Techniques in Representing High Dimensional Distributions

Dorota Kurowicka

Techniques in representing high dimensional distributions

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Chapter 1

Introduction

1.1 Representing high dimensional distributions

In analyzing physical systems our gaol is to capture the relationship between input and output of a model. Usually, we represent model as a vector function

$$Y = [Y_1, Y_2, \dots, Y_m]$$

with input vector

$$X = [X_1, X_2, \dots, X_n].$$

Values of X propagated through the model give us the corresponding value of Y. Usually models are very complex and dimensions of vectors X and Y can be lage. Moreover, since we cannot specify values of X precisely in most analysis, the input vector is considered as a random vector and a distribution is assigned to X. Exact analysis of such a model requires finding a joint distribution of X to capture possible relationships and dependencies between elements of input vector. For complex problems, finding such the joint distribution is very difficult. Many approaches are then possible:

1. We can simplify our model by assuming independence between components of X. Then the joint distribution of input vector is equal to product of the distributions assigned to the components of X.

2. We can simplify our model by simplifying mapping function from the analysis input to the corresponding analysis results (e.g. considering linear models).

3. Graphical models can be used to represent high dimensional distribution (e.g. Markov trees and its generalization to influence diagrams or new graphical model called vines).

A well known example where first independence between variables was assumed and then results were generalized to the dependent case, is theory of extreme values. It is shown there that if the components of X are independent and identically distributed and if the mapping function is minimum (maximum) of these variables, then as dimension of the vector X becomes large we obtain one of three possible distribution classes for the output vector Y. The domains of attraction of the possible distribution classes are known, that is, necessary and sufficient conditions for components of X so distribution of Y is one of the tree-element class of possible distributions (Gnedenko [17], Haan [9]). The classical extreme value theory is generalized by permitting dependence (e.g. stationarity or the Markovian dependence) or allowing components to have different distributions (Lindgren, Leadbetter and Rootzen [44]). This theory was intensively studied for many years and was applied in various problems (e.g strength of materials testing, wave or flood data analyzing (Lindgren, Leadbetter and Rootzen [44], Castillo [4], Gumbel [18]).

The first part of this thesis contains a contribution to the extreme value theory. Double indexed, independent and identically distributed variables

$$X_{11},\ X_{12},\ X_{22},\ X_{21},\ X_{13},\ X_{23},\ X_{33},\ X_{32},\ X_{31},\ldots$$

are placed in rectangular matrix $[X_{ij}]$. Output variable is defined as

$$Y = \min_i \max_j X_{ij}$$

or

$$Y = \max_i \min_j X_{ij}$$

then ten possible limit survive function classes (called reliability function classes) of Y under linear normalization are determined (Kolowrocki [29],[28]). Possible distributions depends on shape of the matrix $[X_{ij}]$, that is, on relationships between numbers of elements in rows and columns of this matrix. In this thesis the domains of attraction of these limit distribution classes are determined (Chapter 2). In Chapter 3 it is shown how the theorems about domains of attraction can be used to find possible limit distribution for non-homogeneous minmax models (where X_{ij} have different distributions).

Simple dependent models can be constructed by the linear transformation of the independent variables with given marginals. This approach is due to Steffensen [49].

One of the more common ways to define a high dimensional distribution is to transform each of the input variables to univariate normal, and then to take the multivariate normal distribution to introduce dependence between the variables (Lauritzen [40], Muirhead [45]).

Graphical models seem to be a very convenient way of representing high dimensional distributions. They can visually represent a given model and help to describe its dependencies. An important property of graphical models is their ability to describe complex structures in modular way, combining dependencies between adjacent elements. The best known approach in this context is a tree structure. A tree on N variables specifies at most $N-1$ edges between the variables. Each edge may be associated with a copula, that is a distribution on $[0,1]^2$ with uniform marginals. Popular copulae are the diagonal band (Cooke and Waij [7]) and the minimum information copulae (Meeuwissen and Bedford [42]).

In the last chapter of this thesis, Chapter 6, the new copula, elliptical copula is introduced, and its properties are studied. The elliptical copula is continuous and can realize any correlation value in $(-1,1)$. In constructing this copulae properties of elliptically contoured and rotationally invariant random vectors were used (Harding [5], Misiewicz [43]). A density function of the elliptical copula with correlation $\rho \in (-1,1)$ is following

$$f_\rho(x,y) = \begin{cases} \frac{1}{\pi\sqrt{1-\rho^2}} \frac{1}{\sqrt{\frac{1}{4}-x^2-\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)^2}} & (x,y) \in B \\ 0 & (x,y) \notin B \end{cases}$$

where

$$B = \left\{ (x,y) |\; x^2 + \left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)^2 < \frac{1}{4} \right\}.$$

The elliptical copula has linear regression property.

Given any tree on N variables with copulae assigned to the edges, a joint distribution can always be constructed satisfying the tree-copulae specification. Moreover, it is shown in (Cooke [6]) that there is a unique minimum information joint distribution satisfying the tree-copulae specification and under this distribution the tree becomes a Markov tree. Distributions specified in this way can be sampled on the fly. The tree-copulae method of specifying a joint distribution is limited by the fact that there can be at most $N-1$ edges on the tree so tree full of constraints must be specified.

One generalization of the Markov trees are belief nets and influence diagrams which use directed acyclic graphs as a representation of conditional independence relationships. These structures have been used in Bayesian inference and decision analysis.

A new class of models called vines was introduced in (Cooke [6]). A vine on N variables is a set of trees, where the edges of tree j are the nodes of tree $j+1$, and each tree has the maximum number of edges. A regular vine on N variables is a vine in which two edges in tree j are joined by an edge in tree $j+1$ only if these edges share a common node. The difference between Markov trees and vines is that the conditional independence from Markov trees is replaced by conditional dependence, with given conditional correlation coefficient.

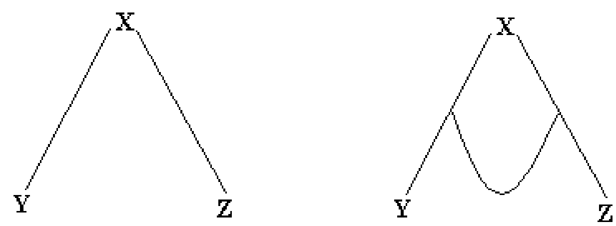


Figure 1. A Markov tree (left) and a vine (right) on 3 elements

Figure 1 shows examples of a Markov tree and a vine on three variables. In the Markov tree variables Y and Z are conditionally independent given X, in the vine Y and Z are not conditionally independent. It is shown in Chapter 5 that the partial correlation between Y and Z with X may be large even in case when Y and Z are conditionally independent given X.

Partial correlations or conditional (rank) correlations can be assigned to the edges of the regular vine. There are $\binom{n}{2}$ edges in regular vine on n elements and there is a bijection from $(-1,1)^{\binom{n}{2}}$ to the set of full rank correlation matrices (Bedford and Cooke [3]). Thus we can specify a full rank correlation matrix with $\binom{n}{2}$ numbers which need not satisfy any algebraic constraints (e.g. positive definiteness).

Using regular vines with conditional rank correlations we can determine a convenient way of representing high dimensional distribution to realize a correlation matrix and sample from this distribution on the fly.

In Chapter 5 of this thesis the relationship between partial and conditional correlation is studied with particular attention to copulae used in high dimensional graphical models. Sufficient and, in some cases, necessary conditions for equality of partial and constant conditional correlations are obtained. Numerical results show that the difference between partial and conditional correlation is small for the minimum information copula with given product moment correlation. When approximate equality holds, regular vines enable us to specify a correlation structure without algebraic constraints (e.g. positive definiteness).

In Chapter 4 techniques based on the properties of regular vines are used to tackle a number problems relating to positive definiteness of a matrix. Define a proto correlation matrix as symmetric real matrix with elements in the interval (-1,1) and with "1"'s on the main diagonal. We can determine whether a proto correlation matrix is positive definite simply by calculating partial correlation assigned to the edges of regular vine. If we find partial correlation on a regular vine which is not in the interval (-1,1) then considered matrix is not positive definite. The speed of this algorithm appears to be comparable to that of previous algorithms. With this algorithm non-positive definite matrix can be transformed into a positive definite matrix simply by changing values of these partial correlations on regular vine which are less then -1 or garter than 1 and recalculating respective correlations in initial proto correlation matrix. With the new algorithm these alterations have a clear probabilistic interpretation. In complex problems many entries in the correlation matrix may be unspecified, and this partially specified matrix must be extended to a positive definite matrix so completion problem must be solved (Laurent [38]). In Chapter 5 we show how regular vine can be used to determine whether a partially specified matrix can be extended to a correlation matrix. This approach can be useful where a high dimensional correlation matrix should be specified (e.g. dependent Monte Carlo simulations).

1.2 Outline of thesis

This Ph.D. Thesis consists of two parts. First contains results of the work I have been doing in Gdynia Maritime Academy, Poland, second shows the results of the research carried out in Delft University of Technology. There are five main chapters of this thesis. They are largely self contained and they can be read individually. At the moment all chapters are being published or are submitted to publication in scientific journals or refereed conferences proceedings. It is explained in Section 1.1 how they fit to the topic of this thesis.

Part I

Chapter 2

Domains of attraction of limit reliability functions

Dorota Kurowicka

Abstract: The problem of domains of attraction of limit reliability functions $R = 1 - F$ for series-parallel and parallel-series systems is presented. The necessary and sufficient conditions for reliability function of the particular components of the system are established so the limit reliability function is one of the ten-element class of possible limit distribution. Moreover, some examples are presented.

Keywords: extreme value theory, reliability, limit reliability functions, domains of attraction

2.1 Introduction

Classical extreme value theory is concerned with properties of distributions of the maximum

$$X_n = \max\{\xi_1, \xi_2, \ldots \xi_n\}$$

of n independent identically distributed random variables, as n becomes large. The basic classical results states that if for some sequences of normalizing constants $a_n > 0,\ b_n \in (-\infty, \infty)$

$$\frac{X_n - b_n}{a_n}$$

has a non-degenerate limiting distribution function, then this function must have one of the three possible forms as follows:

$$\begin{aligned}
\Phi_\alpha(x) &= \begin{cases} 0, & x \le 0, \alpha > 0 \\ \exp[-x^{-\alpha}], & x > 0 \end{cases} \\
\Psi_\alpha(x) &= \begin{cases} \exp[-(-x)^{\alpha}], & x \le 0, \alpha > 0 \\ 1, & x > 0 \end{cases} \qquad (2.1) \\
\Lambda(x) &= \exp[-\exp(-x)], \quad x \in (-\infty, \infty)
\end{aligned}$$

Frechet, Fisher and Tippett discovered these three possible distribution functions. We call these distributions Weibull, Frechet and Gumbel respectively. In 1936, von Mises [51] gave sufficient conditions under which the three asymptotic distributions are valid.

Theorem 2.1 (von Mises conditions)

(A) *Suppose F has a positive density F' for all $x \ge x_1$. If for some $\alpha > 0$*

$$\lim_{x\to\infty} \frac{xF'(x)}{1 - F(x)} = \alpha$$

then F belong to the domain of attraction of Φ_α.

(B) *Suppose F has a density F' which is positive in some interval (x_1, x_0) and vanishes for $x > x_0$. If for some $\alpha > 0$*

$$\lim_{x \uparrow x_0} \frac{(x_0 - x)F'(x)}{1 - F(x)} = \alpha$$

then F belong to the domain of attraction of Ψ_α.

(C) *Suppose F has non-negative second derivative F" for all x in some interval (x_1, x_0) and let F' vanishes for $x \geq x_0$ where x_0 may be finite or infinite. If*

$$\lim_{x \uparrow x_0} \frac{F"(x)(1 - F(x))}{(F'(x))^2} = -1$$

then F belong to the domain of attraction of Λ .

Gnedenko [17] has established necessary and sufficient conditions for domains of attraction of these functions.

Theorem 2.2 (Gnedenko conditions)

(A) *A necessary and sufficient conditions for the common distribution function (cdf) F to belong to the domain of attraction of Φ_α are*

(a) *there exists x_0 such that $F(x_0) = 1$ and $F(x_0 - \epsilon) < 1$ for all $\epsilon > 0$,*

(b) $\lim_{x \to 0^-} \frac{1-F(kx+x_0)}{1-F(x+x_0)} = k^\alpha$ *for $k > 0$.*

(B) *A necessary and sufficient condition for the cdf F to belong to the domain of attraction of Ψ_α is*

$$\lim_{x \to \infty} \frac{1 - F(x)}{1 - F(kx)} = k^\alpha \text{ for } k > 0.$$

(C) *A necessary and sufficient conditions for the cdf F to belong to the domain of attraction of Λ is there exists a continuous function $A(z)$ such that $A(z) \to 0$ as $z \to x_0$ and*

$$\lim_{z \to x_0^-} \frac{1 - F(z + A(z)x)}{1 - F(z)} = e^{-x}$$

where $x_0 \leq \infty$, $F(x_0) = 1$ and $F(x) < 1$ for all $x < x_0$.

De Haan [9] formulated and proved the theorems about domains of attraction using the theory of the regularly varying functions. He presented among other things the following results:

Theorem 2.3 (De Haan conditions)

(A) *A distribution function F belongs to the domain of attraction of Φ_α if and only if $1-F$ is $(-\alpha)$ -varying at infinity*[1].

(B) *A distribution function F belongs to the domain of attraction of Ψ_α if and only if F has a finite endpoint*[2] *x_0 and the function $U(x) = 1 - F(x_0 - x^{-1})$ for all $x \in \Re^+$ is $(-\alpha)$-varying at infinity.*

(C) *A distribution function F belongs to the domain of attraction of Λ if and only if*

$$\lim_{t \downarrow 0} \frac{U(tx) - U(t)}{U(ty) - U(y)} = \frac{\log x}{\log y}$$

for all positive x and y $(y \neq 1)$, where $U : \Re^+ \to \Re$ is defined by

$$U(x) = \inf\{y : 1 - F(y) \leq x\}.$$

The three limit distributions with parameters a_n, b_n can be written in von Mises form as follows

$$H_c(x) \quad = \quad \exp\left\{ -\left[1 + c\left(\frac{x-\lambda}{\delta} \right) \right]^{-\frac{1}{c}} \right\}, \quad 1 + c\left(\frac{x-\lambda}{\delta} \right) \geq 0.$$

For $c > 0, c < 0, c = 0$ (in the case $c = 0$ interpreted in the limit sense) we get the Frechet, Weibull and Gumbel families respectively. For above representation of limit distributions the following necessary and sufficient condition was presented in Castillo [4].

[1] A function $U : \Re^+ \to \Re$ *varies regularly at infinity* if there exists a $\rho \in \Re$ such that for all $x \in \Re^+$

$$\lim_{t \to \infty} \frac{U(tx)}{U(t)} = x^\rho.$$

[2] A point $x_0 \leq \infty$ is called *endpoint* of the distribution function F if $x_0 = \sup\{x | F(x) < 1\}$.

Theorem 2.4 (Castillo conditions)
Let p_1, p_2, p_3, p_4 be four real numbers in the interval $(0,1)$. A necessary and sufficient condition for a continuous cdf F to belong to the domain of attraction of maxima of H_c is given by

$$\lim_{n\to\infty} \frac{F^{-1}(p_1^{\frac{1}{n}}) - F^{-1}(p_2^{\frac{1}{n}})}{F^{-1}(p_3^{\frac{1}{n}}) - F^{-1}(p_4^{\frac{1}{n}})} = \frac{(-\log p_1)^{-c} - (-\log p_2)^{-c}}{(-\log p_3)^{-c} - (-\log p_4)^{-c}}.$$

A lot of peoples dealt with this problem. De Haan and Resnic [10] studied continuous and ones differentiable domains of attraction. They gave sufficient conditions and rate of convergence. They also gave results concerning Lp convergence. Sweeting [50] dealt with uniform local convergence of densities of absolutely continuous distributions. Pikands III [20] characterised the domains of attraction in terms of inverse cumulative hazard function. Galambos and Obretenov [16] established the necessary and sufficient conditions for Λ in term of the expected residual life and hazard rate.

The limit theorems and domains of attraction in the case of dependence sequences were in interest of a lot of authors. The summarisation can be found for instance in the book of Leabetter, Lindgren and Rootzen [44].

The problem of finding limit distributions of regular homogeneous series-parallel systems is directly related to the classic extreme value theory. With references to this theory the problem can be formulated as follows. Let $[X_{ij}]$ be a double array of independent, identically distributed random variables. For the sequences of natural numbers k_n and l_n we define an array of random variables

$$[Y_{ni} : n = 1, 2, ..., i = 1, 2, ..., k_n]$$

where

$$Y_{ni} = \min_{1\le j\le l_i} \{X_{ij}\}$$

then $[Y_{ni}]$ is a row-wise independent identically distributed double array with the distribution function $1 - (1-F)^{l_n}$. The problem of possible limit distributions of suitably normed maxim

$$X_n = \max_{1\le k\le k_n} \{X_{nk}\}$$

is discussed in Kolowrocki[28],[25] and summarised in Kolowrocki [29]. In these papers it is shown that there are ten possible limit distributions. This class of limit laws is more extensive than up to now known three-element class. It is interesting that these laws strictly depend on the shape of the system structure. In the case of square systems it was found three possible limit distributions the same as in the case of maximum. Chernoff and Teicher [19], in this case, established the domains of attraction in terms of certain limiting functions.

In this chapter the problem of domains of attraction of possible limit reliability functions for series-parallel and parallel-series functions is solved.

The chapter is organized as follows. In Section 2.2 the essential notion required for the paper will be given. Well known definitions and formulae will be recall. The lemmas that will be necessary in the next sections are presented and proved, or references are given to the papers or books, where their proofs can be found. In Section 2.3 ten possible limit reliability functions for series-parallel systems are presented. In Section 2.4 for the certain limit reliability functions the theorem of the domains of attraction are established, that is, the necessary and sufficient conditions which the reliability function of the particular component should satisfy so the limit is one of the possible distributions. The statements and proofs of the results about domains of attraction differ for the individual cases. Section 2.5 consist of theorems about domains of attraction for parallel-series systems and in Section 2.6 the examples are presented.

2.2 Essential notions and theorems

Suppose that $E_i, i = 1, 2, \ldots, n, n \in$ N are components of the system S and X_i are lifetimes of E_i. Moreover, suppose that X_i are independent random variables.

Definition 2.1 (Series system)
A system S is called series if its lifetime X is given by

$$X = \min_{1 \le i \le n} \{X_i\}.$$

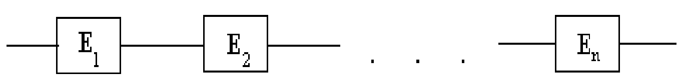


Figure 2.1 The shape of series system.

The sequence of reliability function of the series system is given by

$$\overline{\Re}_n(x) \quad = \quad \prod_{i=1}^{n} R_i(x), \quad x \in (-\infty, \infty), \;\; n \in \mathrm{N}.$$

Definition 2.2 (Parallel system)
A system S is called parallel if its lifetime X is given by

$$X \quad = \quad \max_{1 \le i \le n} \{X_i\}.$$

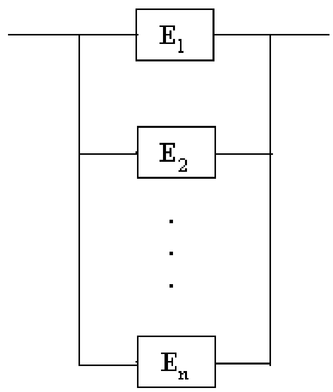


Figure 2.2 The shape of parallel system.

