	50	72	128
1.21	3	4.63	8	10	15	19.67	28	36	50	72	128
1.35	3	4.75	8	10	15	20	28.01	36	50	73	128
1.71	3	5	8	10	15	20	28.56	37	50	73.06	128
2	3	5	8	10	15.07	20	29	38	50.18	75	135
2	3	5	8	10	15.09	20.10	29.02	38	50.18	77	142.6
2	3	5	8	11	15.09	20.10	30	38	51	77	168
2	3	5	8	11	16.23	21	30	38	51	77	182
2	3	5	8	11	16.42	22	31	40	51	78	203
2	3	5	8.02	11	16.5	22	31	40	52.58	81	

Table 1. Compressor Unit Data without competing risk

1	2	3.01	5	8.02	<u>11</u>	17	<u>22</u>	<u>31</u>	<u>40</u>	<u>54.56</u>	81
1	2	3.01	5	8.02	12	17	<u>22.42</u>	<u>31</u>	<u>40</u>	55	<u>83.03</u>
1	2	3.04	5.08	8.02	<u>12</u>	17	<u>22.42</u>	<u>31</u>	41	55	<u>84.69</u>
1	2	3.17	<u>5.08</u>	8.63	<u>12</u>	17	23	<u>31</u>	41	<u>55.02</u>	<u>85</u>
1	<u>2</u>	<u>3.40</u>	<u>5.63</u>	8.63	<u>12</u>	<u>17</u>	<u>23</u>	32	41	<u>56</u>	<u>85</u>
1	<u>2</u>	<u>3.48</u>	<u>6</u>	9	<u>12</u>	<u>17</u>	25	32	41.01	58	86
1	<u>2</u>	3.96	<u>6</u>	9	<u>12</u>	<u>17</u>	25	32	<u>42</u>	59	<u>86</u>
1	<u>2</u>	4	7	<u>9</u>	<u>12</u>	18	26	32	42.25	<u>59.4</u>	<u>86</u>
1	<u>2</u>	4	7	<u>9</u>	13	18	26	<u>33</u>	43	60	<u>86</u>
<u>1</u>	<u>2</u>	4	7	<u>9</u>	13	18	26	<u>33.67</u>	43	61.29	87
<u>1</u>	<u>2</u>	4	7	<u>9</u>	<u>13</u>	18	26	34	43	62	91
<u>1</u>	<u>2</u>	4	7	<u>9</u>	<u>13</u>	<u>18</u>	26	<u>34</u>	44	62	91
<u>1</u>	<u>2</u>	4	7	<u>9</u>	<u>13</u>	<u>18</u>	<u>26</u>	<u>34</u>	<u>44</u>	63	91
<u>1</u>	<u>2.63</u>	4	7	<u>9</u>	13.25	<u>18</u>	<u>26</u>	<u>34</u>	<u>44</u>	<u>63</u>	91
<u>1.01</u>	3	<u>4</u>	7	<u>9.08</u>	<u>13.35</u>	19	<u>26.38</u>	<u>34</u>	45.71	<u>63</u>	100
<u>1.01</u>	3	<u>4</u>	<u>7</u>	9.42	14	19	<u>27</u>	<u>34</u>	46	65	100
1.01	3	<u>4.01</u>	<u>7</u>	10	14	19	<u>27</u>	35	<u>48</u>	<u>65</u>	101
<u>1.04</u>	3	4.02	7.83	10	<u>14.01</u>	19	<u>27</u>	35	<u>48</u>	<u>67</u>	105
<u>1.08</u>	3	4.07	8	10	15	19	28	<u>35.04</u>	48.35	70	110
<u>1.17</u>	3	4.19	8	10	15	19.21	<u>28</u>	36	<u>49</u>	72	<u>127</u>
1.21	3	4.25	8	<u>10</u>	15	<u>19.48</u>	28	36	50	<u>72</u>	<u>128</u>
<u>1.21</u>	3	4.63	8	<u>10</u>	15	19.67	<u>28</u>	<u>36</u>	<u>50</u>	<u>72</u>	<u>128</u>
<u>1.35</u>	3	<u>4.75</u>	<u>8</u>	<u>10</u>	<u>15</u>	20	28.01	<u>36</u>	<u>50</u>	73	<u>128</u>
1.71	3	5	<u>8</u>	<u>10</u>	<u>15</u>	20	28.56	37	<u>50</u>	73.06	<u>128</u>
2	3	5	<u>8</u>	<u>10</u>	15.07	20	<u>29</u>	38	50.18	75	135
2	<u>3</u>	5	<u>8</u>	<u>10</u>	<u>15.09</u>	20.10	<u>29.02</u>	<u>38</u>	50.18	77	<u>142.6</u>
2	<u>3</u>	5	<u>8</u>	11	<u>15.09</u>	20.10	30	<u>38</u>	51	<u>77</u>	<u>168</u>
2	<u>3</u>	5	<u>8</u>	11	<u>16.23</u>	21	30	<u>38</u>	51	<u>77</u>	<u>182</u>
2	<u>3</u>	5	<u>8</u>	11	16.42	<u>22</u>	31	<u>40</u>	51	<u>78</u>	203
2	<u>3</u>	5	8.02	11	16.5	<u>22</u>	31	<u>40</u>	52.58	81	