The sequence of reliability function of the parallel system is given by

$$\Re_n(x) \quad = \quad 1 - \prod_{i=1}^{n} F_i(x), \quad x \in (-\infty, \infty), \;\; n \in \mathrm{N}.$$

Definition 2.3 (Homogeneous system)
We call the system S homogeneous if the random variables $X_i, i = 1, 2, \ldots, n$, have the same cdf $F(x) = P(X_i \geq x)$, that is, if the components E_i have the same reliability function $R = 1 - F$.

It means that the sequence of reliability functions of the homogeneous series system is

$$\overline{\Re}_n(x) \quad = \quad (R_i(x))^n, \quad x \in (-\infty, \infty) \qquad (2.2)$$

and for the homogeneous parallel system

$$\Re_n(x) \quad = \quad 1 - (F_i(x))^n, \quad x \in (-\infty, \infty). \tag{2.3}$$

Suppose that E_{ij}, where $i = 1, 2, \ldots, k_n, j = 1, 2, \ldots, l_i$, are components of the system S and X_{ij} are lifetimes of E_{ij}. Moreover, suppose that X_{ij} are independent random variables.

Definition 2.4 (Series-parallel system)
A system S is called series-parallel if its lifetime X is given by

$$X \quad = \quad \max_{1 \le i \le k} \{ \min_{1 \le j \le l_i} \{X_{ij}\}\}.$$

Definition 2.5 (Parallel-series system)
A system S is called parallel-series if its lifetime X is given by

$$X \quad = \quad \min_{1 \le i \le k} \{ \max_{1 \le j \le l_i} \{X_{ij}\}\}.$$

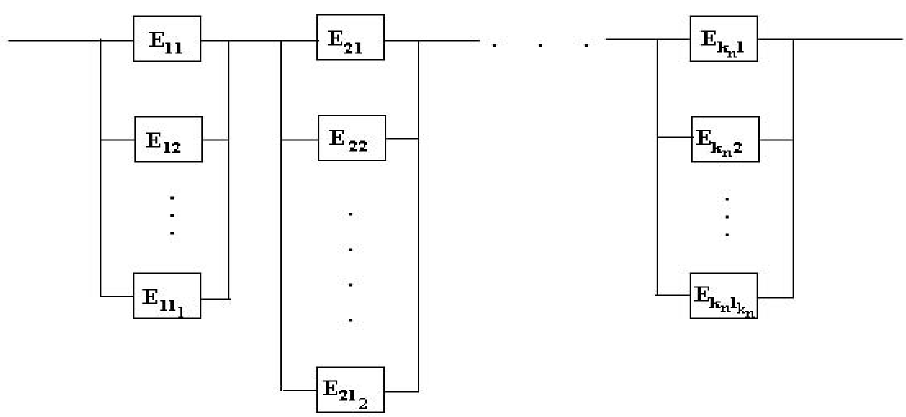


Figure 2.3 The shape of parallel-series system.

Definition 2.6 (Regular system)
A system S is called regular if

$$l_1 = l_2 = \ldots = l_{k_n} = l_n, \;\; l_n \in \mathrm{N}$$

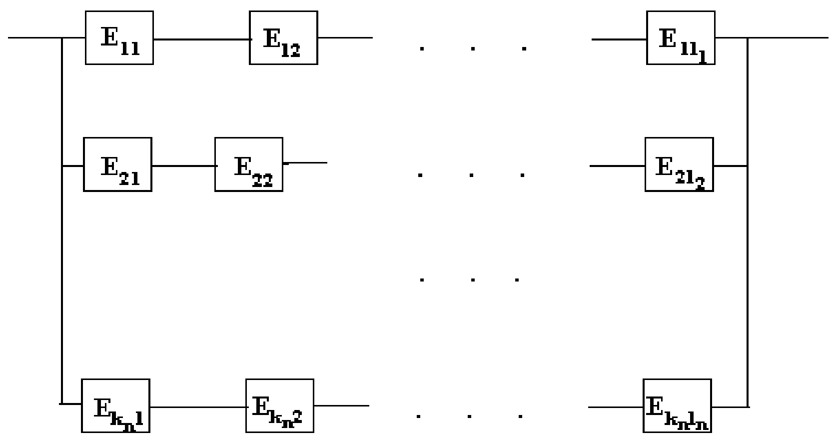


Figure 2.4 The shape of regular series-parallel system.

Definition 2.7 (Homogeneous system)
A regular system S is called homogeneous if the random variables $X_{ij}, i = 1, 2, \ldots, k_n$, $j = 1, 2, \ldots, l_n$ have the same distribution functions F, that is, if the components E_{ij} have the same reliability function R.

It is well known that the sequence of reliability functions of regular homogeneous series-parallel systems is given by

$$\Re_{l_n,k_n}(x) = 1 - [1 - (R(x))^{l_n}]^{k_n}, \;\; x \in (-\infty, \infty), \;\; n \in \mathrm{N} \tag{2.4}$$

and the sequence of reliability functions of regular homogeneous parallel-series systems is given by

$$\overline{\Re}_{l_n,k_n}(x) = [1 - (F(x))^{l_n}]^{k_n}, \;\; x \in (-\infty, \infty), \;\; n \in \mathrm{N} \tag{2.5}$$

Next replacing n by a positive real number t and assuming that k_t and l_t are positive real numbers, we obtain families of the regular systems corresponding to the pair (k_t, l_t). For these families of the systems there exist families of reliability functions.

The family of reliability functions of the homogeneous series system is given by

$$\overline{\Re}_t(x) = (R(x))^t, \quad x \in (-\infty, \infty), \; t \in (0, \infty) \tag{2.6}$$

and for the homogeneous parallel system

$$\Re_t(x) = 1 - (F(x))^t, \quad x \in (-\infty, \infty), \; t \in (0, \infty) \tag{2.7}$$

The family of reliability functions of regular homogeneous series-parallel systems is given by

$$\Re_{l_t, k_t}(x) = 1 - [1 - (R(x))^{l_t}]^{k_t}, \quad x \in (-\infty, \infty), \; t \in (-\infty, \infty) \tag{2.8}$$

and for regular homogeneous parallel-series systems

$$\overline{\Re}_{l_t, k_t}(x) = [1 - (F(x))^{l_t}]^{k_t}, \quad x \in (-\infty, \infty), \; t \in (-\infty, \infty) \tag{2.9}$$

Let us assume that the lifetime distribution do not necessarily have to be concentrated on the interval $[0, \infty)$. Then the reliability function does not have to satisfy the usually demanded condition

$$R(x) = 1 \text{ for } x < 0.$$

This is a generalization of the commonly used concept of reliability functions. This generalization is convenient in theoretical considerations. At the same time, from the achieved results about the generalized reliability functions, the same properties of usually used reliability functions appear. Hence we get that a reliability function R is non-increasing, right-continuous $R(-\infty) = 1$ and $R(+\infty) = 0$.

Definition 2.8 (Degenerate relability function)
A reliability function R is called degenerate if there exists $x_0 \in (-\infty, \infty)$ such that

$$R(x) = \begin{cases} 1, & x < x_0 \\ 0, & x \geq x_0. \end{cases}$$

Corollary 2.1 *A function*

$$R(x) \quad = \quad 1 - \exp[-V(x)], \ \ x \in (-\infty, \infty), \tag{2.10}$$

is a reliability function if and only if function V is non-negative, non-increasing, right-continuous function, $V(-\infty) = \infty, V(\infty) = 0$ and besides V may be identically ∞ in an interval.

Agreement

In our further considerations, if we use V we always mean a function with the properties specified in Corollary 2.1. If V is identically equal to ∞ in an interval, we define that $\exp[-\infty] = 0$. If we say that V is a non-negative, non-increasing and right-continuous functions, we only mean that these properties hold on the interval where $V \neq \infty$. Moreover, we denote the set of continuity points of the reliability function R by C_R and the set composed of continuity points of V and points such that $V = \infty$ by C_V .

Definition 2.9 (Degenerate function)

A function V defined for $x \in (-\infty, \infty)$ non-negative, non-increasing, right-continuous $V(-\infty) = \infty, V(\infty) = 0$ is called degenerate if there exists $x_0 \in (-\infty, \infty)$ such that

$$V(x) \quad = \quad \begin{cases} \infty, & x < x_0 \\ 0, & x \geq x_0. \end{cases}$$

Now, the following corollary is clear.

Corollary 2.2 *A reliability function R given by (2.10) is degenerate if and only if a function V is degenerate.*

We shall investigate limit distributions of standardized random variable

$$\frac{X - b_t}{a_t},$$

where $a_t = a(t) > 0$ and $b_t = b(t) \in (-\infty, \infty)$ are some suitably chosen functions. And, since

$$P\left(\frac{X - b_t}{a_t} > x\right) = P(X > a_t x + b_t) = \Re_{l_t, k_t}(a_t x + b_t)$$

then we introduce the following definition.

Definition 2.10 (Limit reliability function, domains of attraction) *A reliability function $\mathcal{R}$ is called a limit reliability function of the family $\Re_{l_t,k_t}$ given by (2.8) or an asymptotic reliability function of the series-parallel system if there exist functions $a_t > 0$ and $b_t \in (-\infty, \infty)$ such that*

$$\lim_{t\to\infty} \Re_{l_t,k_t}(a_t x + b_t) = \lim_{t\to\infty} 1 - (1 - (R(a_t x + b_t))^{l_t})^{k_t} = \mathcal{R}(x) \text{ for } x \in C_{\mathcal{R}}$$

The pair (a_t, b_t) is called a norming function pair.
We say that a reliability function R belongs to the domain of attraction $(D_{\mathcal{R}})$ of $\mathcal{R}$.

We can formulate similar definitions for series, parallel and parallel-series systems.

Definition 2.11 *The reliability functions R_0 and R are said to be of the same type if there exist numbers $a > 0$ and $b \in (-\infty, \infty)$ such that*

$$R_0(x) = R(ax + b) \text{ for } x \in (-\infty, \infty). \tag{2.11}$$

Definition 2.12 *The reliability functions V_0 and V are said to be of the same type if there exist numbers $a > 0$ and $b \in (-\infty, \infty)$ such that*

$$V_0(x) = V(ax + b) \text{ for } x \in (-\infty, \infty).$$

First it will be in convenient to obtain some useful results necessary in the next investigations. We will introduce the lemmas, which gives the equivalent conditions of convergence to the non-degenerate limit reliability function. The following two lemmas can be found in Kolowrocki [28].

Lemma 2.1 *If*

(a) *the reliability function $\mathcal{R}$ is given by (2.10),*

(b) *the family $\Re_{l_t,k_t}$ is given by (2.8),*

(c) $\lim_{t\to\infty} k_t = \infty$,

(d) $a_t > 0, b_t \in (-\infty, \infty)$ *are some functions*

then

$$\lim_{t\to\infty} \Re_{l_t,k_t}(a_t x + b_t) = \mathcal{R}(x) \text{ for } x \in C_{\mathcal{R}}$$

is equivalent to the assertion

$$\lim_{t\to\infty} k_t(R(a_tx+b_t))^{l_t} = V(x) \quad \textit{for } x \in C_V.$$

Lemma 2.2 *If $\Re_t$ is a reliability function's family such that for some functions $a_t > 0, b_t \in (-\infty,\infty)$*

$$\lim_{t\to\infty} \Re_t(a_tx+b_t) = \mathcal{R}_0(x) \ \ \textit{for } x \in C_{\mathcal{R}_0},$$

where $\mathcal{R}_0$ is non-degenerate reliability function, then the assertion

$$\lim_{t\to\infty} \Re_t(\alpha_tx+\beta_t) = \mathcal{G}_0(x) \ \ \textit{for } x \in C_{\mathcal{G}_0}$$

where $\mathcal{G}_0$ is non-degenerate reliability function and $\alpha_t > 0, \beta_t \in (-\infty,\infty)$, are some functions, holds if and only if there exist constants $a > 0, b \in (-\infty,\infty)$ such that

$$\lim_{t\to\infty} \frac{\alpha_t}{a_t} = a \ \ \textit{and} \ \lim_{t\to\infty} \frac{\beta_t - b_t}{a_t} = b$$

Moreover, $\mathcal{G}_0$ and $\mathcal{R}_0$ are of the same type, that is

$$\mathcal{G}_0(x) = \mathcal{R}_0(ax+b) \ \textit{for } x \in (-\infty,\infty).$$

By Lemma 2.1 and Lemma 2.2 we can very easily obtain the following lemma.

Lemma 2.3 *If R is a reliability function such that for some functions $a_t > 0, b_t \in (-\infty,\infty)$*

$$\lim_{t\to\infty} k_t(R(a_tx+b_t))^{l_t} = V_0(x) \ \ \textit{for } x \in C_{V_0},$$

where V_0 is the function with properties like in Corollary 2.2, then the assertion

$$\lim_{t\to\infty} k_t(R(\alpha_tx+\beta_t))^{l_t} = V_1(x) \ \ \textit{for } x \in C_{V_1}$$

where V_1 is the function with properties like in Corollary 2.2 and $\alpha_t > 0, \beta_t \in (-\infty,\infty)$, are some functions, holds if and only if there exist constants $a > 0, b \in (-\infty,\infty)$ such that

$$\lim_{t\to\infty} \frac{\alpha_t}{a_t} = a \ \ \textit{and} \ \lim_{t\to\infty} \frac{\beta_t - b_t}{a_t} = b$$

Moreover, V_0 and V_1 are of the same type, that is

$$V_0(x) = V_1(ax+b) \ \textit{for } x \in (-\infty,\infty).$$

The following lemma is a generalization of the Lemma 1 in Chernoff and Teicher [19].

Lemma 2.4 *The condition*

$$\lim_{t\to\infty} t(R(a_t x + b_t))^{l_t} = V(x) \quad \textit{for } x \in C_V \tag{2.12}$$

is satisfied if and only if
there exist an extended, real-valued, non-decreasing, right-continuous function $\gamma(x)$ *with* $\gamma(-\infty) = -\infty, \gamma(\infty) = \infty$ *such that for* x *in the set of points at which* $\gamma(x)$ *is continuous and finite we get*

$$R(a_t x + b_t) = \exp\left[-\frac{\log t + \gamma(x) + o(1)}{l_t}\right]. \tag{2.13}$$

Proof. Let (2.12) holds. Then for all x such that $V(x) \neq \infty$ we get

$$t(R(a_t x + b_t))^{l_t} = (1 + o(1))V(x).$$

Hence if

$$\gamma(x) = -\log(V(x))$$

then

$$\log t + l_t \log R(a_t x + b_t) = -\gamma(x) + o(1)$$

what proves (2.13). In analogous way we can show that (2.13) converges to

$$V(x) = \exp(-\gamma(x))$$

for all continuity points where γ is finite. □

All results obtained for series-parallel lead to the analogous results for parallel-series systems.

Lemma 2.5 *If a reliability function* $\mathcal{R}$ *is an asymptotic reliability function of the regular homogeneous series-parallel system with the reliability function of particular component* R *then a reliability function*

$$\overline{\mathcal{R}}(x) = 1 - \mathcal{R}(-x) \textit{ for } x \in C_{\mathcal{R}}$$

is an asymptotic reliability function of the regular homogeneous parallel-series system with the reliability function of particular components

$$\overline{R}(x) = 1 - R(-x).$$

Notation:
In the whole paper, we use the following notation:
If $x(t)$ and $y(t)$ are positive functions then

(a) $x(t) \ll y(t)$ means that $\lim_{t\to\infty} \frac{x(t)}{y(t)} = 0$,

(b) $x(t) \approx y(t)$ means that $\lim_{t\to\infty} \frac{x(t)}{y(t)} = 1$,

(c) $x(t) \gg y(t)$ means that $\lim_{t\to\infty} \frac{x(t)}{y(t)} = \infty$,

(d) $x(t) \underset{\approx}{\ll} y(t)$ means that $x(t)$ and $y(t)$ are such that (a) or (b) hold,

(e) $x(t) \pm y(t)$ means either $x(t) + y(t)$ or $x(t) - y(t)$.

2.3 Limit reliability functions

In the following theorem, which can be found in Kolowrocki [25], under a few important assumptions related with the shape of the rectangular system, the three possible limit reliability functions are determined.

Theorem 2.5 *Suppose*

$$k_t = t,\ l_t = c(\log t)^{\rho(t)},\ \ t \in (0,\infty),\ \ c > 0,$$

where $\rho(t)$ *has the properties defined below.*
Define the following three cases:

Case 1

$$\log(\log t) \ll |l_t - c\log t| \text{ and } \rho(t) \ll (\log t)^{\lambda} \text{ for every } \lambda > 0$$

and

$$|\rho(\tau_\nu) - \rho(t)| \underset{\approx}{\ll} \frac{\delta \log \nu}{\log t[\log(\log t)]}$$

for every natural $\nu > 1$, *where* $0 < \delta \neq 1$ *and* $\tau_\nu = \tau_\nu(t)$, $t \in (0,\infty)$, *is given by*

$$\frac{\tau_\nu}{t} = \nu^{\frac{1}{1-\rho(t)}}.$$

Case 2

$$s \ll |l_t - c\log t| \lessapprox C\log(logt), \quad s>0; \quad C>0,$$

and

$$|\rho(\tau_\nu) - \rho(t)| \approx \frac{\delta \log \nu}{\log t[\log(\log t)]}$$

for every natural $\nu > 1$, *where* $0 < \delta$ *and* $\tau_\nu = \tau_\nu(t)$, $t \in (0,\infty)$, *is given by*

$$\frac{\tau_\nu}{t} = \nu^{\frac{1}{(1-\rho(t))\log(\log t)}}.$$

Case 3

$$\rho(t) \gtrapprox (\log t)^\lambda, \quad \lambda > 0$$

and

$$|\rho(\tau_\nu) - \rho(t)| \approx \frac{\delta \log \nu}{\log t[\log(\log t)]}$$

for every natural $\nu > 1$, *where* $0 < \delta$ *and* $\tau_\nu = \tau_\nu(t)$, $t \in (0,\infty)$, *is given by*

$$\frac{\tau_\nu}{t} = \nu^{\frac{1}{(1-\rho(t))A(t)}}$$

where

$$A(t) \approx \prod_{i=1}^{n} f_i(\rho(t))$$

and the sequence f_i *is defined inductively by*

$$f_1(x) = \log x, \; f_i(x) = \log(f_i - 1(x))$$

and n *is such that*

$$f_{n+1}(\rho(t)) \ll A\log(\log t), \quad A > 0.$$

Under the Cases $1, 2$ *and* 3 *the only possible, non-degenerate, limit reliability functions are as follows:*

$$\begin{aligned} \mathcal{R}_1(x) &= \begin{cases} 1, & x < 0 \\ 1 - \exp[-x^{-\alpha}], & x \geq 0 \end{cases} \\ \mathcal{R}_2(x) &= \begin{cases} 1 - \exp[-(-x)^{\alpha}], & x < 0 \\ 0, & x \geq 0 \end{cases} \\ \mathcal{R}_3(x) &= 1 - \exp[-\exp(-x)], \quad x \in (-\infty, \infty). \end{aligned} \tag{2.14}$$

We will present briefly the ideas of the proof of this theorem.

Sketch of the proof. First it can be shown that if function $\mathcal{R}$ given by (2.10) is a limit reliability function of the series-parallel system and the assumptions of the Cases $1, 2, 3$ are satisfied then the following estimations hold: In the Case 1 of Theorem 2.5, we get

$$\frac{l_{\tau_\nu}}{l_t} = 1 + \frac{\log \frac{\tau_\nu}{t}}{\log t} - \frac{\log \nu}{\log t} \pm r\left(\frac{\delta \log \nu}{\log t}\right) \text{ where } r(s) \lessapprox s, \tag{2.15}$$

$$\lim_{t\to\infty} \log t \left[-\frac{\log \nu}{\log t} \pm r\left(\frac{\delta \log \nu}{\log t}\right)\right] = -\log \nu^{\mu}, \quad \mu = 1 \pm \delta \text{ or } \mu = 1. \tag{2.16}$$

When the assumptions of the Case 2 are satisfied we obtain

$$\frac{l_{\tau_\nu}}{l_t} = 1 + \frac{\log \frac{\tau_\nu}{t}}{\log t} - \frac{\log \nu}{\log t \log(\log t)} \pm w\left(\frac{\delta \log \nu}{\log t}\right) \text{ where } w(s) \approx s \tag{2.17}$$

$$\lim_{t\to\infty} \log t \left[-\frac{\log \nu}{\log t \log(\log t)} \pm w\left(\frac{\delta \log \nu}{\log t}\right)\right] = -\log \nu^{\mu}, \quad \mu = \pm\delta. \tag{2.18}$$

If the assumptions from Case 3 of Theorem 2.5 hold then

$$\frac{l_{\tau_\nu}}{l_t} = 1 + \frac{\log \frac{\tau_\nu}{t}}{\log t} - \frac{\log \nu}{A(t) \log t} \pm w\left(\frac{\delta \log \nu}{\log t}\right), \tag{2.19}$$

$$\lim_{t\to\infty} \log t \left[-\frac{\log \nu}{A(t) \log t} \pm w\left(\frac{\delta \log \nu}{\log t}\right)\right] = -\log \nu^{\mu}, \quad \mu = \pm\delta. \tag{2.20}$$

If $\mathcal{R}$ is a limit reliability function then by Lemma 2.1, we get

$$\lim_{t\to\infty} k_t(R(a_t x + b_t))^{l_t} = V(x) \quad \text{for } x \in C_V.$$

Since in the Case 1, 2, 3 of Theorem 2.5 we get $\tau_\nu = \tau_\nu(t) \to \infty$ as $t \to \infty$ then from the above

$$\lim_{t\to\infty} \tau_\nu \left(R(a_{\tau_\nu}x + b_{\tau_\nu})\right)^{l_{\tau_\nu}} = V(x) \quad \text{for } x \in C_V. \tag{2.21}$$

We can write (2.21) in the following form

$$\lim_{t\to\infty} \tau_\nu t^{-l_{\tau_\nu}/l_t} \left[t\left(R(a_{\tau_\nu}x + b_{\tau_\nu})\right)^{l_t}\right]^{l_{\tau_\nu}/l_t} = V(x).$$

From the estimations (2.15) - (2.20) and by Definition 2.12 we can obtain the functional equation

$$\nu^\mu V(\alpha_\nu x + \beta_\nu) = V(x) \tag{2.22}$$

for $x \in (-\infty, \infty)$ and for any $\nu > 1$ and some $\mu \neq 0$.
The possible solutions of this equation determine the possible limit reliability functions. □

Remark 2.1 *It will be of convenience to modify, slightly, the assumption of Theorem 2.5. In defining the sequences $\tau_\nu = \tau_\nu(t)$, $t \in (0, \infty)$ in Cases $1, 2$ and 3, the possibility $\nu = 1$ was excluded. This case however, is trivial as we now explain.*
First note that $\tau_\nu(t) = t$ when $\nu = 1$. In the case $\nu = 1$ the estimations (2.15) - (2.20) are also true and the case $\nu = 1$ does not impose extra restrictions on V as that equation (2.22) is always valid (with $\alpha_1 = 1, \beta_1 = 0$). Hence Theorem 2.5 holds also for $\nu = 1$.

In the following theorem in the case when the number of series components has the order of logarithm of number of parallel components, the four new limit reliability functions were obtained. The full proof of this theorem can be found in Kolowrocki [28].

Theorem 2.6 *If*

$$k_t = t,\ (l_t - c\log t) \approx s,\ \text{where } s \in (-\infty, \infty),\ c > 0 \tag{2.23}$$

then the only possible, non-degenerate, limit reliability functions of a series-parallel system are of the following types:

$$\mathcal{R}_4(x) = \begin{cases} 1, & x < 0 \\ 1 - \exp[-\exp(-x^\alpha - \frac{s}{c})], & x \geq 0 \end{cases}$$

$$\mathcal{R}_5(x) = \begin{cases} 1-\exp[-\exp((-x)^{\alpha}-\frac{s}{c})], & x<0 \\ 0, & x \geq 0 \end{cases}$$

$$\mathcal{R}_6(x) = \begin{cases} 1-\exp[-\exp(\beta(-x)^{\alpha}-\frac{s}{c})], & x<0 \\ 1-\exp[-\exp(-x^{\alpha}-\frac{s}{c})], & x \geq 0 \end{cases}$$

$$\mathcal{R}_7(x) = \begin{cases} 1, & x<x_1 \\ 1-\exp[-\exp(-\frac{s}{c})], & x_1 \leq x < x_2 \\ 0, & x \geq x_2,\ x_1<x_2. \end{cases}$$

Proof. In this case we define sequence $\tau_\nu = t^\nu$ for all $\nu > 1$. Clearly $\tau_\nu \to \infty$ as $t \to \infty$. Hence by Lemma 2.1 we get for $x \in C_V$

$$\lim_{t\to\infty} \tau_\nu \left(R(a_{\tau_\nu}x+b_{\tau_\nu})\right)^{l_{\tau_\nu}} = V(x)$$

and

$$\lim_{t\to\infty} \tau_\nu t^{-l_{\tau_\nu}/l_t} \left[t\left(R(a_{\tau_\nu}x+b_{\tau_\nu})\right)^{l_t}\right]^{l_{\tau_\nu}/l_t} = V(x).$$

Since in this case then $l_t = c\log t + s(t)$ where $s(t) \approx s$ then

$$\lim_{t\to\infty} \frac{l_{\tau_\nu}}{l_t} = \lim_{t\to\infty} \frac{c\log\tau_\nu + s(\tau_\nu)}{c\log t + s(t)} = \lim_{t\to\infty} \frac{c\nu\log t + s(\tau_\nu)}{c\log t+s(t)} = \lim_{t\to\infty}\frac{\nu+o(1)}{1+o(1)} = \nu$$

and

$$\begin{aligned} \lim_{t\to\infty} \tau_\nu t^{-l_{\tau_\nu}/l_t} &= \lim_{t\to\infty} \exp\left[\log\tau_\nu - \frac{l_{\tau_\nu}}{l_t}\log t\right] = \lim_{t\to\infty}\exp\left[\frac{s(t)\nu\log t + s(\tau_\nu)\log t}{c\log t + s(t)}\right] \\ &= \lim_{t\to\infty}\exp\left[\frac{s}{c}\frac{\nu\frac{s(t)}{s} - \frac{s(\tau_\nu)}{s}}{1+o(1)}\right] = \exp\left[\frac{s}{c}(\nu-1)\right]. \end{aligned}$$

From the above and by Definition 2.12 the possible limit reliability functions were determined by finding the solution of the equation

$$[V(\alpha_\nu x+\beta_\nu)]^\nu = V(x)$$

for $x \in (-\infty,\infty)$ and for any $\nu > 1$. □

The following theorem we can obtain almost immediately from the well-known extreme value theory results for series systems.

Theorem 2.7 *If*

$$\lim_{t\to\infty} k_t = k, \quad \lim_{t\to\infty} l_t = \infty \tag{2.24}$$

then the only possible non-degenerate limit reliability functions are one of the following types:

$$\mathcal{R}_8(x) = \begin{cases} 1-[1-\exp[-(-x)^{-\alpha}]]^k, & x<0 \\ 0, & x\geq 0 \end{cases}$$

$$\mathcal{R}_9(x) = \begin{cases} 1, & x<0 \\ 1-[1-\exp[-x^{\alpha}]]^k, & x\geq 0 \end{cases}$$

$$\mathcal{R}_{10}(x) = 1-[1-\exp[-\exp x]]^k, \quad x\in(-\infty,\infty).$$

Proof. A function $\mathcal{R}$ is a limit reliability function of the series-parallel system if and only if there exist functions $a_t>0$ and $b_t\in(-\infty,\infty)$ such that

$$\mathcal{R}(x) = \lim_{t\to\infty} \mathcal{R}_{l_t,k_t}(a_t x + b_t) = \lim_{t\to\infty} 1-[1-(R(a_t x+b_t))^{l_t}]^{k_t} \text{ for } x\in C_{\mathcal{R}}.$$

Since $k_t\to k$ and the well-known results for the series systems are true is the case of the real index then the only possible non-degenerate limit reliability functions of the family $\mathcal{R}_{l_t,k_t}$ are of the form

$$\mathcal{R}(x) = 1-[1-\overline{\mathcal{R}}_i(x)]^k, \quad i=1,2,3$$

where $\overline{\mathcal{R}}_i,\ i=1,2,3$ are given by

$$\overline{\mathcal{R}}_1(x) = \begin{cases} \exp[-(-x)^{-\alpha}], & x<0 \\ 0, & x\geq 0 \end{cases}$$

$$\overline{\mathcal{R}}_2(x) = \begin{cases} 1, & x<0 \\ \exp[-x^{\alpha}], & x\geq 0 \end{cases}$$

$$\overline{\mathcal{R}}_3(x) = \exp[-\exp x], \quad x\in(-\infty,\infty).$$

which concludes the proof. □

2.4 Domains of attraction of limit reliability functions for series-parallel systems

In this section we present theorems which give sufficient and necessary conditions for function R to be in domain of attraction of determined in Section 2.4

possible limit reliability functions.

Let G be the functional inverse of R, $G : [0,1] \to \Re$, $G(u) = \inf\{x : R(x) \leq u\}$ and define function h as follows

$$\lim_{t\to\infty} \frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\gamma+\epsilon_t}{l_t}}\right) - b_t\right] = h(\gamma). \tag{2.25}$$

The existence of this limit follows directly from Lemma 2.4.

Proposition 2.1 *Under the assumption of Theorem 2.5 the function h takes one of the following forms:*

$$\begin{cases} h(\gamma) = \gamma & \gamma(x) = x & x \in \Re & (\textit{in the case } \mathcal{R}_3) \\ h(\gamma) = e^{\frac{\gamma}{\alpha}} & \gamma(x) = \alpha\log x & x > 0 & (\textit{in the case } \mathcal{R}_1) \\ h(\gamma) = -e^{-\frac{\gamma}{\alpha}} & \gamma(x) = -\alpha\log(-x) & x < 0 & (\textit{in the case } \mathcal{R}_2) \end{cases} \tag{2.26}$$

Proof. By Theorem 2.5 we get that the only possible solutions of the functional equation (2.22) are functions

$$\begin{array}{lll} V_3(x) = \exp(-x) & \text{for} & x \in \Re, \\ V_1(x) = x^{-\alpha} & \text{for} & x > 0,\ \alpha > 0, \\ V_2(x) = (-x)^{\alpha} & \text{for} & x < 0,\ \alpha > 0. \end{array}$$

Since by Lemma 2.4

$$\gamma(x) = -\log(V(x))$$

then the only possible forms of the functions γ and h are given by (2.26). □

Proposition 2.2 *If the assumptions of Theorem 2.5 are satisfied then there exists $\mu \neq 0$ such that for all natural ν (see Remark 2.1) and those x for which γ is continuous and finite, we get*

$$R(a_{\tau_\nu}x + b_{\tau_\nu}) \quad = \quad \exp\left[-\frac{\log t + \log\mu^{\nu} + \gamma(x) + o(1)}{l_t}\right]. \tag{2.27}$$

Proof. Since, $\tau_\nu \to \infty$ as $t \to \infty$ then by Lemma 2.4 for x where γ is continuous and finite, we get

$$R(a_{\tau_\nu}x + b_{\tau_\nu}) \quad = \quad \exp\left[-\frac{\log\tau_\nu + \gamma(x) + o(1)}{l_{\tau_\nu}}\right].$$

Hence

$$
\begin{aligned}
R(a_{\tau_\nu}x + b_{\tau_\nu}) &= \exp\left[-\frac{\log\tau_\nu + \gamma(x) + o(1)}{l_t\frac{l_{\tau_\nu}}{l_t}}\right] \\
&= \exp\left[-\frac{(\log\tau_\nu + \gamma(x) + o(1))\frac{l_t}{l_{\tau_\nu}}}{l_t}\right].
\end{aligned}
$$

Since by (2.15), (2.17) and (2.19) we get $\frac{l_{\tau_\nu}}{l_t} \to 1$ as $t \to \infty$ then

$$
R(a_{\tau_\nu}x + b_{\tau_\nu}) = \exp\left[-\frac{\log\tau_\nu + \gamma(x) + o(1) + \log\frac{\tau_\nu^{l_t/l_{\tau_\nu}}}{t}}{l_t}\right].
$$

Now, it is enough to show that for all cases of Theorem 2.5

$$
\log\frac{\tau_\nu^{l_t/l_{\tau_\nu}}}{t} = \log\nu^\mu + o(1).
$$

By (2.16)

$$
\begin{aligned}
\log\frac{\tau_\nu^{l_t/l_{\tau_\nu}}}{t} &= \log\left(\exp\left[\frac{\log\tau_\nu}{\frac{l_{\tau_\nu}}{l_t}} - \log t\right]\right) = \frac{\log\tau_\nu}{1 + \frac{\log\frac{\tau_\nu}{t}}{\log t} - \frac{\log\nu}{\log t} \pm r\left(\frac{\delta\log\nu}{\log t}\right)} - \log t \\
&= \frac{\log\tau_\nu - \log t - \log\frac{\tau_\nu}{t} + \log\nu^\mu + o(1)}{1 + o(1)} = \log\nu^\mu + o(1).
\end{aligned}
$$

Similar calculations can be done for Case 2 and Case 3 of Theorem 2.5 using estimation (2.18) and (2.20). This concludes the proof. □

Define a_t^* as follows

$$
a_t^* = G\left(e^{-\frac{\log t + 1}{l_t}}\right) - G\left(e^{-\frac{\log t}{l_t}}\right). \tag{2.28}
$$

By (2.27) for fixed $\mu \neq 0$, we get

$$
a_{\tau_\nu}^* = G\left(e^{-\frac{\log t + \mu\log\nu + 1 + \epsilon_t'}{l_t}}\right) - G\left(e^{-\frac{\log t + \mu\log\nu + \epsilon_t}{l_t}}\right).
$$

Define function $r_t(\beta, \nu)$ as follows

$$
\begin{aligned}
r_t(\beta, \nu) &= r_t(\beta, \nu, \epsilon_t', \epsilon_t) \\
&= \frac{1}{a_t^*}\left[G\left(e^{-\frac{\log t + \mu\log\nu + \beta + \epsilon_t'}{l_t}}\right) - G\left(e^{-\frac{\log t + \mu\log\nu + \epsilon_t}{l_t}}\right)\right]
\end{aligned} \tag{2.29}
$$

where $t \in (0, \infty)$, $\beta \in \Re$, ν is a natural number and $\epsilon'_t = o(1)$, $\epsilon_t = o(1)$.

Now we can formulate theorems about domains of attraction of limit reliability functions given by (2.14). The theorems characterise the domain of attraction in terms of the value of $\lim_{t\to\infty} r_t(1,\nu)$.

Theorem 2.8 *If $R \in D_{\mathcal{R}_3}$ and we choose $a_t = a_t^*$ then*

$$\lim_{t\to\infty} r_t(\beta,\nu) = \beta \tag{2.30}$$

for all $\beta \in \Re$, $\nu \in \mathrm{N}$, fixed $\mu \neq 0$ and $\epsilon'_t = o(1)$, $\epsilon_t = o(1)$.
Conversely, if $\lim_{t\to\infty} r_t(\beta,\nu) = \beta$ exists for all $\beta \in \Re$, $\nu \in \mathrm{N}$, fixed $\mu \neq 0$ and $\epsilon'_t = o(1)$, $\epsilon_t = o(1)$ and if for all natural ν, $\epsilon'_t = o(1)$, $\epsilon_t = o(1)$

$$\lim_{t\to\infty} r_t(1,\nu) = 1 \tag{2.31}$$

then $R \in D_{\mathcal{R}_3}$ with $a_t = a_t^*$ and $b_t = G\left(e^{-\frac{\log t}{l_t}}\right)$.

Proof. If $R \in D_{\mathcal{R}_3}$ with $a_t = a_t^*$ then by (2.29) and from (2.25) and (2.26) we get, for fixed $\mu \neq 0$

$$\begin{aligned}\lim_{t\to\infty} r_t(\beta,\nu) &= \lim_{t\to\infty} \frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\beta+\epsilon'_t}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t}{l_t}}\right)\right] \\ &= h(\mu\log\nu+\beta) - h(\mu\log\nu) = \mu\log\nu+\beta-\mu\log\nu = \beta.\end{aligned}$$

Now we will show the sufficiency. Let $a_t = a_t^*$ and $\mu \neq 0$ then for all natural ν, for all positive, natural numbers k and $\epsilon_t^{(k)}, \epsilon_t^{(0)}$

$$\begin{aligned}&\lim_{t\to\infty} r_t(k,\nu) = \\ &= \lim_{t\to\infty} \tfrac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+k+\epsilon_t^{(k)}}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t^{(0)}}{l_t}}\right)\right] \\ &= \textstyle\sum_{i=0}^{k-1} \lim_{t\to\infty} \tfrac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+(k-i-1)+1+\epsilon_t^{(k-i)}}{l_t}}\right) - \right. \\ &\qquad \left. - G\left(e^{-\frac{\log t+\mu\log\nu+(k-i-1)+\epsilon_t^{(k-i-1)}}{l_t}}\right)\right].\end{aligned} \tag{2.32}$$

By (2.31) we get

$$\lim_{t\to\infty} r_t(k,\nu) = \sum_{i=0}^{k-1} \lim_{t\to\infty} r_t(1,\nu) = \sum_{i=0}^{k-1} 1 = k.$$

For negative integers we have

$$\lim_{t\to\infty} r_t(-k,\nu) = -\lim_{t\to\infty} \frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+(-k)+k+\epsilon_t^{(0)}}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu+(-k)+\epsilon_t^{(k)}}{l_t}}\right)\right]$$

and so from the above we obtain

$$\lim_{t\to\infty} r_t(-k,\nu) \quad = \quad -k.$$

Hence for all integer k, for all natural ν and ϵ_t', ϵ_t

$$\lim_{t\to\infty} r_t(k,\nu) \quad = \quad k. \tag{2.33}$$

Hence for $k = 0$ we get

$$\begin{aligned}
\lim\nolimits_{t\to\infty} \quad & \tfrac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t'}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t}{l_t}}\right)\right] = \\
& = \lim\nolimits_{t\to\infty} \tfrac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t'}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu+k}{l_t}}\right)\right. \\
& \qquad \left. + G\left(e^{-\frac{\log t+\mu\log\nu+k}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t}{l_t}}\right)\right] = \\
& = -k + k = 0.
\end{aligned} \tag{2.34}$$

for all natural ν and all $\epsilon_t' = o(1), \epsilon_t = o(1)$. This shows that $\lim_{t\to\infty} r_t(\beta,\nu,\epsilon_t',\epsilon_t)$ is independent of ϵ_t', ϵ_t.

The next step is to show that this limit does not depend on ν. Choose an arbitrary integer $\nu' > 0$ and consider the reparametrization $r_{t'}(\beta,\nu)$ of $r_t(\beta,\nu)$ where $t' = \tau_{\nu'}$. Function $r_{t'}(\beta,\nu)$ can be written

$$\frac{1}{a_{\tau_{\nu'}}}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\mu\log\nu'+\beta+\epsilon_t^{*'}}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu+\mu\log\nu'+\epsilon_t^{*}}{l_t}}\right)\right]$$

where $\epsilon_t^{*'} = o(1), \epsilon_t^* = o(1)$. We can write it like that because μ is the same for all ν.