Table 2 Competing risk data for “Failure Mode” field; Risk1: LEK, IST, UST, VIB, Risk2: LIG, STG, VAR, OVH, ANN (Risk2 is underlined)

<u>1</u>	<u>2</u>	<u>3.01</u>	<u>5</u>	8	<u>11</u>	<u>17</u>	22	<u>31</u>	<u>40</u>	54.56	<u>81</u>
<u>1</u>	<u>2</u>	<u>3.01</u>	<u>5</u>	8.02	<u>12</u>	<u>17</u>	<u>22.42</u>	<u>31</u>	<u>40</u>	55	<u>83.03</u>
<u>1</u>	<u>2</u>	<u>3.04</u>	<u>5.08</u>	8.02	<u>12</u>	<u>17</u>	<u>22.42</u>	<u>31</u>	<u>41</u>	<u>55</u>	84.69
<u>1</u>	<u>2</u>	<u>3.17</u>	<u>5.08</u>	<u>8.63</u>	<u>12</u>	<u>17</u>	23	31	41	<u>55.02</u>	<u>85</u>
<u>1</u>	<u>2</u>	<u>3.40</u>	<u>5.63</u>	<u>8.63</u>	<u>12</u>	17	23	<u>32</u>	41	<u>56</u>	<u>85</u>
<u>1</u>	<u>2</u>	<u>3.48</u>	<u>6</u>	<u>9</u>	<u>12</u>	17	<u>25</u>	<u>32</u>	41.01	58	<u>86</u>
<u>1</u>	<u>2</u>	<u>3.96</u>	<u>6</u>	<u>9</u>	<u>12</u>	17	<u>25</u>	<u>32</u>	42	59	<u>86</u>
<u>1</u>	<u>2</u>	<u>4</u>	<u>7</u>	<u>9</u>	<u>12</u>	<u>18</u>	<u>26</u>	<u>32</u>	<u>42.25</u>	<u>59.4</u>	<u>86</u>
1	<u>2</u>	<u>4</u>	<u>7</u>	<u>9</u>	<u>13</u>	<u>18</u>	<u>26</u>	<u>33</u>	<u>43</u>	<u>60</u>	<u>86</u>
1	<u>2</u>	<u>4</u>	<u>7</u>	<u>9</u>	<u>13</u>	<u>18</u>	<u>26</u>	<u>33.67</u>	<u>43</u>	61.29	<u>87</u>
1	<u>2</u>	<u>4</u>	<u>7</u>	<u>9</u>	<u>13</u>	<u>18</u>	<u>26</u>	<u>34</u>	<u>43</u>	<u>62</u>	<u>91</u>
1	<u>2</u>	<u>4</u>	<u>7</u>	<u>9</u>	<u>13</u>	18	<u>26</u>	<u>34</u>	<u>44</u>	<u>62</u>	<u>91</u>
1	<u>2</u>	<u>4</u>	<u>7</u>	<u>9</u>	<u>13</u>	18	<u>26</u>	<u>34</u>	<u>44</u>	<u>63</u>	<u>91</u>
1	2.63	<u>4</u>	<u>7</u>	<u>9</u>	13.25	18	<u>26</u>	<u>34</u>	<u>44</u>	<u>63</u>	<u>91</u>
<u>1.01</u>	<u>3</u>	<u>4</u>	<u>7</u>	<u>9.08</u>	<u>13.35</u>	<u>19</u>	<u>26.38</u>	<u>34</u>	<u>45.71</u>	<u>63</u>	<u>100</u>
<u>1.01</u>	<u>3</u>	<u>4</u>	<u>7</u>	<u>9.42</u>	<u>14</u>	<u>19</u>	<u>27</u>	<u>34</u>	<u>46</u>	65	100
<u>1.01</u>	<u>3</u>	<u>4.01</u>	<u>7</u>	<u>10</u>	<u>14</u>	<u>19</u>	<u>27</u>	<u>35</u>	<u>48</u>	65	<u>101</u>
<u>1.04</u>	<u>3</u>	<u>4.02</u>	<u>7.83</u>	<u>10</u>	<u>14.01</u>	19	<u>27</u>	<u>35</u>	<u>48</u>	<u>67</u>	<u>105</u>
<u>1.08</u>	<u>3</u>	<u>4.07</u>	<u>8</u>	<u>10</u>	<u>15</u>	19	<u>28</u>	<u>35.04</u>	48.35	<u>70</u>	110
<u>1.17</u>	<u>3</u>	<u>4.19</u>	<u>8</u>	<u>10</u>	<u>15</u>	19.21	<u>28</u>	<u>36</u>	<u>49</u>	<u>72</u>	<u>127</u>
<u>1.21</u>	<u>3</u>	<u>4.25</u>	<u>8</u>	<u>10</u>	<u>15</u>	19.48	28	<u>36</u>	<u>50</u>	<u>72</u>	<u>128</u>
<u>1.21</u>	<u>3</u>	<u>4.63</u>	<u>8</u>	<u>10</u>	<u>15</u>	19.67	28	<u>36</u>	<u>50</u>	<u>72</u>	<u>128</u>
<u>1.35</u>	<u>3</u>	<u>4.75</u>	<u>8</u>	<u>10</u>	<u>15</u>	<u>20</u>	<u>28.01</u>	<u>36</u>	<u>50</u>	<u>73</u>	<u>128</u>
<u>1.71</u>	<u>3</u>	<u>5</u>	<u>8</u>	<u>10</u>	15	<u>20</u>	<u>28.56</u>	<u>37</u>	<u>50</u>	<u>73.06</u>	<u>128</u>
<u>2</u>	<u>3</u>	<u>5</u>	<u>8</u>	<u>10</u>	<u>15.07</u>	<u>20</u>	<u>29</u>	<u>38</u>	50.18	75	<u>135</u>
<u>2</u>	<u>3</u>	<u>5</u>	<u>8</u>	<u>10</u>	<u>15.09</u>	<u>20.10</u>	<u>29.02</u>	<u>38</u>	50.18	<u>77</u>	<u>142.6</u>
<u>2</u>	<u>3</u>	<u>5</u>	<u>8</u>	<u>11</u>	<u>15.09</u>	<u>20.10</u>	<u>30</u>	<u>38</u>	<u>51</u>	<u>77</u>	168
<u>2</u>	<u>3</u>	<u>5</u>	<u>8</u>	<u>11</u>	16.23	<u>21</u>	<u>30</u>	<u>38</u>	<u>51</u>	<u>77</u>	<u>182</u>
<u>2</u>	<u>3</u>	<u>5</u>	<u>8</u>	<u>11</u>	<u>16.42</u>	<u>22</u>	<u>31</u>	<u>40</u>	<u>51</u>	<u>78</u>	<u>203</u>
<u>2</u>	<u>3</u>	<u>5</u>	<u>8</u>	<u>11</u>	<u>16.5</u>	<u>22</u>	<u>31</u>	<u>40</u>	<u>52.58</u>	<u>81</u>	