From (2.31) and (2.34) this function has the same limit as $r_t(\beta, \nu\nu', \epsilon_t', \epsilon_t)$ as $t \to \infty$.

$$\lim_{t\to\infty} r_t(\beta,\nu) = \lim_{t\to\infty} r_t(\beta,\nu\nu') = \lim_{t\to\infty} r_t(\beta,\nu')$$

which proves that $\lim_{t\to\infty} r_t(\beta,\nu)$ is independent of ν.
We now want to show that $\lim_{t\to\infty} r_t(\beta,\nu) = \beta$ for all real β and start with the case $\beta = \frac{1}{k}$ for some positive integer k.

$$\begin{aligned}
k \quad & \lim_{t\to\infty} r_t(\tfrac{1}{k},\nu) = \\
= \quad & k \lim_{t\to\infty} \tfrac{1}{a_t} \left[G\left(e^{-\frac{\log t + \mu \log \nu + \frac{1}{k}}{l_t}} \right) - G\left(e^{-\frac{\log t + \mu \log \nu}{l_t}} \right) \right] \\
= \quad & \textstyle\sum_{i=0}^{k-1} \lim_{t\to\infty} \tfrac{1}{a_t} \left[G\left(e^{-\frac{\log t + \mu \log \nu + \frac{i+1}{k}}{l_t}} \right) - G\left(e^{-\frac{\log t + \mu \log \nu + \frac{i}{k}}{l_t}} \right) \right] \\
= \quad & \lim_{t\to\infty} \tfrac{1}{a_t} \left[G\left(e^{-\frac{\log t + \mu \log \nu + 1}{l_t}} \right) - G\left(e^{-\frac{\log t + \mu \log \nu}{l_t}} \right) \right] \\
= \quad & 1.
\end{aligned}$$

Hence we get

$$\lim_{t\to\infty} r_t\left(\frac{1}{k},\nu\right) = \frac{1}{k}$$

which by the argument used for (2.32), implies for all integers j, k and $\nu > 1$,$\epsilon_t' = o(1)$, $\epsilon_t = o(1)$ that

$$\lim_{t\to\infty} \frac{1}{a_t} \left[G\left(e^{-\frac{\log t + \mu \log \nu + \frac{j}{k} + \epsilon_t'}{l_t}} \right) - G\left(e^{-\frac{\log t + \mu \log \nu + \epsilon_t}{l_t}} \right) \right] = \frac{j}{k}. \qquad (2.35)$$

Let $r_t', (r_t")$ be the monotone, rational functions converging as $t \to \infty$ from below (above) to the real number β. Then from (2.35), and using the monotonicity of G

$$r_t' \le \lim_{t\to\infty} \frac{1}{a_t} \left[G\left(e^{-\frac{\log t + \mu \log \nu + \beta + \epsilon_t'}{l_t}} \right) - G\left(e^{-\frac{\log t + \mu \log \nu + \epsilon_t}{l_t}} \right) \right] \le r_t".$$

Consequently

$$\lim_{t\to\infty} r_t(\beta,\nu) \quad = \quad \beta$$

for all $\beta \in \Re$, $\nu \in \mathrm{N}$, fixed $\mu \neq 0$ and $\epsilon_t' = o(1)$, $\epsilon_t = o(1)$. We now show that $R \in D_{\mathcal{R}_3}$ by applying Lemma 2.4 and will show that we may take $\gamma(x) = x$

there. Since $\lim_{t\to\infty} r_t(\beta,\nu)$ is independent of and of ϵ_t, ϵ_t' and of ν then we get for all real x

$$\lim_{t\to\infty} \frac{1}{a_t}\left[G\left(e^{-\frac{\log t + x + \epsilon_t}{l_t}}\right) - G\left(e^{-\frac{\log t}{l_t}}\right)\right] = x.$$

If $b_t = G\left(e^{-\frac{\log t}{l_t}}\right)$ then from the above we obtain

$$\lim_{t\to\infty} \frac{1}{a_t}\left[G\left(e^{-\frac{\log t + x + \epsilon_t}{l_t}}\right) - b_t\right] = x.$$

Hence for any $\epsilon > 0$ and $t \in (0,\infty)$

$$a_t(x-\epsilon) + b_t \leq G\left(e^{-\frac{\log t + x + \epsilon_t}{l_t}}\right) \leq a_t(x+\epsilon) + b_t$$

and

$$R(a_t(x+\epsilon) + b_t) \leq e^{-\frac{\log t + x + \epsilon_t}{l_t}} \leq R(a_t(x+\epsilon) + b_t).$$

Hence for sufficiently large t

$$e^{-\frac{\log t + x + 2\epsilon + \epsilon_t}{l_t}} \leq R(a_t x + b_t) \leq e^{-\frac{\log t + x - 2\epsilon + \epsilon_t}{l_t}}$$

which is tantamount to (2.12) for $\gamma(x) = x$ and by Lemma 2.4 completes the proof of sufficiency. □

Theorem 2.9 *If $R \in D_{\mathcal{R}_1}$ and we choose $a_t = a_t^*(e^{\frac{1}{\alpha}} - 1)^{-1}$ then*

$$\lim_{t\to\infty} r_t(\beta,\nu) = \nu^{\frac{\mu}{\alpha}}(e^{\frac{\beta}{\alpha}} - 1)(e^{\frac{1}{\alpha}} - 1)^{-1} \tag{2.36}$$

for all $\beta \in \Re$, $\nu \in \mathrm{N}$, fixed $\mu \neq 0$ and $\epsilon_t' = o(1)$, $\epsilon_t = o(1)$.
Conversely, if $\lim_{t\to\infty} r_t(\beta,\nu) = \beta$ exists for all $\beta \in \Re$, $\nu \in \mathrm{N}$, fixed $\mu \neq 0$ and $\epsilon_t' = o(1)$, $\epsilon_t = o(1)$ and if for all natural ν, $\epsilon_t' = o(1)$, $\epsilon_t = o(1)$

$$\lim_{t\to\infty} r_t(1,\nu) = \nu^{\frac{\mu}{\alpha}} \tag{2.37}$$

then $R \in D_{\mathcal{R}_1}$ with $a_t = a_t^*(e^{\frac{1}{\alpha}} - 1)^{-1}$ and $b_t = G\left(e^{-\frac{\log t}{l_t}}\right) - a_t$.

Proof. If $R \in D_{\mathcal{R}_1}$ with $a_t = a_t^*(e^{\frac{1}{\alpha}} - 1)^{-1}$ then by (2.29) and from (2.25) and (2.26) we get, for fixed $\mu \neq 0$

$$\begin{aligned}
\lim_{t\to\infty} r_t(\beta,\nu) &= \lim_{t\to\infty} \frac{1}{a_t^*}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\beta+\epsilon_t'}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t}{l_t}}\right)\right] \\
&= (e^{\frac{1}{\alpha}}-1)^{-1}[h(\mu\log\nu+\beta)-h(\mu\log\nu)] \\
&= (e^{\frac{1}{\alpha}}-1)^{-1}[e^{\frac{\mu\log\nu+\beta}{\alpha}} - e^{\frac{\mu\log\nu}{\alpha}}] \\
&= (e^{\frac{1}{\alpha}}-1)^{-1}\nu^{\frac{\mu}{\alpha}}[e^{\frac{\beta}{\alpha}}-1]
\end{aligned}$$

which proves (2.36).
Now we will show the sufficiency. Let $a_t = a_t^*(e^{\frac{1}{\alpha}} - 1)^{-1}$ then by (2.37)

$$\begin{aligned}
\lim_{t\to\infty} r_t(1,\nu) &= \lim_{t\to\infty} \frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+k+\epsilon_t'}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t}{l_t}}\right)\right] \\
&= \nu^{\frac{\mu}{\alpha}}(e^{\frac{1}{\alpha}}-1).
\end{aligned}$$

Hence for all positive, natural numbers k

$$\begin{aligned}
\lim_{t\to\infty} r_t(k,\nu) &= \lim_{t\to\infty} \frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+k+\epsilon_t^{(k)}}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t^{(0)}}{l_t}}\right)\right] \\
&= \sum_{i=0}^{k-1} \lim_{t\to\infty} \frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+(k-i-1)+1+\epsilon_t^{(k-i)}}{l_t}}\right)\right. \\
&\qquad\left. -G\left(e^{-\frac{\log t+\mu\log\nu+(k-i-1)+\epsilon_t^{(k-i-1)}}{l_t}}\right)\right].
\end{aligned}$$

By (2.37) we get

$$\begin{aligned}
\lim_{t\to\infty} r_t(k,\nu) &= \sum_{i=0}^{k-1} \lim_{t\to\infty} \exp\left[\frac{\mu\log\nu+k-i-1}{\alpha}\right](e^{\frac{1}{\alpha}}-1) \\
&= \nu^{\frac{\mu}{\alpha}} e^{\frac{k}{\alpha}} e^{-\frac{1}{\alpha}}(e^{\frac{1}{\alpha}}-1)\sum_{i=0}^{k-1}(e^{-\frac{1}{\alpha}}) \\
&= \nu^{\frac{\mu}{\alpha}} e^{\frac{k}{\alpha}} e^{-\frac{1}{\alpha}}(e^{\frac{1}{\alpha}}-1)\frac{1-e^{-\frac{k}{\alpha}}}{1-e^{-\frac{1}{\alpha}}} = \nu^{\frac{\mu}{\alpha}}(e^{-\frac{k}{\alpha}}-1).
\end{aligned}$$

For negative integers k we have

$$\begin{aligned}\lim_{t\to\infty} r_t(-k,\nu) &= -\lim_{t\to\infty}\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+(-k)+k+\epsilon_t^{(0)}}{l_t}}\right)\right.\\ &\qquad\left.-G\left(e^{-\frac{\log t+\mu\log\nu+(-k)+\epsilon_t^{(k)}}{l_t}}\right)\right] =\\ &= -\nu^{\frac{\mu}{\alpha}}e^{-\frac{k}{\alpha}}(e^{\frac{k}{\alpha}}-1) = \nu^{\frac{\mu}{\alpha}}(e^{-\frac{k}{\alpha}}-1).\end{aligned}$$

Hence for all integer k, for all natural ν and ϵ_t', ϵ_t

$$\lim_{t\to\infty} r_t(k,\nu) = \nu^{\frac{\mu}{\alpha}}(e^{\frac{k}{\alpha}}-1). \tag{2.38}$$

Hence for $k=0$ we get

$$\begin{aligned}\lim_{t\to\infty} &\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t'}{l_t}}\right)-G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t}{l_t}}\right)\right] =\\ &= \lim_{t\to\infty}\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t'}{l_t}}\right)-G\left(e^{-\frac{\log t+\mu\log\nu+k}{l_t}}\right)\right.\\ &\qquad\left.+G\left(e^{-\frac{\log t+\mu\log\nu+k}{l_t}}\right)-G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t}{l_t}}\right)\right] =\\ &= -\nu^{\frac{\mu}{\alpha}}(e^{\frac{k}{\alpha}}-1)+\nu^{\frac{\mu}{\alpha}}(e^{\frac{k}{\alpha}}-1) = 0.\end{aligned} \tag{2.39}$$

for all natural ν and all $\epsilon_t' = o(1), \epsilon_t = o(1)$, which means that $\lim_{t\to\infty} r_t(\beta,\nu,\epsilon_t',\epsilon_t)$ is independent of ϵ_t', ϵ_t.
The next step is to show that this limit does not depend on ν. Choose an arbitrary integer $\nu' > 0$ and consider the reparametrization $r_{t'}(\beta,\nu)$ of $r_t(\beta,\nu)$ where $t' = \tau_{\nu'}$. Function $r_{t'}(\beta,\nu)$ can be written

$$\frac{1}{a_{\tau_{\nu'}}}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\mu\log\nu'+\beta+\epsilon_t^{*'}}{l_t}}\right)-G\left(e^{-\frac{\log t+\mu\log\nu+\mu\log\nu'+\epsilon_t^{*}}{l_t}}\right)\right]$$

where $\epsilon_t^{*'} = o(1), \epsilon_t^* = o(1)$. We can write it like that because μ is the same for all ν.
From (2.37) and (2.38) this function has the same limit as $\nu^{-\frac{\mu}{\alpha}} r_t(\beta,\nu\nu',\epsilon_t',\epsilon_t)$ as $t\to\infty$. Hence

$$\lim_{t\to\infty}\nu^{-\frac{\mu}{\alpha}} r_t(\beta,\nu) = \lim_{t\to\infty}\nu^{-\frac{\mu}{\alpha}} r_t(\beta,\nu\nu') = \lim_{t\to\infty}\nu^{-\frac{\mu}{\alpha}} r_t(\beta,\nu')$$

which proves that $\lim_{t\to\infty} \nu^{-\frac{\mu}{\alpha}} r_t(\beta,\nu)$ is independent of ν.
We now want to show that

$$\lim_{t\to\infty} \frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t'}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t}{l_t}}\right)\right] = \nu^{\frac{\mu}{\alpha}}(e^{\frac{\beta}{\alpha}}-1)$$

for all real β and start with the case $\beta = \frac{1}{k}$ for some positive integer k. From (2.38)

$$\begin{aligned}
\nu^{\frac{\mu}{\alpha}}(e^{\frac{1}{\alpha}}-1) &= \lim_{t\to\infty}\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+1}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu}{l_t}}\right)\right] \\
&= \lim_{t\to\infty}\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\frac{k}{k}}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu+\frac{k-1}{k}}{l_t}}\right)\right. \\
&\qquad +G\left(e^{-\frac{\log t+\mu\log\nu+\frac{k-1}{k}}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu+\frac{k-2}{k}}{l_t}}\right) \\
&\qquad\qquad \cdot\quad\cdot\quad\cdot\quad\cdot \\
&\qquad \left.+G\left(e^{-\frac{\log t+\mu\log\nu+\frac{k-1}{k}}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu+\frac{k-2}{k}}{l_t}}\right)\right] \\
&= \lim_{t\to\infty}\frac{1}{a_t}\sum_{i=0}^{k-1}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\frac{i+1}{k}}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu+\frac{i}{k}}{l_t}}\right)\right] \\
&= \left[\sum_{i=0}^{k-1}\nu^{\frac{\mu}{\alpha}}(e^{\frac{1}{k\alpha}})^i\right]\lim_{t\to\infty}\frac{1}{a_t}\nu^{-\frac{\mu}{\alpha}}\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\frac{1}{k}}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu}{l_t}}\right)\right] \\
&= \nu^{\frac{\mu}{\alpha}}\frac{1-e^{\frac{1}{\alpha}}}{1-e^{\frac{1}{k\alpha}}}\nu^{-\frac{\mu}{\alpha}}\lim_{t\to\infty}\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\frac{1}{k}}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu}{l_t}}\right)\right]
\end{aligned}$$

Hence for all natural k, fixed $\mu \neq 0$ and all natural ν and $\epsilon_t' = o(1), \epsilon_t = o(1)$, we obtain

$$\lim_{t\to\infty}\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\frac{1}{k}+\epsilon_t'}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t}{l_t}}\right)\right] = \nu^{\frac{\mu}{\alpha}}\left(e^{\frac{1}{k\alpha}}-1\right).$$

For rational r this condition takes form

$$\lim_{t\to\infty}\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+r+\epsilon_t'}{l_t}}\right) - G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t}{l_t}}\right)\right] = \nu^{\frac{\mu}{\alpha}}\left(e^{\frac{r}{\alpha}}-1\right).$$

As in the proof of Theorem 2.8, we can take monotone, rational functions converging from below (above) to the real number x and by monotonicity of G get

$$\lim_{t\to\infty}\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+x+\epsilon'_t}{l_t}}\right)-G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t}{l_t}}\right)\right]=\nu^{\frac{\mu}{\alpha}}\left(e^{\frac{x}{\alpha}}-1\right).$$

Since $\lim_{t\to\infty}\nu^{-\frac{\mu}{\alpha}}r_t(\beta,\nu)$ is independent of ν and ϵ'_t,ϵ_t then for sufficiently large t and for any $\epsilon>0$

$$e^{\frac{x}{\alpha}}-1-\epsilon\le\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+x+\epsilon_t}{l_t}}\right)-G\left(e^{-\frac{\log t}{l_t}}\right)\right]\le e^{\frac{x}{\alpha}}-1+\epsilon.$$

Let $b_t=G\left(e^{-\frac{\log t}{l_t}}\right)-a_t$ then we obtain

$$a_t(e^{\frac{x}{\alpha}}-\epsilon)+b_t\le G\left(e^{-\frac{\log t+x+\epsilon_t}{l_t}}\right)\le a_t(e^{\frac{x}{\alpha}}+\epsilon)+b_t$$

which is tantamount to (2.12) for $\gamma(x)=\alpha\log x,\ \alpha>0,\ x>0$ and by Lemma 2.4 completes the proof of sufficiency. □

Theorem 2.10 *If $R\in D_{\mathcal{R}_2}$ and we choose $a_t=a^*_t(1-e^{-\frac{1}{\alpha}})^{-1}$ then*

$$\lim_{t\to\infty} r_t(\beta,\nu)=\nu^{-\frac{\mu}{\alpha}}(e^{-\frac{\beta}{\alpha}}-1)(e^{-\frac{1}{\alpha}}-1)^{-1} \tag{2.40}$$

for all $\beta\in\Re,\ \nu\in\mathrm{N}$, fixed $\mu\neq 0$ and $\epsilon'_t=o(1),\ \epsilon_t=o(1)$.
Conversely, if $\lim_{t\to\infty}\ r_t(\beta,\nu)=\beta$ exists for all $\beta\in\Re,\ \nu\in\mathrm{N}$, fixed $\mu\neq 0$ and $\epsilon'_t=o(1),\ \epsilon_t=o(1)$ and if for all natural ν, $\epsilon'_t=o(1),\ \epsilon_t=o(1)$

$$\lim_{t\to\infty}\ r_t(1,\nu)=\nu^{-\frac{\mu}{\alpha}} \tag{2.41}$$

then $R\in D_{\mathcal{R}_2}$ with $a_t=a^*_t(1-e^{-\frac{1}{\alpha}})^{-1}$ and $b_t=G\left(e^{-\frac{\log t}{l_t}}\right)+a_t$.