Table 3 Competing risk data for “Action Taken” field; Risk1: AKU, Risk2: AKR, JUS, OVH, MOD, ANN (Risk2 is underlined)

1	<u>2</u>	<u>3.01</u>	<u>5</u>	8.02	<u>11</u>	<u>17</u>	<u>22</u>	<u>31</u>	<u>40</u>	<u>54.56</u>	<u>81</u>
1	<u>2</u>	<u>3.01</u>	<u>5</u>	8.02	<u>12</u>	<u>17</u>	<u>22.42</u>	<u>31</u>	<u>40</u>	<u>55</u>	<u>83.03</u>
1	<u>2</u>	<u>3.04</u>	<u>5.08</u>	8.02	<u>12</u>	<u>17</u>	<u>22.42</u>	<u>31</u>	<u>41</u>	<u>55</u>	<u>84.69</u>
1	<u>2</u>	<u>3.17</u>	<u>5.08</u>	8.63	<u>12</u>	<u>17</u>	23	<u>31</u>	<u>41</u>	<u>55.02</u>	<u>85</u>
<u>1</u>	<u>2</u>	<u>3.40</u>	<u>5.63</u>	8.63	<u>12</u>	<u>17</u>	23	<u>32</u>	<u>41</u>	<u>56</u>	<u>85</u>
<u>1</u>	<u>2</u>	<u>3.48</u>	<u>6</u>	9	<u>12</u>	<u>17</u>	25	<u>32</u>	<u>41.01</u>	<u>58</u>	<u>86</u>
<u>1</u>	<u>2</u>	<u>3.96</u>	<u>6</u>	9	<u>12</u>	<u>17</u>	<u>25</u>	<u>32</u>	<u>42</u>	<u>59</u>	<u>86</u>
<u>1</u>	<u>2</u>	<u>4</u>	<u>7</u>	9	<u>12</u>	<u>18</u>	<u>26</u>	<u>32</u>	<u>42.25</u>	<u>59.4</u>	<u>86</u>
<u>1</u>	<u>2</u>	<u>4</u>	<u>7</u>	9	<u>13</u>	<u>18</u>	<u>26</u>	<u>33</u>	<u>43</u>	<u>60</u>	<u>86</u>
<u>1</u>	<u>2</u>	<u>4</u>	<u>7</u>	9	<u>13</u>	<u>18</u>	<u>26</u>	<u>33.67</u>	<u>43</u>	<u>61.29</u>	<u>87</u>
<u>1</u>	<u>2</u>	<u>4</u>	<u>7</u>	9	<u>13</u>	<u>18</u>	<u>26</u>	<u>34</u>	<u>43</u>	<u>62</u>	<u>91</u>
<u>1</u>	<u>2</u>	<u>4</u>	<u>7</u>	9	<u>13</u>	<u>18</u>	<u>26</u>	<u>34</u>	<u>44</u>	<u>62</u>	<u>91</u>
<u>1</u>	<u>2</u>	<u>4</u>	<u>7</u>	9	<u>13</u>	<u>18</u>	<u>26</u>	<u>34</u>	<u>44</u>	<u>63</u>	<u>91</u>
<u>1</u>	<u>2.63</u>	<u>4</u>	<u>7</u>	9	<u>13.25</u>	<u>18</u>	<u>26</u>	<u>34</u>	<u>44</u>	<u>63</u>	<u>91</u>
<u>1.01</u>	<u>3</u>	<u>4</u>	<u>7</u>	<u>9.08</u>	<u>13.35</u>	19	<u>26.38</u>	<u>34</u>	<u>45.71</u>	<u>63</u>	<u>100</u>
<u>1.01</u>	<u>3</u>	<u>4</u>	<u>7</u>	9.42	<u>14</u>	19	<u>27</u>	<u>34</u>	<u>46</u>	<u>65</u>	<u>100</u>
<u>1.01</u>	<u>3</u>	<u>4.01</u>	<u>7</u>	10	<u>14</u>	19	<u>27</u>	<u>35</u>	<u>48</u>	<u>65</u>	<u>101</u>
<u>1.04</u>	<u>3</u>	<u>4.02</u>	<u>7.83</u>	10	<u>14.01</u>	19	<u>27</u>	<u>35</u>	<u>48</u>	<u>67</u>	<u>105</u>
<u>1.08</u>	<u>3</u>	<u>4.07</u>	<u>8</u>	10	<u>15</u>	19	28	<u>35.04</u>	<u>48.35</u>	<u>70</u>	<u>110</u>
<u>1.17</u>	<u>3</u>	<u>4.19</u>	<u>8</u>	10	<u>15</u>	19.21	28	<u>36</u>	<u>49</u>	<u>72</u>	<u>127</u>
<u>1.21</u>	<u>3</u>	<u>4.25</u>	<u>8</u>	10	<u>15</u>	<u>19.48</u>	28	<u>36</u>	<u>50</u>	<u>72</u>	<u>128</u>
<u>1.21</u>	<u>3</u>	<u>4.63</u>	<u>8</u>	10	<u>15</u>	<u>19.67</u>	28	<u>36</u>	<u>50</u>	<u>72</u>	<u>128</u>
<u>1.35</u>	<u>3</u>	<u>4.75</u>	<u>8</u>	10	<u>15</u>	<u>20</u>	28.01	<u>36</u>	<u>50</u>	<u>73</u>	<u>128</u>
1.71	<u>3</u>	<u>5</u>	<u>8</u>	10	<u>15</u>	<u>20</u>	<u>28.56</u>	<u>37</u>	<u>50</u>	<u>73.06</u>	<u>128</u>
2	<u>3</u>	<u>5</u>	<u>8</u>	10	<u>15.07</u>	<u>20</u>	<u>29</u>	<u>38</u>	<u>50.18</u>	<u>75</u>	<u>135</u>
2	<u>3</u>	<u>5</u>	<u>8</u>	10	<u>15.09</u>	20.10	<u>29.02</u>	<u>38</u>	<u>50.18</u>	<u>77</u>	<u>142.6</u>
2	<u>3</u>	<u>5</u>	<u>8</u>	11	<u>15.09</u>	20.10	30	<u>38</u>	<u>51</u>	<u>77</u>	<u>168</u>
<u>2</u>	<u>3</u>	<u>5</u>	<u>8</u>	11	<u>16.23</u>	<u>21</u>	<u>30</u>	<u>38</u>	<u>51</u>	<u>77</u>	<u>182</u>
<u>2</u>	<u>3</u>	<u>5</u>	<u>8</u>	11	<u>16.42</u>	22	31	<u>40</u>	<u>51</u>	<u>78</u>	<u>203</u>
<u>2</u>	<u>3</u>	<u>5</u>	8.02	<u>11</u>	<u>16.5</u>	<u>22</u>	<u>31</u>	<u>40</u>	<u>52.58</u>	<u>81</u>	