Proof. If $R\in D_{\mathcal{R}_2}$ with $a_t=a^*_t(1-e^{-\frac{1}{\alpha}})^{-1}$ then by (2.29) and from (2.25) and (2.26) we get, for fixed $\mu\neq 0$

$$\begin{aligned}\lim_{t\to\infty}\ r_t(\beta,\nu) &= \lim_{t\to\infty}\frac{1}{a^*_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\beta+\epsilon'_t}{l_t}}\right)-G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t}{l_t}}\right)\right]\\ &= (1-e^{-\frac{1}{\alpha}})^{-1}[h(\mu\log\nu+\beta)-h(\mu\log\nu)]\\ &= (1-e^{-\frac{1}{\alpha}})^{-1}[-e^{-\frac{\mu\log\nu+\beta}{\alpha}}-e^{-\frac{\mu\log\nu}{\alpha}}]\\ &= (e^{-\frac{1}{\alpha}}-1)^{-1}\nu^{-\frac{\mu}{\alpha}}[e^{-\frac{\beta}{\alpha}}-1]\end{aligned}$$

which proves (2.40).
Now we will show the sufficiency condition. Set $a_t = a_t^*(1-e^{-\frac{1}{\alpha}})^{-1}$. Then, as in the proof of Theorem 2.8 a simple consequence of (2.41) is

$$\lim_{t\to\infty}\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+k+\epsilon_t'}{l_t}}\right)-G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t}{l_t}}\right)\right]=\nu^{-\frac{\mu}{\alpha}}(1-e^{\frac{k}{\alpha}}) \quad (2.42)$$

for all integer k, for all natural ν and ϵ_t', ϵ_t. When we consider the case $k=0$ then analogous way like in the proof of Theorem 2.8 we can have that $r_t(\beta,\nu,\epsilon_t',\epsilon_t)$ is independent of ϵ_t', ϵ_t.
As in Theorem 2.8, we can show that $\lim_{t\to\infty}\nu^{\frac{\mu}{\alpha}}r_t(\beta,\nu)$ is independent of ν.
We now want to show that

$$\lim_{t\to\infty}\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t'}{l_t}}\right)-G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t}{l_t}}\right)\right]=\nu^{-\frac{\mu}{\alpha}}(1-e^{-\frac{\beta}{\alpha}}).$$

for all real β. In the case $\beta=\frac{1}{k}$ for some positive integer k we get. From (2.40)

$$\begin{aligned}
\nu^{-\frac{\mu}{\alpha}}(1-e^{-\frac{1}{\alpha}}) &= \lim_{t\to\infty}\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+1}{l_t}}\right)-G\left(e^{-\frac{\log t+\mu\log\nu}{l_t}}\right)\right]\\
&= \lim_{t\to\infty}\frac{1}{a_t}\sum_{i=0}^{k-1}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\frac{i+1}{k}}{l_t}}\right)-G\left(e^{-\frac{\log t+\mu\log\nu+\frac{i}{k}}{l_t}}\right)\right]\\
&= \left[\sum_{i=0}^{k-1}\nu^{-\frac{\mu}{\alpha}}(e^{-\frac{1}{k\alpha}})^i\right]\lim_{t\to\infty}\nu^{\frac{\mu}{\alpha}}\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\frac{1}{k}}{l_t}}\right)-G\left(e^{-\frac{\log t+\mu\log\nu}{l_t}}\right)\right]\\
&= \nu^{-\frac{\mu}{\alpha}}\frac{1-e^{-\frac{1}{\alpha}}}{1-e^{-\frac{1}{k\alpha}}}\nu^{\frac{\mu}{\alpha}}\lim_{t\to\infty}\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\frac{1}{k}}{l_t}}\right)-G\left(e^{-\frac{\log t+\mu\log\nu}{l_t}}\right)\right]
\end{aligned}$$

Hence for all natural k, fixed $\mu\neq 0$ and all natural ν and $\epsilon_t'=o(1), \epsilon_t=o(1)$, we obtain

$$\lim_{t\to\infty}\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+\frac{1}{k}+\epsilon_t'}{l_t}}\right)-G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t}{l_t}}\right)\right]=\nu^{-\frac{\mu}{\alpha}}\left(1-e^{-\frac{1}{k\alpha}}\right).$$

For rational r this condition takes form

$$\lim_{t\to\infty}\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+r+\epsilon_t'}{l_t}}\right)-G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t}{l_t}}\right)\right]=\nu^{-\frac{\mu}{\alpha}}\left(1-e^{-\frac{r}{\alpha}}\right).$$

As in the proof of Theorem 2.8, we can take monotone , rational functions converging from below (above) to the real number x and by monotonicity of G get

$$\lim_{t\to\infty}\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+\mu\log\nu+x+\epsilon_t'}{l_t}}\right)-G\left(e^{-\frac{\log t+\mu\log\nu+\epsilon_t}{l_t}}\right)\right]=\nu^{-\frac{\mu}{\alpha}}\left(1-e^{-\frac{x}{\alpha}}\right).$$

Hence for sufficiently large t and for any $\epsilon>0$

$$1-e^{-\frac{x}{\alpha}}-\epsilon\le\frac{1}{a_t}\left[G\left(e^{-\frac{\log t+x+\epsilon_t}{l_t}}\right)-G\left(e^{-\frac{\log t}{l_t}}\right)\right]\le 1-e^{-\frac{x}{\alpha}}+\epsilon.$$

Taking $b_t=G\left(e^{-\frac{\log t}{l_t}}\right)+a_t$ we obtain

$$a_t(-e^{-\frac{x}{\alpha}}-\epsilon)+b_t\le G\left(e^{-\frac{\log t+x+\epsilon_t}{l_t}}\right)\le a_t(-e^{-\frac{x}{\alpha}}+\epsilon)+b_t$$

which is tantamount to (2.12) for $\gamma(x)=-\alpha\log(-x),\ \alpha>0,\ x<0$ and by Lemma 2.4 completes the proof of sufficiency. □

Theorem 2.11 *$R\in D_{\mathcal{R}_4}$ if and only if*

(a) *there exists an unique y such that $R(y)=e^{-\frac{1}{c}}$,*

(b) $\lim_{r\downarrow 0}\frac{1+c\log R(rx+y)}{1+c\log R(r+y)}=x^{\alpha},\ x>0,$
$\lim_{r\downarrow 0}\frac{1+c\log R(rx+y)}{1+c\log R(r+y)}=-\infty,\ x<0.$

Proof. Let the reliability function R belongs to the domain of attraction of $\mathcal{R}_4$. By Lemma 2.1 there exist $a_t>0$ and $b_t\in(-\infty,\infty)$ such that

$$\lim_{t\to\infty}t(R(a_tx+b_t))^{l_t}=V^*(x)=\begin{cases}\infty, & x<0\\ e^{-x^{\alpha}-\frac{s}{c}}, & x\ge 0.\end{cases}$$

Since in this case $l_t=c\log t+s(t)$ where $s(t)\approx s$ then

$$t\left(R(a_tx+b_t)\right)^{l_t}=t^{1-\frac{c\log t+s(t)}{c\log t}}\left[t\left(R(a_tx+b_t)\right)^{c\log t}\right]^{\frac{c\log t+s(t)}{c\log t}}. \tag{2.43}$$

Moreover we get

$$t^{1-\frac{c\log t+s(t)}{c\log t}}\to e^{-\frac{s}{c}}\text{ and }\frac{c\log t+s(t)}{c\log t}\to 1\text{ as }t\to\infty.$$

Therefore

$$\lim_{t\to\infty} t\left(R(a_t x + b_t)\right)^{c\log t} = V(x) \tag{2.44}$$

where

$$V(x) \quad = \quad \begin{cases} \infty, & x < 0 \\ e^{-x^{\alpha}}, & x \geq 0. \end{cases}$$

We get for $x \geq 0$ as $t \to \infty$

$$\left[t\left(R(a_t x + b_t)\right)^{c\log t}\right]^2 \to e^{-2x^{\alpha}}$$

and further

$$t^2\left(R(a_t x + b_t)\right)^{c\log t^2} \to e^{-(\gamma x)^{\alpha}} \tag{2.45}$$

where

$$\gamma = 2^{\frac{1}{\alpha}}.$$

Replacing in (2.45) γx by x, we get

$$t^2\left(R(\frac{a_t}{\gamma}x + b_t)\right)^{c\log t^2} \to e^{-x^{\alpha}}.$$

By Lemma 2.2, we get

$$a_{t^2} = \frac{a_t}{\gamma}, \quad b_{t^2} = b_t.$$

We can repeat this process and this way, for all natural k, we get

$$a_{t^{2^k}} = \frac{a_t}{\gamma^k}, \quad b_{t^{2^k}} = b_{t^{2^{k-1}}} = \ldots = b_{t^2} = b_t. \tag{2.46}$$

Hence $a_t \to 0$ as $t \to \infty$ and b_t is a constant, say y. For $x = 0$, we obtain from (2.48)

$$t\left(R(b_t)\right)^{c\log t} \to 1 \text{ and } e^{\log t(1+c\log R(b_t))} \to 1 \text{ as } t \to \infty.$$

Hence

$$R(y) = e^{-\frac{1}{c}}. \tag{2.47}$$

We can show that y is unique.
Let there exists $y' < y$ such that $R(y') = e^{-\frac{1}{c}}$ then since $a_t \to 0$ we get

$$\forall_{x<0}\ \exists_{T_x}\ \forall_{t>T_x}\ \ a_t x + b_t \in <y', y) \text{ and } R(a_t x + b_t) = e^{-\frac{1}{c}}.$$

Hence

$$\forall_{x<0}\ t\left(R(a_t x + b_t)\right)^{c\log t} \to e^0 = 1$$

which is inconsistent with (2.44).
In analogous way we can show that there is no $y" > y$, such that $R(y") = e^{-\frac{1}{c}}$. Therefore the condition (a) is proved.
Hence and by (2.45), we get

$$\lim_{t\to\infty} \log t\,(1 + c\log R(a_t x + b_t)) = \log V(x) = \begin{cases} \infty, & x < 0 \\ -x^{\alpha}, & x \geq 0. \end{cases} \tag{2.48}$$

Let $r \to 0^+$. Since $a_t \to 0$ as $t \to \infty$ then for sufficiently small $r > 0$ we can choose sufficiently large t, such that

$$a_t \leq r \leq a_{t+1} \ \text{ if } \ a_t \leq a_{t+1} \tag{2.49}$$

or

$$a_{t+1} \leq r \leq a_t \ \text{ if } \ a_{t+1} \leq a_t. \tag{2.50}$$

Hence, for such that (2.49) holds, we get

$$1 + c\log R(a_{t+1} + y) \leq 1 + c\log R(r + y) \leq 1 + c\log R(a_t + y).$$

For $x > 0$ we also get

$$1 + c\log R(a_{t+1}x + y) \leq 1 + c\log R(rx + y) \leq 1 + c\log R(a_t x + y)$$

and for $x < 0$

$$1 + c\log R(a_t x + y) \leq 1 + c\log R(rx + y) \leq 1 + c\log R(a_{t+1}x + y).$$

This leads to

$$\frac{1 + c\log R(a_{t+1}x + y)}{1 + c\log R(a_t + y)} \leq \frac{1 + c\log R(rx + y)}{1 + c\log R(r + y)} \leq \frac{1 + c\log R(a_t x + y)}{1 + c\log R(a_{t+1} + y)} \tag{2.51}$$

for $x > 0$ and for $x < 0$ we get

$$\frac{1 + c\log R(a_t x + y)}{1 + c\log R(a_t + y)} \leq \frac{1 + c\log R(rx + y)}{1 + c\log R(r + y)} \leq \frac{1 + c\log R(a_{t+1}x + y)}{1 + c\log R(a_{t+1} + y)} \tag{2.52}$$

Since

$$\frac{\log(t+1)}{\log t} \to 1 \text{ as } t \to \infty$$

then from (2.48), it follows that, denominators of right and left sides of that inequality (2.51) are convergent to -1 and numerators to $-x^\alpha$ for $x > 0$. By (2.48) we also get that dominators of right and left sides of inequality (2.52) are convergent to -1 and numerators to ∞ for $x < 0$. In analogous way, we can consider the case when (2.50) holds and we conclude the proof of necessity.
Now, suppose that (a) and (b) hold. We will show that R belongs to the domain of attraction of $\mathcal{R}_4$. Define for $x > 0$

$$\begin{aligned} a_t = \inf\{x : 1 + c\log R(x(1+0) + y) \quad &\leq -\tfrac{1}{\log t} \leq \\ &\leq 1 + c\log R(x(1-0) + y)\} \end{aligned} \tag{2.53}$$

Since, according to (a) and (b) $a_t \to 0$ as $t \to \infty$ then by (b), for $\epsilon \in (0,1)$ and for $x > 0$ if $t \to \infty$, we have

$$\frac{1 + c\log R(a_t x + y)}{1 + c\log R(a_t(1-\epsilon) + y)} = \frac{1 + c\log R(a_t(1-\epsilon)\frac{x}{1-\epsilon} + y)}{1 + c\log R(a_t(1-\epsilon) + y)} \to \left(\frac{x}{1-\epsilon}\right)^\alpha,$$
$$\frac{1 + c\log R(a_t x + y)}{1 + c\log R(a_t(1+\epsilon) + y)} = \frac{1 + c\log R(a_t(1+\epsilon)\frac{x}{1+\epsilon} + y)}{1 + c\log R(a_t(1+\epsilon) + y)} \to \left(\frac{x}{1+\epsilon}\right)^\alpha.$$

For $x < 0$, we get

$$\frac{1 + c\log R(a_t x + y)}{1 + c\log R(a_t(1-\epsilon) + y)} \to -\infty, \quad \frac{1 + c\log R(a_t x + y)}{1 + c\log R(a_t(1+\epsilon) + y)} \to -\infty.$$

The left sides of the above relations are continuous functions of ϵ and monotone and the right sides are continuous functions. Therefore the convergence is uniform. Now, for $x > 0$, we can write that for $t \to \infty$

$$\frac{1 + c\log R(a_t x + y)}{1 + c\log R(a_t(1-0) + y)} \to x^\alpha, \quad \frac{1 + c\log R(a_t x + y)}{1 + c\log R(a_t(1+0) + y)} \to x^\alpha \tag{2.54}$$

and moreover for $x < 0$

$$\frac{1 + c\log R(a_t x + y)}{1 + c\log R(a_t(1-0) + y)} \to -\infty, \quad \frac{1 + c\log R(a_t x + y)}{1 + c\log R(a_t(1+0) + y)} \to -\infty. \tag{2.55}$$

Form definition (2.53), for all x, we have

$$\frac{1 + c \log R(a_t x + y)}{1 + c \log R(a_t (1-0) + y)} \leq - \log t \, (1 + c \log R(a_t x + y)) \leq \frac{1 + c \log R(a_t x + y)}{1 + c \log R(a_t (1+0) + y)}$$

By (2.54) and (2.55)

$$\log t \, (1 + c \log R(a_t x + y)) \to \begin{cases} \infty, & x < 0 \\ -x^{\alpha}, & x > 0. \end{cases}$$

For $x = 0$ we get

$$\log t \, (1 + c \log R(a_t x + y)) \to 0$$

hence

$$\lim_{t \to \infty} t \, (R(a_t x + b_t))^{c \log t} = V(x)$$

where

$$V(x) = \begin{cases} \infty, & x < 0 \\ e^{-x^{\alpha}}, & x \geq 0. \end{cases}$$

Considering the beginning of the proof

$$\lim_{t \to \infty} t \, (R(a_t x + b_t))^{l_t} = V^*(x)$$

where

$$V^*(x) = \begin{cases} \infty, & x < 0 \\ e^{-x^{\alpha} - \frac{s}{c}}, & x \geq 0. \end{cases}$$

By Lemma 2.1, we conclude that R belongs to the domain of attraction of $\mathcal{R}_4$. $\square$

Theorem 2.12 *$R \in D_{\mathcal{R}_5}$ if and only if*

(a) *there exists an unique y such that $R(y-0) = e^{-\frac{1}{c}}$,*

(b) $\lim_{r \uparrow 0} \frac{1 + c \log R(rx+y)}{1 + c \log R(r+y)} = x^{\alpha}$, $x > 0$,
$\lim_{r \uparrow 0} \frac{1 + c \log R(rx+y)}{1 + c \log R(r+y)} = -\infty$, $x < 0$.

Proof. Let a reliability function R belongs to the domain of attraction of $\mathcal{R}_5$. By Lemma 2.1 there exist $a_t > 0$ and $b_t \in (-\infty, \infty)$ such that

$$\lim_{t\to\infty} t(R(a_t x + b_t))^{l_t} = e^{-\frac{s}{c}} V(x) = \begin{cases} e^{(-x)^\alpha - \frac{s}{c}}, & x < 0 \\ 0, & x \geq 0. \end{cases}$$

and

$$\lim_{t\to\infty} \log t\,(1 + c \log R(a_t x + b_t)) = \log V(x) = \begin{cases} (-x)^\alpha, & x < 0 \\ -\infty, & x \geq 0. \end{cases} \qquad (2.56)$$

Proceeding in the same way as in the proof of the Theorem 2.11, we can construct a_t and b_t such that

$$a_t \to 0 \text{ as } t \to \infty \text{ and } b_t = y$$

From (2.56), we have

$$\forall_{x<0} \quad \log t\,(1 + c \log R(a_t x + b_t)) \to (-x)^\alpha.$$

Hence

$$\lim_{x\uparrow y} R(x) = e^{-\frac{1}{c}}.$$

As in the proof of Theorem 2.11 we can show that y is unique. Therefore part (a) is proved.
Further

$$\lim_{t\to\infty} \log t\,(1 + c \log R(a_t x + y)) = \log V(x) = \begin{cases} (-x)^\alpha, & x < 0 \\ -\infty, & x \geq 0. \end{cases} \qquad (2.57)$$

Let $r \to 0^-$. Since $a_t \to 0$ as $t \to \infty$ then for sufficiently small $-r > 0$ we can choose sufficiently large t, such that

$$-a_t \leq r \leq -a_{t+1} \text{ if } a_{t+1} \leq a_t \qquad (2.58)$$

or

$$-a_{t+1} \leq r \leq -a_t \text{ if } a_t \leq a_{t+1}. \qquad (2.59)$$

As in the proof of Theorem 2.11, in the case when (2.58) holds, we get

$$\frac{1 + c \log R(-a_{t+1} x + y)}{1 + c \log R(-a_t + y)} \leq \frac{1 + c \log R(rx + y)}{1 + c \log R(r + y)} \leq \frac{1 + c \log R(-a_t x + y)}{1 + c \log R(-a_{t+1} + y)}$$

for $x > 0$ and

$$\frac{1+c\log R(-a_t x+y)}{1+c\log R(-a_t+y)} \le \frac{1+c\log R(rx+y)}{1+c\log R(r+y)} \le \frac{1+c\log R(-a_{t+1}x+y)}{1+c\log R(-a_{t+1}+y)}.$$

for $x < 0$. We can consider also the case when (2.59) holds. This way we obtain that (b) is satisfied.
Now, suppose that (a) and (b) hold. We will show that R belongs to the domain of attraction of $\mathcal{R}_5$. Define for $x > 0$

$$\begin{aligned} a_t = \inf\{x : 1+c\log R(-x(1-0)+y) \quad &\le \tfrac{1}{\log t} \le \\ &\le 1+c\log R(-x(1+0)+y)\} \end{aligned} \tag{2.60}$$

Since, according to (a) and (b) $a_t \to 0$ as $t \to \infty$ then by (b), for $\epsilon \in (0,1)$ and for $x < 0$ if $t \to \infty$, we have

$$\frac{1+c\log R(a_t x+y)}{1+c\log R(-a_t(1-\epsilon)+y)} = \frac{1+c\log R(-a_t(1-\epsilon)\frac{-x}{1-\epsilon}+y)}{1+c\log R(a_t(1-\epsilon)+y)} \to \left(-\frac{x}{1-\epsilon}\right)^{\alpha},$$
$$\frac{1+c\log R(a_t x+y)}{1+c\log R(-a_t(1+\epsilon)+y)} = \frac{1+c\log R(a_t(1+\epsilon)\frac{-x}{1+\epsilon}+y)}{1+c\log R(-a_t(1+\epsilon)+y)} \to \left(-\frac{x}{1+\epsilon}\right)^{\alpha}.$$

For $x > 0$, we get

$$\frac{1+c\log R(a_t x+y)}{1+c\log R(-a_t(1-\epsilon)+y)} \to -\infty, \quad \frac{1+c\log R(a_t x+y)}{1+c\log R(-a_t(1+\epsilon)+y)} \to -\infty.$$

The left sides of the above relations are continuous functions of ϵ and monotone and the right sides are continuous functions. Therefore the convergence is uniform. Now, for $x < 0$, we can write that for $t \to \infty$

$$\frac{1+c\log R(a_t x+y)}{1+c\log R(-a_t(1-0)+y)} \to (-x)^{\alpha}, \tag{2.61}$$
$$\frac{1+c\log R(a_t x+y)}{1+c\log R(a_t(1+0)+y)} \to (-x)^{\alpha} \tag{2.62}$$

and moreover for $x > 0$

$$\frac{1+c\log R(a_t x+y)}{1+c\log R(-a_t(1-0)+y)} \to -\infty, \tag{2.63}$$
$$\frac{1+c\log R(a_t x+y)}{1+c\log R(-a_t(1+0)+y)} \to -\infty. \tag{2.64}$$

Form definition (2.60), for all x, we have

$$\frac{1+c\log R(a_t x+y)}{1+c\log R(-a_t(1-0)+y)} \le \log t\,(1+c\log R(a_t x+y)) \le \frac{1+c\log R(a_t x+y)}{1+c\log R(-a_t(1+0)+y)}$$

By (2.61),2.62) and (2.63), (2.64)

$$\log t\,(1+c\log R(a_t x+y)) \to \begin{cases} (-x)^\alpha, & x<0 \\ -\infty, & x\ge 0 \end{cases}$$

and

$$\lim_{t\to\infty} t\,(R(a_t x+b_t))^{c\log t} = V(x)$$

where

$$V(x) \quad = \quad \begin{cases} e^{(-x)^\alpha}, & x<0 \\ 0, & x\ge 0. \end{cases}$$

Finally we get

$$\lim_{t\to\infty} t(R(a_t x+b_t))^{l_t} = e^{-\frac{s}{c}}V(x) = \begin{cases} e^{(-x)^\alpha-\frac{s}{c}}, & x<0 \\ 0, & x\ge 0. \end{cases}$$

By Lemma 2.1, we conclude that R belongs to the domain of attraction of $\mathcal{R}_5$. $\square$

Theorem 2.13 *$R\in D_{\mathcal{R}_6}$ if and only if*

(a) *there exists an unique y such that $R(y)=e^{-\frac{1}{c}}$,*

(b) $\lim_{r\to 0}\frac{1+c\log R(rx+y)}{1+c\log R(r+y)} = x^\alpha$, $x>0$.