Table 4 Competing risk data for “Degree of Failure” field; Risk1: critical failure, Risk2: non-critical failure (Risk2 is underlined)

1	2	3.01	<u>5</u>	8.02	<u>11</u>	17	22	<u>31</u>	<u>40</u>	54.56	81
1	2	3.01	<u>5</u>	<u>8.02</u>	12	<u>17</u>	<u>22.42</u>	<u>31</u>	<u>40</u>	55	<u>83.03</u>
1	2	<u>3.04</u>	<u>5.08</u>	<u>8.02</u>	12	<u>17</u>	<u>22.42</u>	<u>31</u>	41	55	<u>84.69</u>
1	2	3.17	<u>5.08</u>	8.63	12	<u>17</u>	23	<u>31</u>	<u>41</u>	<u>55.02</u>	<u>85</u>
1	2	<u>3.40</u>	5.63	<u>8.63</u>	12	<u>17</u>	23	<u>32</u>	<u>41</u>	56	<u>85</u>
1	2	3.48	6	9	12	<u>17</u>	25	<u>32</u>	<u>41.01</u>	58	86
1	2	<u>3.96</u>	6	9	<u>12</u>	<u>17</u>	25	<u>32</u>	<u>42</u>	<u>59</u>	86
1	<u>2</u>	4	7	9	<u>12</u>	18	26	<u>32</u>	<u>42.25</u>	<u>59.4</u>	<u>86</u>
1	<u>2</u>	4	7	9	13	18	26	33	43	<u>60</u>	<u>86</u>
1	<u>2</u>	4	7	9	13	18	26	33.67	43	61.29	<u>87</u>
1	<u>2</u>	4	7	9	13	<u>18</u>	26	34	<u>43</u>	<u>62</u>	91
<u>1</u>	<u>2</u>	<u>4</u>	7	<u>9</u>	<u>13</u>	<u>18</u>	26	34	44	<u>62</u>	91
<u>1</u>	<u>2</u>	<u>4</u>	7	<u>9</u>	<u>13</u>	<u>18</u>	<u>26</u>	34	44	63	91
<u>1</u>	2.63	<u>4</u>	<u>7</u>	<u>9</u>	13.25	<u>18</u>	<u>26</u>	34	<u>44</u>	63	91
1.01	3	<u>4</u>	<u>7</u>	9.08	13.35	19	26.38	34	45.71	63	100
1.01	3	<u>4</u>	<u>7</u>	<u>9.42</u>	14	19	27	34	<u>46</u>	65	100
<u>1.01</u>	3	<u>4.01</u>	<u>7</u>	10	<u>14</u>	19	27	<u>35</u>	<u>48</u>	65	101
<u>1.04</u>	3	<u>4.02</u>	7.83	10	<u>14.01</u>	19	27	<u>35</u>	<u>48</u>	<u>67</u>	105
1.08	3	4.07	8	10	15	<u>19</u>	28	35.04	<u>48.35</u>	70	110
1.17	3	4.19	8	10	15	19.21	28	36	49	72	<u>127</u>
1.21	3	<u>4.25</u>	8	10	15	19.48	28	36	50	72	<u>128</u>
<u>1.21</u>	3	4.63	8	10	15	<u>19.67</u>	28	36	50	72	<u>128</u>
<u>1.35</u>	<u>3</u>	4.75	8	<u>10</u>	15	20	<u>28.01</u>	<u>36</u>	<u>50</u>	73	<u>128</u>
<u>1.71</u>	<u>3</u>	5	8	<u>10</u>	15	20	28.56	<u>37</u>	<u>50</u>	<u>73.06</u>	<u>128</u>
2	<u>3</u>	5	8	<u>10</u>	<u>15.07</u>	20	29	38	<u>50.18</u>	75	<u>135</u>
2	<u>3</u>	5	8	<u>10</u>	15.09	<u>20.10</u>	<u>29.02</u>	38	<u>50.18</u>	77	<u>142.6</u>
2	<u>3</u>	5	8	11	15.09	<u>20.10</u>	30	38	51	77	168
2	<u>3</u>	<u>5</u>	<u>8</u>	11	16.23	21	<u>30</u>	38	51	<u>77</u>	<u>182</u>
2	<u>3</u>	<u>5</u>	<u>8</u>	11	16.42	22	31	40	<u>51</u>	78	<u>203</u>
2	<u>3</u>	<u>5</u>	8.02	11	16.5	22	<u>31</u>	40	<u>52.58</u>	81	

Table 5 Competing risk data for “System” field; Risk1: SPE, ELM, GEA, Risk2: SMO, KOM, INS (Risk2 is underlined)

1	2	3.01	<u>5</u>	<u>8.02</u>	<u>11</u>	17	<u>22</u>	<u>31</u>	<u>40</u>	<u>54.56</u>	81
1	<u>2</u>	3.01	5	8.02	<u>12</u>	17	<u>22.42</u>	<u>31</u>	<u>40</u>	55	<u>83.03</u>
1	<u>2</u>	3.04	<u>5.08</u>	<u>8.02</u>	<u>12</u>	17	<u>22.42</u>	<u>31</u>	41	55	<u>84.69</u>
1	<u>2</u>	3.17	<u>5.08</u>	<u>8.63</u>	<u>12</u>	17	<u>23</u>	<u>31</u>	41	<u>55.02</u>	<u>85</u>
1	<u>2</u>	<u>3.40</u>	<u>5.63</u>	<u>8.63</u>	<u>12</u>	<u>17</u>	<u>23</u>	<u>32</u>	<u>41</u>	<u>56</u>	<u>85</u>
1	<u>2</u>	<u>3.48</u>	<u>6</u>	9	<u>12</u>	<u>17</u>	25	<u>32</u>	<u>41.01</u>	58	<u>86</u>
1	<u>2</u>	<u>3.96</u>	<u>6</u>	9	<u>12</u>	<u>17</u>	<u>25</u>	<u>32</u>	42	59	<u>86</u>
<u>1</u>	<u>2</u>	4	7	9	<u>12</u>	18	26	<u>32</u>	<u>42.25</u>	<u>59.4</u>	<u>86</u>
<u>1</u>	<u>2</u>	4	7	9	<u>13</u>	18	26	<u>33</u>	43	60	<u>86</u>
<u>1</u>	<u>2</u>	4	7	9	<u>13</u>	18	26	<u>33.67</u>	43	<u>61.29</u>	<u>87</u>
<u>1</u>	<u>2</u>	4	7	9	<u>13</u>	18	26	<u>34</u>	<u>43</u>	<u>62</u>	91
<u>1</u>	<u>2</u>	4	7	9	<u>13</u>	18	26	<u>34</u>	<u>44</u>	<u>62</u>	91
<u>1</u>	<u>2</u>	4	7	9	<u>13</u>	18	26	<u>34</u>	<u>44</u>	63	91
<u>1</u>	<u>2.63</u>	4	7	9	<u>13.25</u>	18	26	<u>34</u>	<u>44</u>	<u>63</u>	91
<u>1.01</u>	3	4	7	<u>9.08</u>	<u>13.35</u>	19	<u>26.38</u>	<u>34</u>	45.71	<u>63</u>	100
<u>1.01</u>	3	4	7	<u>9.42</u>	14	<u>19</u>	<u>27</u>	<u>34</u>	46	65	<u>100</u>
<u>1.01</u>	3	<u>4.01</u>	7	10	14	<u>19</u>	<u>27</u>	<u>35</u>	<u>48</u>	<u>65</u>	101
<u>1.04</u>	3	<u>4.02</u>	<u>7.83</u>	<u>10</u>	<u>14.01</u>	19	<u>27</u>	<u>35</u>	<u>48</u>	<u>67</u>	<u>105</u>
<u>1.08</u>	3	4.07	8	<u>10</u>	15	<u>19</u>	28	<u>35.04</u>	48.35	70	110
<u>1.17</u>	3	4.19	8	<u>10</u>	15	<u>19.21</u>	<u>28</u>	<u>36</u>	<u>49</u>	<u>72</u>	<u>127</u>
1.21	3	4.25	8	<u>10</u>	15	<u>19.48</u>	28	<u>36</u>	50	<u>72</u>	<u>128</u>
<u>1.21</u>	3	4.63	8	<u>10</u>	15	19.67	<u>28</u>	<u>36</u>	<u>50</u>	<u>72</u>	<u>128</u>
<u>1.35</u>	<u>3</u>	<u>4.75</u>	8	<u>10</u>	15	<u>20</u>	<u>28.01</u>	<u>36</u>	<u>50</u>	73	<u>128</u>
<u>1.71</u>	<u>3</u>	5	8	<u>10</u>	15	<u>20</u>	28.56	<u>37</u>	<u>50</u>	<u>73.06</u>	<u>128</u>
2	<u>3</u>	5	8	<u>10</u>	<u>15.07</u>	<u>20</u>	<u>29</u>	<u>38</u>	<u>50.18</u>	75	<u>135</u>
2	<u>3</u>	5	8	<u>10</u>	<u>15.09</u>	<u>20.10</u>	<u>29.02</u>	<u>38</u>	<u>50.18</u>	77	<u>142.6</u>
2	<u>3</u>	5	8	11	<u>15.09</u>	<u>20.10</u>	<u>30</u>	<u>38</u>	51	<u>77</u>	<u>168</u>
2	<u>3</u>	<u>5</u>	8	11	<u>16.23</u>	21	<u>30</u>	<u>38</u>	<u>51</u>	<u>77</u>	<u>182</u>
2	<u>3</u>	<u>5</u>	8	11	<u>16.42</u>	<u>22</u>	31	<u>40</u>	<u>51</u>	<u>78</u>	203
2	<u>3</u>	5	<u>8.02</u>	<u>11</u>	16.5	<u>22</u>	31	<u>40</u>	52.58	81	