Proof. Let $R\in D_{\mathcal{R}_6}$. From Theorem 2.12, we have

$$\exists_{y_2} \quad R(y_2-0)=e^{-\frac{1}{c}} \tag{2.65}$$

and

$$\lim_{r\uparrow 0}\frac{1+c\log R(r(\beta^{\frac{1}{\alpha}}x)+y_2)}{1+c\log R(r+y_2)} = \beta x^\alpha, \;\; x>0,\; \beta>0$$

that is

$$\lim_{r\uparrow 0}\frac{1+c\log R(rx+y)}{1+c\log R(r+y)}=x^{\alpha},\ \ x>0 \tag{2.66}$$

and

$$a_t\to 0,\ \ b_t\to y_2.$$

From Theorem 2.11, we get

$$\exists_{y_1}\ \ R(y_1)=e^{-\frac{1}{c}} \tag{2.67}$$

and

$$\lim_{r\downarrow 0}\frac{1+c\log R(rx+y_1)}{1+c\log R(r+y_1)}=x^{\alpha},\ \ x>0,\ \beta>0 \tag{2.68}$$

and

$$a_t\to 0,\ \ b_t\to y_1.$$

Hence $y_1=y_2=y$. By (2.65) and (2.67), we get (a) and by (2.66), (2.68), we get (b).
Now, suppose that (a) and (b) hold. To fix the form of limit reliability function $\mathcal{R}$ for $x<0$, we may notice that

$$\exists_{y}\ \ \ R(y)=e^{-\frac{1}{c}} \text{ hence } R(y-0)=e^{-\frac{1}{c}}. \tag{2.69}$$

By (b) for $\beta_1>0$, we get

$$\lim_{r\uparrow 0}\frac{1+c\log R(r(\beta_1^{\frac{1}{\alpha}}x)+y_2)}{1+c\log R(r+y_2)}=\beta_1 x^{\alpha},\ \ x>0. \tag{2.70}$$

By (2.69) and (2.70), considering Theorem 2.12, we get

$$\mathcal{R}(x)=1-\exp[-\exp[\beta_1(-x)^{\alpha}-\frac{s}{c}]]\quad\text{for}\quad x<0. \tag{2.71}$$

To fix the form of the limit reliability function $\mathcal{R}$ for $x\geq 0$, we may notice that

$$\exists_{y}\ \ R(y)=e^{-\frac{1}{c}}. \tag{2.72}$$

By (b) for $\beta_2 > 0$, we get

$$\lim_{r\uparrow 0} \frac{1 + c\log R(r(\beta_2^{\frac{1}{\alpha}} x) + y_2)}{1 + c\log R(r + y_2)} = \beta_2 x^\alpha, \quad x > 0. \tag{2.73}$$

By (2.72) and (2.73), considering Theorem 2.11, we get

$$\mathcal{R}(x) = 1 - \exp[-\exp[\beta_2 x^\alpha - \frac{s}{c}]] \quad \text{for} \quad x \geq 0. \tag{2.74}$$

Combining (2.72) and (2.74)

$$\mathcal{R}(x) \quad = \quad \begin{cases} 1 - \exp[-\exp[\beta_1(-x)^\alpha - \frac{s}{c}]], & \text{for} \quad x < 0 \\ 1 - \exp[-\exp[\beta_2 x^\alpha - \frac{s}{c}]], & \text{for} \quad x \geq 0. \end{cases}$$

This las result, according to Definition 2.11, means that $R \in D_{\mathcal{R}_6}$. □

Theorem 2.14 $R \in D_{\mathcal{R}_7}$ *if and only if there exist* $a > 0$ *and* b *such that*

$$\forall_{y \in < ax_1+b, ax_2+b)}\ R(y) = e^{-\frac{1}{c}} \text{ and } \forall_{y \notin < ax_1+b, ax_2+b)}\ R(y) \neq e^{-\frac{1}{c}} \tag{2.75}$$

Proof. Let (2.75) holds. We will show that $R \in D_{\mathcal{R}_7}$. Since

$$t\left(R(a_t x + b_t)\right)^{l_t} \quad = \quad e^{\log t(1+c\log R(a_t x + b_t))} e^{-\frac{s}{c}}$$

then assuming $a_t = a$ and $b_t = b$, we get

$$\begin{array}{lll} \forall_{x=\frac{y-b}{a} \in < x_1, x_2)} & & e^{\log t(1+c\log R(a_t x+b_t))} e^{-\frac{s}{c}} = e^{-\frac{s}{c}}, \\ \forall_{x=\frac{y-b}{a} < x_1} & \exists_{\epsilon > 0} & e^{\log t(1+c\log R(a_t x+b_t))} e^{-\frac{s}{c}} = e^{\log t(1+c\log(e^{-\frac{1}{c}+\epsilon}))} e^{-\frac{s}{c}} \to \infty, \\ \forall_{x=\frac{y-b}{a} \geq x_2} & \exists_{\epsilon > 0} & e^{\log t(1+c\log R(a_t x+b_t))} e^{-\frac{s}{c}} = e^{\log t(1+c\log(e^{-\frac{1}{c}-\epsilon}))} e^{-\frac{s}{c}} \to 0 \end{array}$$

as $t \to \infty$. It means that there exist $a_t > 0$ and $b_t \in (-\infty, \infty)$ such that

$$\lim_{t\to\infty} t\left(R(a_t x + b_t)\right)^{l_t} \quad = \quad e^{-\frac{s}{c}} V(x)$$

where

$$V(x) \quad = \quad \begin{cases} \infty, & x < x_1 \\ 1, & x_1 \leq x < x_2 \\ 0, & x \geq x_2 \end{cases}$$

Hence, by Lemma 2.1, $R \in D_{\mathcal{R}_7}$.
Now, suppose that $R \in D_{\mathcal{R}_7}$. Since there exist $a_t > 0$ and $b_t \in (-\infty, \infty)$ such that

$$\lim_{t\to\infty} e^{\log t(1+c\log R(a_t x+b_t))} e^{-\frac{s}{c}} \quad = \quad e^{-\frac{s}{c}} V(x) \tag{2.76}$$

where

$$V(x) \quad = \quad \begin{cases} \infty, & x < x_1 \\ 1, & x_1 \le x < x_2 \\ 0, & x \ge x_2 \end{cases}$$

then

$$\forall_{x\in<x_1,x_2)} \;\; R(a_t x + b_t) = e^{-\frac{1}{c}}. \tag{2.77}$$

We will discuss separately the following three possible cases:
Case 1:

$$\exists_{y_0\in C_R} \;\; R(y_0) = e^{-\frac{1}{c}} \quad \text{and} \quad \forall_{y\neq y_0} \;\; R(y) \neq e^{-\frac{1}{c}}.$$

In this case by (2.77), we get

$$\forall_{x\in<x_1,x_2)} \;\; a_t x + b_t \to y_0 \text{ as } t \to \infty$$

and

$$R(a_t x + b_t) = e^{-\frac{1}{c} \pm o(\frac{1}{\log t})}.$$

Hence we get $a_t \to 0$, $b_t \to y_0$ and further

$$\forall_{x\in(-\infty,x_1)\cup<x_2,\infty)} \;\; a_t x + b_t \to y_0 \text{ as } t \to \infty.$$

Since R is continuous in y_0, then

$$\forall_{x\in(-\infty,x_1)\cup<x_2,\infty)} \;\; R(a_t x + b_t) = e^{-\frac{1}{c} \pm o(\frac{1}{\log t})}$$

which is inconsistent with (2.76).
Case 2:

$$\exists_{y_0} \;\; R(y_0) = e^{-\frac{1}{c}} \;\; R(y_0 - 0) = e^{-\frac{1}{c}} \;\; \forall_{y\neq y_0} \;\; R(y) \neq e^{-\frac{1}{c}}.$$

In this case by (2.77), we get

$$\forall_{x\in<x_1,x_2)}\ \ a_t x + b_t \to y_0 \text{ as } t \to \infty$$

and

$$R(a_t x + b_t) = e^{-\frac{1}{c} - o(\frac{1}{\log t})}.$$

Hence we get $a_t \to 0$, $b_t \to y_0$ and further

$$\forall_{x\in<x_2,\infty)}\ \ a_t x + b_t \to y_0 \text{ as } t \to \infty.$$

Since R is right continuous in y_0, then

$$\forall_{x\in<x_2,\infty)}\ \ R(a_t x + b_t) = e^{-\frac{1}{c} - o(\frac{1}{\log t})}$$

which is inconsistent with (2.76).
Case 3:

$$\exists_{y_1,y_2}\ \ \forall_{y\in<y_1,y_2>} R(y) = e^{-\frac{1}{c}}\ \ \forall_{y\notin<y_1,y_2>}\ \ R(y) \neq e^{-\frac{1}{c}}.$$

From the above and by (2.76), we have

$$\forall_{x\in<x_1,x_2)}\ \ \exists_{y\in<y_1,y_2>}\ \ a_t x + b_t \to y \text{ as} t \to \infty, \tag{2.78}$$

which holds either

(a) if $a_t \to 0$ and there exists $y_0 \in< y_1, y_2 >$ such that $b_t \to y_0$
or

(b) if there exist $a > 0$ and b such that $y = ax + b \in< y_1, y_2 >$ and $a_t \to a, b_t \to b$.
If the condition (a) holds then by the same arguments to that used in Case 1 and Case 2 we can find contradiction with (2.76). If the condition (b) holds then by Lemma 2.2 we get $a_t = a, b_t = b$. Since (2.77) holds then

$$< ax_1 + b, ax_2 + b) \subset< y_1, y_2 >$$

and

$$\forall_{y\in<ax_1+b,ax_2+b)}\ \ R(y) = e^{-\frac{1}{c}}.$$

If there exists $y_0 < ax_1 + b$ such that $R(y_0) = e^{-\frac{1}{c}}$ then since $a_t = a, b_t = b$ and $\frac{y_0-b}{a} < x_1$ we get

$$e^{\log t(1+c\log R(a_t(\frac{y_0-b}{a})+b_t))} = e^{\log t(1+c\log R(y_0))} = 1,$$

which is inconsistent with (2.76).
Similar way we can show that there is no $y_0 \geq ax_2 + b$ such that $R(y_0) = e^{-\frac{1}{c}}$.
From the above and since R is reliability function we get (2.75). □

Theorem 2.15 *$R \in D_{\mathcal{R}_8}$ if and only if*

$$\lim_{r\to-\infty} \frac{1-R(r)}{1-R(rx)} = x^{\alpha} \quad \textit{for} \ \ x > 0. \tag{2.79}$$

Proof. $R \in D_{\mathcal{R}_8}$ if and only if there exist $a_t > 0$ and $b_t \in (-\infty, \infty)$ such that

$$\mathcal{R}_8(x) = \lim_{t\to\infty} \mathcal{R}_{l_t,k_t}(a_t x + b_t) = \lim_{t\to\infty} 1 - (1 - (R(a_t x + b_t))^{l_t})^{k_t}$$

for $x \in C_{\mathcal{R}_8}$. Under the assumption (2.24) we get

$$\mathcal{R}_8(x) = \lim_{t\to\infty} 1 - (1 - (R(a_t x + b_t))^{l_t})^k$$

if and only if

$$\lim_{t\to\infty} (R(a_t x + b_t))^{l_t} = \overline{\mathcal{R}}_2(x)$$

which means that $R \in D_{\overline{\mathcal{R}}_2}$. Hence from the well known theorems about domains of attraction of limit reliability functions for series systems we obtain (2.79). □

Theorem 2.16 *$R \in D_{\mathcal{R}_9}$ if and only if*

(a) *there exists y such that $R(y) = 1$ and $R(y+\epsilon) < 1$ for all $\epsilon > 0$,*

(b) $\lim_{r\downarrow 0} \frac{1-R(rx+y)}{1-R(r+y)} = x^{\alpha}$ *for* $x > 0$.

Proof. $R \in D_{\mathcal{R}_9}$ if and only if there exist $a_t > 0$ and $b_t \in (-\infty, \infty)$ such that

$$\mathcal{R}_9(x) = \lim_{t\to\infty} \mathcal{R}_{l_t,k_t}(a_t x + b_t) = \lim_{t\to\infty} 1 - (1 - (R(a_t x + b_t))^{l_t})^{k_t}$$

for $x \in C_{\mathcal{R}_9}$. Under the assumption (2.24) we get

$$\mathcal{R}_9(x) = \lim_{t\to\infty} 1 - (1 - (R(a_t x + b_t))^{l_t})^k$$

if and only if

$$\lim_{t\to\infty} (R(a_t x + b_t))^{l_t} = \overline{\mathcal{R}}_1(x)$$

which means that $R \in D_{\overline{\mathcal{R}}_1}$. Hence from the well known theorems about domains of attraction of limit reliability functions for series systems the proof is concluded. □

Theorem 2.17 *$R \in D_{\mathcal{R}_{10}}$ if and only if the condition*

$$\lim_{t\to\infty} l_t(R(a_t x + b_t)) = e^x \tag{2.80}$$

is satisfied for all x, where b_t and a_t are defined as

$$b_t = \inf\{x : R(x+0) \le 1 - \frac{1}{l_t} \le R(x-0)\} \tag{2.81}$$

$$a_t = \inf\{x : R(x(1+0) + b_t) \le 1 - \frac{e}{l_t} \le R((x(1-0) + b_t)\} \tag{2.82}$$

Proof. $R \in D_{\mathcal{R}_{10}}$ if and only if there exist $a_t > 0$ and $b_t \in (-\infty, \infty)$ such that

$$\mathcal{R}_{10}(x) = \lim_{t\to\infty} \mathcal{R}_{l_t,k_t}(a_t x + b_t) = \lim_{t\to\infty} 1 - (1 - (R(a_t x + b_t))^{l_t})^{k_t}$$

for $x \in C_{\mathcal{R}_9}$. Under the assumption (2.24) we get

$$\mathcal{R}_{10}(x) = \lim_{t\to\infty} 1 - (1 - (R(a_t x + b_t))^{l_t})^k$$

if and only if

$$\lim_{t\to\infty} (R(a_t x + b_t))^{l_t} = \overline{\mathcal{R}}_3(x)$$

which means that $R \in D_{\overline{\mathcal{R}}_3}$. Hence from the well known theorems about domains of attraction of limit reliability functions for series systems we get (2.80), (2.81) and (2.82). □

2.5 Domains of attraction of limit reliability functions for parallel-series systems

Since of duality (see Lemma 2.5) the result obtained for series-parallel systems can be easily transformed on the parallel-series systems. Hence the theorems are presented here without proofs.
First let us define

$$\overline{G} : [0,1] \to (-\infty, \infty), \ \ \overline{G}(u) = \inf\{x | F(x) \geq u\},$$

$$\overline{a^*}_t = \overline{G}\left(e^{-\frac{\log t - 1}{l_t}}\right) - \overline{G}\left(e^{-\frac{\log t}{l_t}}\right),$$

$$\overline{r}_t(\beta, \nu) = \frac{1}{\overline{a^*}_t}\left[\overline{G}\left(e^{-\frac{\log t + \mu \log \nu - \beta + \epsilon'_t}{l_t}}\right) - \overline{G}\left(e^{-\frac{\log t + \mu \log \nu + \epsilon_t}{l_t}}\right)\right].$$

where $t \in (0, \infty)$, $\beta \in \Re$, ν is a natural number and $\epsilon'_t = o(1)$, $\epsilon_t = o(1)$.

Theorem 2.18 *If $R \in D_{\overline{\mathcal{R}}_1}$ and we choose $a_t = \overline{a^*}_t(e^{\frac{1}{\alpha}} - 1)^{-1}$ then*

$$\lim_{t \to \infty} \overline{r}_t(\beta, \nu) = \nu^{-\frac{\mu}{\alpha}}(e^{\frac{\beta}{\alpha}} - 1)(e^{\frac{1}{\alpha}} - 1)^{-1}$$

for all $\beta \in \Re$, $\nu \in \mathrm{N}$, fixed $\mu \neq 0$ and $\epsilon'_t = o(1)$, $\epsilon_t = o(1)$.
Conversely, if $\lim_{t\to\infty} r_t(\beta, \nu) = \beta$ exists for all $\beta \in \Re$, $\nu \in \mathrm{N}$, fixed $\mu \neq 0$ and $\epsilon'_t = o(1)$, $\epsilon_t = o(1)$ and if for all natural ν, $\epsilon'_t = o(1)$, $\epsilon_t = o(1)$

$$\lim_{t \to \infty} \overline{r}_t(1, \nu) = \nu^{-\frac{\mu}{\alpha}}$$

then $R \in D_{\overline{\mathcal{R}}_1}$ with $a_t = \overline{a^*}_t(e^{\frac{1}{\alpha}} - 1)^{-1}$ and $b_t = \overline{G}\left(e^{-\frac{\log t}{l_t}}\right) - a_t$.

Theorem 2.19 *If $R \in D_{\overline{\mathcal{R}}_2}$ and we choose $a_t = \overline{a^*}_t(1 - e^{-\frac{1}{\alpha}})^{-1}$ then*

$$\lim_{t \to \infty} \overline{r}_t(\beta, \nu) = \nu^{\frac{\mu}{\alpha}}(e^{-\frac{\beta}{\alpha}} - 1)(e^{-\frac{1}{\alpha}} - 1)^{-1}$$

for all $\beta \in \Re$, $\nu \in \mathrm{N}$, fixed $\mu \neq 0$ and $\epsilon'_t = o(1)$, $\epsilon_t = o(1)$.
Conversely, if $\lim_{t\to\infty} r_t(\beta, \nu) = \beta$ exists for all $\beta \in \Re$, $\nu \in \mathrm{N}$, fixed $\mu \neq 0$ and $\epsilon'_t = o(1)$, $\epsilon_t = o(1)$ and if for all natural ν, $\epsilon'_t = o(1)$, $\epsilon_t = o(1)$

$$\lim_{t \to \infty} \overline{r}_t(1, \nu) = \nu^{\frac{\mu}{\alpha}}$$

then $R \in D_{\overline{\mathcal{R}}_2}$ *with* $a_t = \overline{a^*}_t(1 - e^{\frac{1}{\alpha}})^{-1}$ *and* $b_t = \overline{G}\left(e^{-\frac{\log t}{l_t}}\right) + a_t$.