Table 6 Competing risk data for “Failure Mode” field; Risk1: LEK, Risk2: VIB, IST, UST, LIG, STG, VAR, OVH, ANN (Risk2 is underlined)

GLOSSARY

Alternative hypothesis - It assumes the opposite of the null hypothesis. It is given the symbol H_1 . It is impossible to locate the sampling distribution for the alternative hypothesis.

Central limit theorem - basic law of statistics that relates different probability distributions to each other. It comprises three basic statements concerning sampling from populations:

(i) As sample size increases, the distribution of scores tends more and more towards the theoretical distribution of the population being sampled.

(ii) As sample size increases, the sampling distribution of any statistic becomes more Normally distributed.

(iii) Given we have a large enough sample size, we can use one probability distribution to model another. There are "pathways" connecting different probability distributions. In particular, we can model virtually any other type of distribution using the Normal distribution.

Competing risk - a failure time is observed together with a cause or risk, the latter being one of several possibilities. In the basic approach only one cause is responsible for the failure, but in complex situation multiple causes are possible.

Component - the smallest partition of a system that is necessary and sufficient to be considered for system analysis.

Copula - the copula of two random variables X and Y with cumulative density functions $F_X(X)$ and $F_Y(Y)$, is the distribution C on the unit square $[0, 1]^2$ of the pair $(F_X(X), F_Y(Y))$. The functional form of $C : [0, 1]^2 \rightarrow \mathbf{R}$ is

$$C(u, v) \equiv H(F_X^{-1}(u), F_Y^{-1}(v)),$$

where H is the joint distribution function of (X, Y) and F_X^{-1} and F_Y^{-1} are the right-continuous inverse functions of F_X and F_Y .

Covariance - the covariance of two random variable X and Y is the real number, if it exists, defined by:

$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])],$$

where $E[X]$ is the expectation of X

Critical event - The system loses one or more essential functions and important

damages to the system and its environment are caused.

Distribution function - Let X be a random variable. The distribution function of X is:

$$F_X(x) = Pr\{X \leq x\}$$

Down time - The component is not capable to accomplish the required function during this period of time

Expectation, mean (of a random variable) - The real number, if it exists, defined:

1. for a discrete random variable X taking values x_i with probability p_i , by the sum

$$E[X] = \sum_i p_i x_i$$

2. for a continuous random variable X with density function $f(x)$, by the integral

$$E[X] = \int x f(x) dx$$

Failure mode - the effect by which a failure is observed

Failure rate - The rate at which failures occur as a function of time. If T denotes the time to failure of an item, the failure rate $r(t)$ is defined as

$$r(t) = \lim_{\Delta t \rightarrow \infty} \frac{Pr\{t < T \leq t + \Delta t | T > t\}}{\Delta t}$$

Hypothesis - a measurable aspect of a theory: see null hypothesis; alternative hypothesis.

Item - All element, component, sub-system, system, socket, equipment, unit, which can be considered individually

Kendall's tau - Let (X_1, Y_1) and (X_2, Y_2) be i.i.d. random copies of the vector (X, Y) , each with joint distribution H , then Kendall's tau is defined as the probability of concordance minus the probability of discordance:

$$\tau(X, Y) = Pr\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - Pr\{(X_1 - X_2)(Y_1 - Y_2) < 0\}$$

or

$$\tau(X, Y) = Pr\{sgn(X_1 - X_2) = sgn(Y_1 - Y_2)\} - Pr\{sgn(X_1 - X_2) \neq sgn(Y_1 - Y_2)\}.$$

Maintenance - combination of all related technical and administrative actions, including supervision and check operations, performed in order to maintain or to put back an item in a state allowing it to accomplish a required function

- Corrective maintenance - Maintenance performed after failure detection and destined to restore an item in a state allowing it to accomplish a required function
- Preventive maintenance - Maintenance performed at regular intervals or following specified criteria and destined to reduce the probability of failure or the degradation of an item
- Scheduled maintenance - Preventive maintenance performed based on a calendar time
- Unscheduled maintenance - Preventive maintenance performed not based on a calendar time, but based on the relative information over the state of the item

Null hypothesis - the hypothesis to be tested by a statistical test. It assumes the opposite of the alternative hypothesis. It is given the symbol H_0 . Only the null hypothesis can be tested statistically because one can only specify the sampling distribution for a statistic assuming a particular value for its location - in most cases, 0 for the null hypothesis.