Theorem 2.20 *If* $R \in D_{\overline{\mathcal{R}}_3}$ *and we choose* $a_t = \overline{a^*}_t$ *then*

$$\lim_{t\to\infty} \overline{r}_t(\beta, \nu) = \beta$$

for all $\beta \in \Re$, $\nu \in \mathrm{N}$, *fixed* $\mu \neq 0$ *and* $\epsilon'_t = o(1)$, $\epsilon_t = o(1)$.
Conversely, if $\lim_{t\to\infty} \overline{r}_t(\beta,\nu) = \beta$ *exists for all* $\beta \in \Re$, $\nu \in \mathrm{N}$, *fixed* $\mu \neq 0$ *and* $\epsilon'_t = o(1)$, $\epsilon_t = o(1)$ *and if for all natural* ν, $\epsilon'_t = o(1)$, $\epsilon_t = o(1)$

$$\lim_{t\to\infty} \overline{r}_t(1, \nu) = 1$$

then $R \in D_{\overline{\mathcal{R}}_3}$ *with* $a_t = \overline{a^*}_t$ *and* $b_t = \overline{G}\left(e^{-\frac{\log t}{l_t}}\right)$.

Theorem 2.21 $R \in D_{\overline{\mathcal{R}}_4}$ *if and only if*

(a) *there exists an unique* y *such that* $R(y-0) = 1 - e^{-\frac{1}{c}}$,

(b) $\lim_{r\uparrow 0} \frac{1+c\log(1-R(rx+y))}{1+c\log(1-R(r+y))} = x^\alpha$, $x > 0$,
$\lim_{r\uparrow 0} \frac{1+c\log(1-R(rx+y))}{1+c\log(1-R(r+y))} = -\infty$, $x < 0$.

Theorem 2.22 $R \in D_{\overline{\mathcal{R}}_5}$ *if and only if*

(a) *there exists an unique* y *such that* $R(y) = 1 - e^{-\frac{1}{c}}$,

(b) $\lim_{r\downarrow 0} \frac{1+c\log(1-R(rx+y))}{1+c\log(1-R(r+y))} = x^\alpha$, $x > 0$,
$\lim_{r\downarrow 0} \frac{1+c\log(1-R(rx+y))}{1+c\log(1-R(r+y))} = -\infty$, $x < 0$.

Theorem 2.23 $R \in D_{\overline{\mathcal{R}}_6}$ *if and only if*

(a) *there exists an unique* y *such that* $R(y) = 1 - e^{-\frac{1}{c}}$,

(b) $\lim_{r\to 0} \frac{1+c\log(1-R(rx+y))}{1+c\log(1-R(r+y))} = x^\alpha$, $x > 0$.

Theorem 2.24 $R \in D_{\overline{\mathcal{R}}_7}$ *if and only if there exist* $a > 0$ *and* b *such that*

$$\forall_{y \in <ax_1+b, ax_2+b)}\ R(y) = 1 - e^{-\frac{1}{c}} \text{ and } \forall_{y \notin <ax_1+b, ax_2+b)}\ R(y) \neq 1 - e^{-\frac{1}{c}}$$

Theorem 2.25 *$R \in D_{\overline{\mathcal{R}}_8}$ if and only if*

(a) *there exists y such that $R(y) = 0$ and $R(y - \epsilon) > 0$ for all $\epsilon > 0$,*

(b) $\lim_{r \uparrow 0} \frac{R(rx+y)}{R(r+y)} = x^{\alpha}$ *for* $x > 0$.

Theorem 2.26 *$R \in D_{\overline{\mathcal{R}}_9}$ if and only if*

$$\lim_{r \to \infty} \frac{R(r)}{R(rx)} = x^{\alpha} \quad \textit{for} \;\; x > 0.$$

Theorem 2.27 *$R \in D_{\mathcal{R}_{10}}$ if and only if the condition*

$$\lim_{t \to \infty} l_t(R(a_t x + b_t)) = e^{-x}$$

is satisfied for all x, where b_t and a_t are defined as

$$b_t = \inf\{x : R(x+0) \leq \frac{1}{l_t} \leq R(x-0)\}$$
$$a_t = \inf\{x : R(x(1+0) + b_t) \leq \frac{}{l_t e} \leq R((x(1-0) + b_t)\}$$

2.6 Examples

Example 1
If the regular homogeneous series-parallel system is such that

$$R(x) = \begin{cases} 1, & x \leq 0 \\ e^{-\lambda x}, & x > 0 \end{cases}$$

and the pair (k_t, l_t) satisfies condition

$$k_t = t, \;\; l_t = (\log t)^t$$

then $R \in D_{\mathcal{R}_3}$.
Justification. Since $\rho(t) = t$ then by Case 3 of Theorem 2.5

$$\tau_\nu = t\nu^{\frac{1}{(1-t)\log t \log(\log t)}}$$

and

$$\begin{aligned} |\rho(\tau_\nu) - \rho(t)| &= |\tau_\nu - t| = t\left|\nu^{\frac{1}{(1-t)\log t \log(\log t)}} - 1\right| = \\ &\approx \left|\frac{\log \nu}{(1-t)\log t \log(\log t)}\right| \approx \frac{\log \nu}{\log t \log(\log t)}. \end{aligned}$$

Hence $\delta = 1$. From estimation (2.20) we get $\mu = 1$.
Now we will find the inverse function of R.

$$G(u) = -\frac{1}{\lambda}\log u.$$

We get

$$a_t^* = G\left(e^{-\frac{\log t+1}{(\log t)^t}}\right) - G\left(e^{-\frac{\log t}{(\log t)^t}}\right) = \frac{1}{\lambda}\frac{\log t+1}{(\log t)^t} - \frac{1}{\lambda}\frac{\log t}{(\log t)^t} = \frac{1}{\lambda}\frac{1}{(\log t)^t}$$

and

$$\begin{aligned} r_t(1,\nu) &= \frac{1}{a_t^*}\left[G\left(e^{-\frac{\log t+1}{(\log t)^t}}\right) - G\left(e^{-\frac{\log t}{(\log t)^t}}\right)\right] \\ &= \lambda(\log t)^t\left[\frac{1}{\lambda}\frac{\log t + \log\nu + 1}{(\log t)^t} - \frac{1}{\lambda}\frac{\log t \log\nu}{(\log t)^t}\right] = 1. \end{aligned}$$

Hence by Theorem 2.8, $R \in D_{\mathcal{R}_3}$ with $a_t = \frac{1}{(\lambda \log t)^t}$ and $b_t = \frac{1}{\lambda}(\log t)^{1-t}$.

Example 2

If the regular homogeneous series-parallel system is such that

$$R(x) = \begin{cases} 1, & x \leq 0 \\ \exp\left[\frac{1}{\log x}\right], & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$$

and the pair (k_t, l_t) satisfies condition

$$k_t = t, \;\; l_t = (\log t)^2$$

then $R \in D_{\mathcal{R}_1}$.
Justification. Since $\rho(t) = 2$ then $\delta = 1$ so $\mu = 1$.
The inverse function of R is given by

$$G(u) = \exp\left[\frac{1}{\log u}\right]$$

and

$$\begin{aligned} a_t^* &= G\left(e^{-\frac{\log t+1}{(\log t)^2}}\right) - G\left(e^{-\frac{\log t}{(\log t)^2}}\right) = \exp\left[-\frac{\log^2 t}{\log t + 1}\right] - \exp\left[-\log t\right] \\ &= e^{-\log t\frac{1}{1+\frac{1}{\log t}}} - e^{-\log t} \approx e^{-\log t(1-\frac{1}{\log t})} - e^{-\log t} = \frac{1}{t}(e-1). \end{aligned}$$

We have

$$\begin{aligned}
r_t(1,\nu) &= \frac{1}{a_t^*}\left[G\left(e^{-\frac{\log t+\log\nu+1}{(\log t)^2}}\right)-G\left(e^{-\frac{\log t+\log\nu}{(\log t)^2}}\right)\right]\\
&= \frac{1}{a_t^*}\left[\exp\left[\frac{1}{\log(1-\frac{\log t+\log\nu+1}{\log^2 t})}\right]-\exp\left[\frac{1}{\log(1-\frac{\log t+\log\nu}{\log^2 t})}\right]\right]\\
&\approx \frac{1}{a_t^*}\left[e^{-\log t(1-\frac{\log\nu+1}{\log t})}-e^{-\log t(1-\frac{\log\nu}{\log t})}\right]=\frac{1}{a_t^*}\left[\frac{1}{t}(e^{-\log\nu+1}-e^{\log\nu})\right]=\\
&= \frac{1}{a_t^*}\left[\frac{1}{t}e^{\log\nu}(e^1-1)\right]\to\nu \text{ as } t\to\infty.
\end{aligned}$$

Hence by Theorem 2.9, $R\in D_{\mathcal{R}_1}$ with $a_t=\frac{1}{t}$ and $b_t=0$.

Example 3
If the regular homogeneous series-parallel system is such that

$$R(x)=\begin{cases}1, & x\le 0\\ 1-x, & 0<x<1\\ 0, & x\ge 1\end{cases}$$

and the pair (k_t,l_t) satisfies condition

$$k_t=t,\;\; l_t=c,\; c>0$$

then $R\in D_{\mathcal{R}_2}$.
Justification. Since $\rho(t)=0$ then $\delta=0$ so $\mu=1$.
The inverse function of R is given by

$$G(u)=1-u.$$

We get

$$a_t^* \;=\; G\left(e^{-\frac{\log t+1}{c}}\right)-G\left(e^{-\frac{\log t}{c}}\right)=1-e^{-\frac{1}{c}(\log t+1)}-1+e^{-\frac{1}{c}\log t}=t^{-\frac{1}{c}(1-e^{-\frac{1}{c}})}$$

and

$$\begin{aligned}
r_t(1,\nu) &= \frac{1}{a_t^*}\left[G\left(e^{-\frac{\log t+\log\nu+1}{c}}\right)-G\left(e^{-\frac{\log t+\log\nu}{c}}\right)\right]\\
&= \frac{1}{a_t^*}\left[1-e^{-\frac{1}{c}(\log t+\log\nu+1)}-1+e^{-\frac{1}{c}(\log t+\log\nu)}\right]=\nu^{-\frac{1}{c}}
\end{aligned}$$

Hence by Theorem 2.10, $R \in D_{\mathcal{R}_2}$ where

$$\mathcal{R}_2(x) = \begin{cases} (-x)^{-c}, & x \leq 0 \\ 0, & x > 0 \end{cases}$$

with $a_t = \frac{1}{t^{1/c}}$ and $b_t = 1$.

Example 4
If the regular homogeneous series-parallel system is such that

$$R(x) = \begin{cases} 1, & x \leq 0 \\ \exp[-A\exp(x)^B], & x \leq 0, A > 0, B > 0 \end{cases}$$

and the shape of the system is described by

$$k_t = t, \;\; l_t = \frac{1}{A}\log t$$

then $R \in D_{\mathcal{R}_4}$.
Justification. We get
for $y = 0, \; R(y) = e^{-A} = e^{-\frac{1}{c}}$
$\lim_{r\downarrow 0} \frac{1+c\log R(rx+y)}{1+c\log R(r+y)} = \lim_{r\downarrow 0} \frac{1-\exp[(rx)^B]}{1-\exp[r^B]} = x^B$ for $x > 0$
$\lim_{r\downarrow 0} \frac{1+c\log R(rx+y)}{1+c\log R(r+y)} = \lim_{r\downarrow 0} \frac{1}{1-\exp[r^B]} = -\infty$ for $x < 0$.
Hence $R \in D_{\mathcal{R}_4}$ where

$$\mathcal{R}_4(x) = \begin{cases} 1, & x < 0 \\ 1-\exp[-\exp(-x^B)], & x \geq 0. \end{cases}$$

Example 5
If the regular homogeneous series-parallel system is such that

$$R(x) = \begin{cases} 1, & x < 0 \\ e^{x(x-2)}, & 0 \leq x < 1, \\ e^{-x}, & x \geq 1 \end{cases}$$

and the shape of the system is described by

$$k_t = t, \;\; l_t = \log t$$

then $R \in D_{\mathcal{R}_5}$.
Justification. We get

for $y=1,\ R(y-0)=e^{-A}=e^{-1}$
$\lim_{r\uparrow 0}\frac{1+c\log R(rx+y)}{1+c\log R(r+y)}=\lim_{r\uparrow 0}\frac{1+(rx+1(rx_1))}{1+(r+1)(r_1)}=x^2$ for $x>0$
$\lim_{r\uparrow 0}\frac{1+c\log R(rx+y)}{1+c\log R(r+y)}=\lim_{r\uparrow 0}\frac{1-(rx+1)}{1+(r+1)(r-1)}=\lim_{r\uparrow 0}\frac{-rx}{r^2}=$
$=\lim_{r\uparrow 0}\frac{-x}{r}=-\infty$ for $x<0$.
Hence $R\in D_{\mathcal{R}_5}$ where

$$\mathcal{R}_5(x)=\begin{cases}1-\exp[-\exp(-x)^2], & x<0\\ 0, & x\geq 0.\end{cases}$$

Example 6
If the regular homogeneous series-parallel system is such that

$$R(x)=\begin{cases}1, & x<-1\\ \frac{1}{2}-\frac{1}{2}x, & -1\leq x<0,\\ \frac{1}{2}-x, & 0\leq x<\frac{1}{2}\\ 0, & x\geq\frac{1}{2}\end{cases}$$

and the shape of the system is described by

$$k_t=t,\ \ l_t=\frac{\log t}{\log 2}$$

then $R\in D_{\mathcal{R}_6}$.
Justification. We get
for $y=0,\ R(y)=\frac{1}{2}=e^{-\log 2}=e^{\frac{-1}{c}}$
$\lim_{r\uparrow 0}\frac{1+c\log R(rx+y)}{1+c\log R(r+y)}=\lim_{r\uparrow 0}\frac{1+\frac{1}{\log 2}\log(\frac{1}{2}-\frac{1}{2}rx)}{1+\frac{1}{\log 2}\log(\frac{1}{2}-\frac{1}{2}r)}=x$ for $x>0$
$\lim_{r\downarrow 0}\frac{1+c\log R(rx+y)}{1+c\log R(r+y)}=\lim_{r\downarrow 0}\frac{1+\frac{1}{\log 2}\log(\frac{1}{2}-rx)}{1+\frac{1}{\log 2}\log(\frac{1}{2}-r)}=x$ for $x<0$
Hence $R\in D_{\mathcal{R}_6}$.

Example 7
If the regular homogeneous series-parallel system is such that

$$R(x)=\begin{cases}1, & x<0\\ e^{-1}, & 0\leq x<A,\\ 0, & 0\leq x\geq A, A>0\end{cases}$$

and the shape of the system is described by

$$k_t=t,\ \ l_t=\log t$$

then $R \in D_{\mathcal{R}_7}$.
Justification. It is easy to notice that above function satisfies condition (2.75) of Theorem 2.14.

Example 8
If the regular homogeneous series-parallel system is such that

$$R(x) = \begin{cases} 1, & x < 0 \\ 1 - \frac{2}{\pi} \arctan x, & x \geq 0 \end{cases}$$

and the shape of the system is described by

$$k_t = k, \quad \lim_{t \to \infty} l_t = \infty$$

then $R \in D_{\mathcal{R}_9}$.
Justification. We get
for $y = 0, R(y) = 1$,

$$\begin{aligned} \lim_{r \downarrow 0} \frac{1 - R(rx + y)}{1 - R(r + y)} &= \lim_{r \downarrow 0} \frac{1 - \frac{2}{\pi} \arctan(rx)}{1 - \frac{2}{\pi} \arctan(r)} \\ &= \lim_{r \downarrow 0} \frac{\frac{x}{1+(xr)^2}}{\frac{1}{1+r^2}} = x \text{ for } x > 0. \end{aligned}$$

2.7 Conclusions

We have presented the necessary and sufficient conditions for the reliability function to be in domain of attraction of certain limit distributions. The obtained results allows us to find the limit distributions of the homogeneous series-parallel and parallel-series systems. These theorems can be used to prepare table which immediately gives the answer what limit reliability function can we get for given shape of the system and given element's distribution. However, they can be useful in finding the limit reliability functions for non-homogeneous and multi-state systems.

Chapter 3

Limit reliability functions for non-homogeneous systems

Dorota Kurowicka, Krzysztof Kolowrocki

Abstract: In this paper we present the way of finding limit reliability functions of non-homogeneous series, parallel and series-parallel systems using theorems about domains of attractions; that is,theorems which give the necessary and sufficient conditions for reliability function of the system's components such that the limit reliability function of this system is determined.

Keywords: extreme value theory, limit distributions, series-parallel systems

3.1 Introduction

In the reliability estimations of large scale systems it is important to know the influence of the type of components' reliability on the reliability of the system. For the series and parallel systems this problem was intensively studied by a lot of authors. These study consisted of finding possible limit distributions of the maximum of n i.i.d. random variables, determining the necessary and sufficient conditions under which these distributions occur (Gnedenko [17]) and (Haan [9]), finding the rate of convergence (Dziubdziela [12]). The theory of i.i.d. sequences was generalized by permitting dependence (e.g for Markov or stationary sequences) or allowing elements to have different distributions. Summarization can found in (Leadbetter, Lindgren and Rootzen [44]). In this chapter we will keep the assumption that variables are independent but they can have different distributions. We will call such a systems *non-homogeneous.* In Section 3.2 we present essential notions, definitions and lemmas necessary in this chapter. In Section 3.3 the possible limit distributions for non-homogeneous series and parallel systems are given and the theorems showing how to find which limit distribution occur are presented. Section 3.4 consists of generalization of obtained for series and parallel systems results on a series-parallel systems. Some examples are also given.

3.2 Essential notions and theorems

Suppose that E_i, $i = 1, 2, \ldots, n$, $n \in \mathrm{N}$ are components of the system S and X_i are lifetimes of E_i. Moreover, suppose that X_i are independent random variables.

Definition 3.1 *We will call the series*[1] *(parallel*[2]*) system non-homogeneous if it is composed of a types elements,* $1 < a \leq n$*, and fraction of i-th kind element is equal to* q_i *, where* $q_i > 0, \sum_{i=1}^{n} q_i = 1$*. Moreover* $R^{(i)}$ *is a reliability function of i-th type elements.*

[1] A system is called *series* if its lifetime X is given by

$$X = \min_{1 \leq i \leq n} \{X_i\}.$$

[2] A system is called *parallel* if its lifetime X is given by

$$X = \max_{1 \leq i \leq n} \{X_i\}.$$

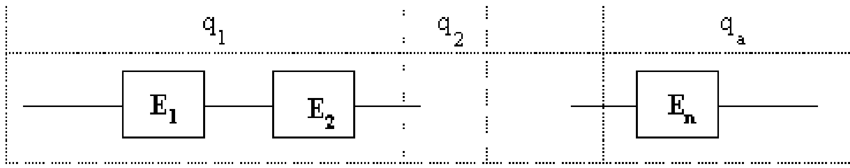


Figure 3.1. The shape of non-homogeneous series system.