One-tailed test - a statistical test in which the statistic to be tested is assumed to be either positive or negative, but not both. Hence, the direction of a relationship between variables (or a difference between groups) is specified. The alternative hypothesis sampling distribution is present only on one side of the null hypothesis sampling distribution; hence the region of rejection of H_0 is placed on only one tail of the null hypothesis sampling distribution.

Operating time - The component accomplishes the required function during this period of time

Percentile - Let X be a random variable with distribution function $F(x)$. The upper $100\alpha\%$ percentile x_α of the distribution $F(x)$ is defined as:

$$Pr\{X > x_\alpha\} = \alpha$$

Probability - the likelihood or chance of an event occurring. It generally ranges from 0 (an impossible event that can never occur) to 1 (a certain event that must always occur). There are four specific definitions of probability:

(i) Subjective probability. This is a rating of the likelihood of an event based on personal experience and opinion, and sometimes on personal knowledge. Example: how likely am I to successfully hail a taxi at 3 a.m. on Stockton High Street?

(ii) Empirical probability. This is an estimate of the likelihood of an event based on experiment. Example: toss a coin 20 times and get 12 heads. The probability of a head is then $12/20 = 0.6$.

(iii) Logical probability. This is calculating the likelihood of an event based on the logical structure of the situation. Example: there are six sides to a die and each one is assumed to have an equal probability of occurrence. Hence, there are six events, each of which has a probability value of $1/6$.

(iv) Axiomatic probability. This is calculating the likelihood of an event by setting up an event space. It uses set theory to represent events and their conjunctions.

Probability density function - Let X be a random variable. The probability density function $f_X(x)$ of X is:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Reliability (performance) - aptitude of an item (system) to accomplish a required function, in given conditions, during a given period of time

Reliability (measure) - probability of an item (system) to accomplish a required function, in given conditions, during a given period of time

Repair - the part of corrective maintenance in which manual actions are performed on the item

Risk - Risk is the potential of loss or damage resulting from exposure to a hazard.

Safety - Safety represents an acceptable level of risk relative to the benefits derived from the hazards-causing activity.

Significance - an arbitrary criterion set by consensus and expressed in terms of a probability value. It tells us how "uncertain" we want to be in conducting statistical analysis.

Spearman's rho - Let X and Y be continuous random variables then the Spearman's rho is defined as the product moment correlation of $F_X(X)$ and $F_Y(Y)$:

$$\rho_r(x, y) = \rho(F_X(X), F_Y(Y)) = \frac{\text{Cov}\{F_X(X), F_Y(Y)\}}{\sqrt{\text{Var}\{F_X(X)\}\text{Var}\{F_Y(Y)\}}}.$$

Two-tailed test - a statistical test in which no assumption about the sign of the statistic to be tested is made. Hence, no direction is specified in the relationship between variables (or the difference between groups). The alternative hypothesis sampling distribution is present on both sides of the null hypothesis sampling distribution; hence the region of rejection of H_0 is placed on both tails of the null hypothesis sampling distribution.

Type I error - the error made in a statistical test when the null hypothesis is rejected even though it is true. Hence, we think we have found support for the alternative hypothesis (i.e. a significant result) when in fact the hypothesis is not true. It is given the symbol α and is equivalent to the significance level we set (i.e. the size of the region of rejection).

Type II error - the error made in a statistical test when the null hypothesis is accepted even though it is false. Hence, we find no support for the alternative hypothesis (i.e. fail to get a significant result) when in fact the hypothesis is true. It is given the symbol β .

Up time - The component is capable to accomplish the required function during this period of time

Variance - Expected value of the square difference between a random variable and its expected value

Curriculum Vitae

Cornel Bunea was born on 9th of April 1975, in Bucharest, Romania. In 1993 he became student at “Politehnica” University of Bucharest, Faculty of Energetic, Department of Thermo-electrical Power Plants. After 5 years, he graduated under supervision of Prof. Dr. Cezar Ionescu, with “Predictive leading of Thermoelectric Power Station with computer assistance” as final thesis. He remained in Faculty to continue his studies in the Department of Reliability for another year. In 1998, he received a Master of Science Diploma in “Reliability, Security and Risk in Energetic Systems”, under the supervision of Prof. Dr. Cezar Ionescu and Dr. Paul Ulmeanu. Beginning with the autumn of 1999, Cornel Bunea worked as a young research fellow in Department of Control Risk, Optimization, Stochastic and System Theory (CROSS) of the Delft University of Technology, until 2003. Under the supervision and with the support of Prof. Dr. Roger Cooke and Prof. Dr. Timothy Bedford, he can now present the Ph.D. Thesis “Mathematical Models for Reliability Data” as a result of his work at Delft University of Technology.

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