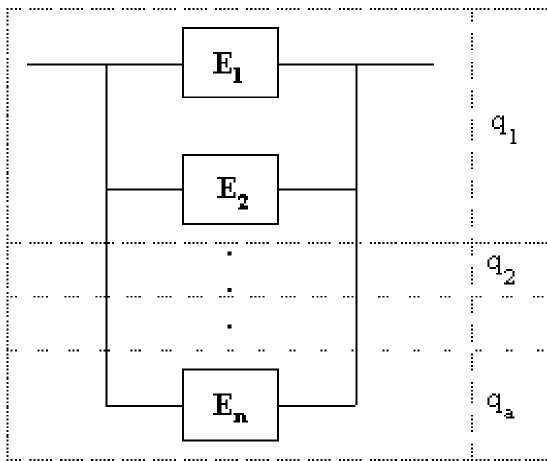


Figure 3.2. The shape of non-homogeneous parallel system.

The sequence of reliability functions of the non-homogeneous series system is

$$\overline{\Re'}_n(x) = \prod_{i=1}^{a} (R(x))^{q_i n}, \;\; x \in (-\infty, \infty), \; n \in \mathrm{N}$$

and for the homogeneous parallel system

$$\Re'_n(x) = 1 - \prod_{i=1}^{a} (F(x))^{q_i n}, \;\; x \in (-\infty, \infty), \; n \in \mathrm{N}.$$

Suppose that E_{ij}, where $i = 1, 2, \ldots, k_n$, $j = 1, 2, ..., l_i$, are components of the system S and X_{ij} are lifetimes of E_{ij}. Moreover, suppose that X_{ij} are independent random variables.

Definition 3.2 *A regular series-parallel*[3] *(parallel-series) system is called non-homogeneous if it is composed of* a*,* $1 < a \leq k_n$*,* $k_n \in \mathrm{N}$*, different kinds of series (parallel) subsystems and the fraction of the i-th kind subsystem in the system is equal to* q_i*, where* $q_i > 0$*,* $\sum_{i=1}^{a} q_i = 1$*. Moreover, the i-th kind series (parallel) subsystem consists of* e_i*,* $1 \leq e_i \leq l_n$*,* $l_n \in \mathrm{N}$*, kinds of components with reliability functions* $R^{(i,j)}$*,* $j = 1, 2, \ldots, e_i$ *and the fraction the j-th kind component in this subsystem is equal to* p_{ij}*, where* $p_{ij} > 0$ *and* $\sum_{j=1}^{e_i} p_{ij} = 1$*.*

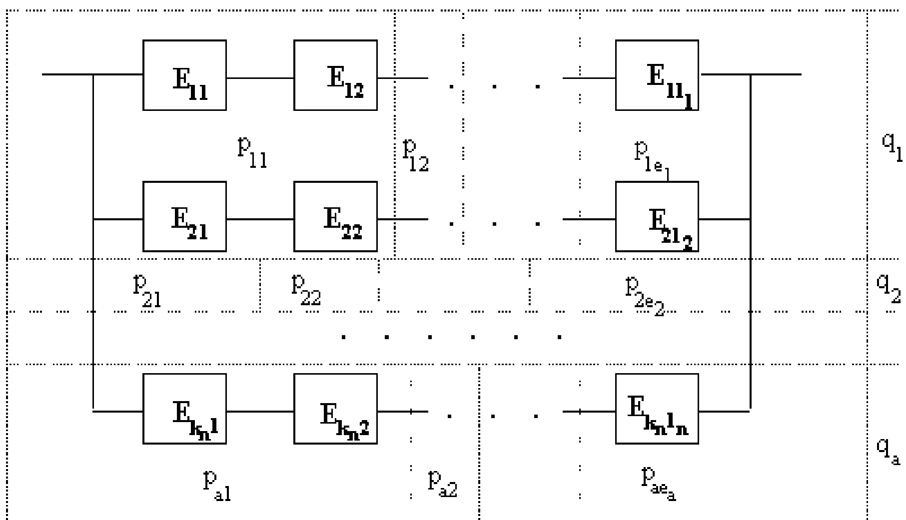


Figure 3.3. The shape of non-homogeneous series-parallel system.

[3] A system is called *regular series-parallel* if its lifetime X is given by

$$X = \max_{1 \leq i \leq k_n} \{ \min_{1 \leq j \leq l_n} \{X_{ij}\}\}.$$

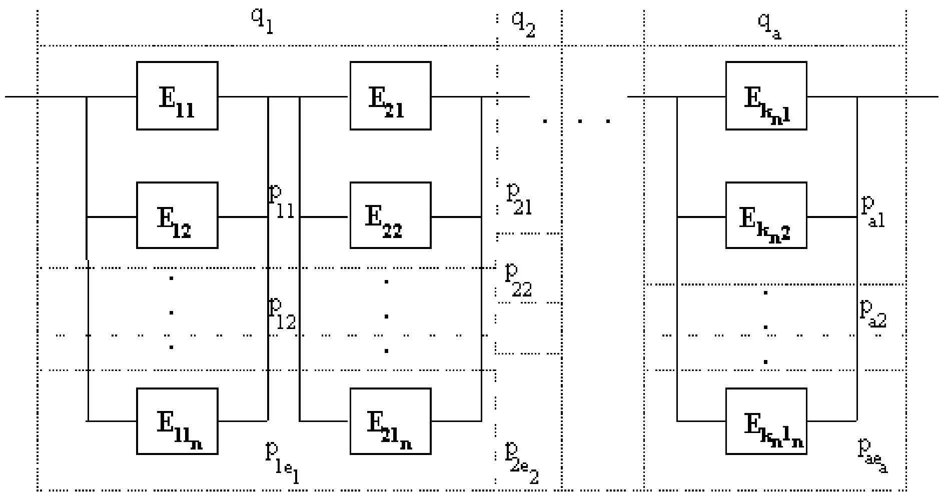


Figure 3.4. The shape of non-homogeneous parallel-series system.

The reliability function of the regular series-parallel non-homogeneous system is given by

$$\Re'_{l_n k_n}(x) = 1 - \prod_{i=1}^{a}[1 - (R^{(i)}(x))^{l_n}]^{q_i k_n},\ x \in (-\infty, \infty)$$

where

$$R^{(i)}(x) = \prod_{j=1}^{e_i}(R^{(i,j)}(x))^{p_{ij}},\ \ i = 1, 2, \ldots, a.$$

The reliability function of the regular parallel-series non-homogeneous system is given by

$$\overline{\Re'}_{l_n k_n}(x) = 1 - \prod_{i=1}^{a}[1 - (F^{(i)}(x))^{l_n}]^{q_i k_n},\ x \in (-\infty, \infty)$$

where

$$F^{(i)}(x) = \prod_{j=1}^{e_i}(F^{(i,j)}(x))^{p_{ij}},\ \ i = 1, 2, \ldots, a.$$

Next replacing n by a positive real number t and assuming that k_t and l_t are positive real numbers, we obtain families of the regular systems corresponding to t and to the pair (k_t, l_t). For these families of the systems there exist families of reliability functions.
The family of reliability functions of the non-homogeneous series system is given by

$$\overline{\Re'}_t(x) = \prod_{i=1}^{a} (R(x))^{q_i t}, \;\; x \in (-\infty, \infty), \; t \in (0, \infty) \tag{3.1}$$

and for the non-homogeneous parallel system by

$$\Re'_t(x) = 1 - \prod_{i=1}^{a} (F(x))^{q_i t}, \;\; x \in (-\infty, \infty), \; n \in (0, \infty). \tag{3.2}$$

The family of reliability function of the regular series-parallel non-homogeneous system is given by

$$\Re'_{l_t k_t}(x) = 1 - \prod_{i=1}^{a} [1 - (R^{(i)}(x))^{l_t}]^{q_i k_t}, \; x \in (-\infty, \infty), \; t \in (0, \infty), \tag{3.3}$$

where

$$R^{(i)}(x) = \prod_{j=1}^{e_i} (R^{(i,j)}(x))^{p_{ij}}, \;\; i = 1, 2, \ldots, a. \tag{3.4}$$

The family of reliability function of the regular parallel-series non-homogeneous system is given by

$$\overline{\Re'}_{l_t k_t}(x) = 1 - \prod_{i=1}^{a} [1 - (F^{(i)}(x))^{l_t}]^{q_i k_t}, \; x \in (-\infty, \infty), \; t \in (0, \infty), \tag{3.5}$$

where

$$F^{(i)}(x) = \prod_{j=1}^{e_i} (F^{(i,j)}(x))^{p_{ij}}, \;\; i = 1, 2, \ldots, a. \tag{3.6}$$

Definition 3.3 (Limit reliability function)
A reliability function $\mathcal{R}$ is called a limit reliability function of the family $\Re_t$ given

by (3.2) ($\Re_{l_t,k_t}$ given by (3.3)) if there exist functions $a_t > 0$ and $b_t \in (-\infty, \infty)$ such that

$$\lim_{t\to\infty} \Re_t(a_t x + b_t) = \mathcal{R}(x) \ \ \textit{for } x \in C_{\mathcal{R}}$$ [4]

$$\left(\lim_{t\to\infty} \Re_{l_t,k_t}(a_t x + b_t) = \mathcal{R}(x) \ \ \textit{for } x \in C_{\mathcal{R}}\right)$$

The pair (a_t, b_t) is called a norming function pair.

Similarly we can introduce the definitions of limit reliability function for series and parallel-series systems.
From the above we get that the exact reliability function in point x is approximately equal to the limit reliability function in the point $\frac{x-b_t}{a_t}$

$$\Re(x) \approx \mathcal{R}\left(\frac{x-b_t}{a_t}\right). \tag{3.7}$$

We can formulate equivalent conditions which should be satisfied to get the weak convergence to the non-degenerate[5] limit reliability function. The lemmas below can be found in Kolowrocki [24]. The proof of the first lemma is presented. The other lemmas can be proven similarly.

Lemma 3.1 *If*

(a) *the reliability function $\overline{\mathcal{R}'}(x) = \exp[-\overline{V'}(x)]$,*

(b) *the family $\overline{\Re'}_t$ is given by (3.1),*

(c) *$a_t > 0, b_t \in (-\infty, \infty)$ are some functions*

then

$$\lim_{t\to\infty} \overline{\Re'}_t(a_t x + b_t) = \overline{\mathcal{R}'}(x) \ \ \textit{for } x \in C_{\overline{\mathcal{R}'}} \tag{3.8}$$

[4] $C_{\mathcal{R}}$ means the set of continuity points of $\mathcal{R}$.

[5] A reliability function R is called *degenerate* if there exists $x_0 \in (-\infty, \infty)$ such that

$$R(x) = \begin{cases} 1, & x < x_0 \\ 0, & x \geq x_0. \end{cases}$$

is equivalent to the assertion

$$\lim_{t\to\infty} t\left(\sum_{i=1}^{a} q_i F^{(i)}(a_t x + b_t)\right) = \overline{V'}(x) \quad \textit{for } x \in C_{\overline{V'}} \tag{3.9}$$

Proof. Suppose that (3.8) is satisfied.Then, for all $x \in C_{\overline{\mathcal{R}'}}$ such that $\overline{\mathcal{R}'} \neq 0$ that is $\overline{V'} \neq \infty$, by (3.1) for $i = 1, 2 \ldots, a$ we have

$$\lim_{t\to\infty} F^{(i)}(a_t x + b_t) = 0. \tag{3.10}$$

Moreover, according to (3.1), the condition (3.8) can be written for $x \in C_{\overline{V'}}$ in the form

$$\begin{aligned} &\lim_{t\to\infty} t \textstyle\sum_{i=1}^{a} q_i \log R^{(i)}(a_t x + b_t) \\ &\quad = \lim_{t\to\infty} t \textstyle\sum_{i=1}^{a} q_i \log[1 - F^{(i)}(a_t x + b_t)] = -\overline{V'}(x). \end{aligned} \tag{3.11}$$

From the expansion

$$\log(1-x) \quad = \quad -x - o(x),$$

where $o(x) \ll x$ and from (3.10), for $i = 1, 2, \ldots, a$ we obtain

$$\begin{aligned} t \log\left[1 - F^{(i)}(a_t x + b_t)\right] &= \left[-tF^{(i)}(a_t x + b_t) - to(F^{(i)}(a_t x + b_t))\right] \\ &= -t\left[F^{(i)}(a_t x + b_t) + \frac{o(F^{(i)}(a_t x + b_t))}{F^{(i)}(a_t x + b_t)}\right]. \end{aligned} \tag{3.12}$$

From the above and (3.11), we get (3.9).
On the other hand, if (3.9) is satisfied, then (3.10) also holds and from (3.12) we obtain (3.11) and next by (3.1) the condition (3.8) holds.
Besides, for all x such that $\overline{\mathcal{R}'} = 0$ that is $\overline{V'} = \infty$ if (3.10) is satisfied, then from the previous performed discussion it follows that conditions

$$\lim_{t\to\infty} \overline{\Re'}_t(a_t x + b_t) = 0 \tag{3.13}$$

and

$$\lim_{t\to\infty} t \sum_{i=1}^{a} q_i F^{(i)}(a_t x + b_t) = \infty \tag{3.14}$$

are equivalent. Otherwise, if (3.10) does not hold, that is, there exists i such that

$$\lim_{t\to\infty} F^{(i)}(a_t x + b_t) \neq 0$$

and (3.13) is satisfied then it follows that (3.14) holds. Now if (3.14) holds then

$$\begin{aligned}\lim_{t\to\infty} t\sum_{i=1}^{a} q_i \log R^{(i)}(a_t x + b_t) &= \lim_{t\to\infty} t\sum_{i=1}^{a} q_i \log\left[1 - F^{(i)}(a_t x + b_t)\right] \\ &\le \lim_{t\to\infty}\left[-t\sum_{i=1} a q_i F^{(i)}(a_t x + b_t)\right] = -\infty\end{aligned}$$

in the case when $F^{(i)}(a_t x + b_t) \neq 1$ for $i = 1, 2, \ldots, a$ and by (3.1)

$$\overline{\Re'}_t(a_t x + b_t) = 0$$

and in the case when there exists i such that $F^{(i)}(a_t x + b_t) = 1$. Hence (3.13) holds which concludes the proof. □

Lemma 3.2 *If*

(a) $\lim_{t\to\infty} k_t = \infty$,

(b) *the reliability function* $\mathcal{R}'(x) = 1 - \exp[-V'(x)]$,

(c) *the family* $\Re'_{l_t k_t}$ *is given by (3.3),*

(d) $a_t > 0, b_t \in (-\infty, \infty)$ *are some functions*

then

$$\lim_{t\to\infty} \Re'_{l_t k_t}(a_t x + b_t) = \mathcal{R}'(x) \text{ for } x \in C_{\mathcal{R}'} \tag{3.15}$$

is equivalent to the assertion

$$\lim_{t\to\infty} k_t\left(\sum_{i=1}^{a} q_i (R^{(i)}(a_t x + b_t))^{l_t}\right) = V'(x) \quad \text{for } x \in C_{V'} \tag{3.16}$$

where $R^{(i)}$ *is given by (3.2).*

Lemma 3.3 *If*

(a) $\lim_{t\to\infty} k_t = k$, $\lim_{t\to\infty} l_t = \infty$,

(b) *the reliability function* $\mathcal{R}'(x) = 1 - \exp[-V'(x)]$,

(c) *the family* $\Re'_{l_t k_t}$ *is given by (3.3),*

(d) $a_t > 0, b_t \in (-\infty, \infty)$ *are some functions*

then

$$\lim_{t\to\infty} \Re'_{l_t k_t}(a_t x + b_t) = \mathcal{R}'(x) \ \ \textit{for } x \in C_{\mathcal{R}'} \tag{3.17}$$

is equivalent to the assertion

$$\lim_{t\to\infty} \left(R^{(i)}(a_t x + b_t))\right)^{l_t} = \Im_0(x) \quad \textit{for } x \in C_{\Im_0} \tag{3.18}$$

where $\Im_0$ *is non-degenerate reliability function.*

Lemma 3.4 *If* $\overline{\mathcal{R}}(x)$ *a limit reliability functions of the series (parallel-series) system then function*

$$\mathcal{R}(x) = 1 - \overline{\mathcal{R}}(-x)$$

is a limit reliability function of parallel (series-parallel) system.
If the pair (a_t, b_t) *is the normalising function pair for series (parallel-series) system then* $(a_t, -b_t)$ *is such a pair for parallel (series-parallel) system.*

3.3 Limit reliability functions for non-homogeneous series and parallel systems

In (Kolowrocki [24]) it is shown that there are three possible limit distribution for non-homogeneous series system as follows:

$$\begin{aligned}
\overline{\mathcal{R}'}_1(x) &= \begin{cases} 1, & x < 0 \\ \exp[-d(x)x^{\alpha}], & x \geq 0 \end{cases} \\
\overline{\mathcal{R}'}_2(x) &= \begin{cases} \exp[-d(x)(-x)^{-\alpha}], & x < 0 \\ 0, & x \geq 0 \end{cases} \\
\overline{\mathcal{R}'}_3(x) &= \exp[-d(x)\exp x], \quad x \in (-\infty, \infty).
\end{aligned} \tag{3.19}$$

and three for non-homogeneous parallel system

$$\begin{aligned}
\mathcal{R}'_1(x) &= \begin{cases} 1, & x < 0 \\ 1 - \exp[-d(x)x^{-\alpha}], & x \geq 0 \end{cases} \\
\mathcal{R}'_2(x) &= \begin{cases} 1 - \exp[-d(x)(-x)^{\alpha}], & x < 0 \\ 0, & x \geq 0 \end{cases} \\
\mathcal{R}'_3(x) &= 1 - \exp[-d(x)\exp(-x)], \quad x \in (-\infty, \infty).
\end{aligned} \tag{3.20}$$

where function d depends on the reliability functions of the particular components and their fraction in the system.
The theorems below show how to find which limit reliability function occur. We can see that in general the procedure of finding limit distribution of non-homogeneous system consist of choosing the "worst" (is sense of condition (c) in the theorems below) reliability function of the elements of the system, such that it belong to the domain of attraction ($R \in D_{\mathcal{R}}$) of respective distribution (condition (b)). The contribution of the reliability functions of the other elements reveals in the function d.

Theorem 3.1 *If*

(a) $R \in \{R^{(1)}, R^{(2)}, \ldots, R^{(a)}\}$,

(b) *there exists* y *such that* $R(y) = 1$ *and* $R(y - \epsilon) < 1$ *for all* $\epsilon > 0$,
$\lim_{r \downarrow 0} \frac{1-R(rx+y)}{1-R(r+y)} = x^{\alpha}, \quad x > 0$

(c) $\lim_{r \downarrow 0} \frac{1-R^{(i)}(rx+y)}{1-R(rx+y)} \leq 1$ *for* $i = 1, 2, \ldots, a,\ x > 0$,
there exists function

$$d(x) = \begin{cases} 0, & x < 0 \\ \lim_{r \downarrow 0} \sum_{i=1}^{a} q_i \frac{1-R^{(i)}(rx+y)}{1-R(r+y)}, & x \geq 0 \end{cases}$$

then $\overline{\mathcal{R}'}_1$ *is a limit reliability function of the non-homogeneous series system.*

Proof. Let (a), (b) and (c) be satisfied. It is shown in the proof of theorem about domain of attraction of limit distribution for series systems (Gnedenko [17])[6] that for normalising functions a_t and b_t defined as follows:

$$a_t = \inf\{x | R(x(1-0)+y) \leq 1 - \frac{1}{t} \leq R(x(1+0)+y)\}, x > 0 \tag{3.21}$$

and

$$b_t = y, \tag{3.22}$$

[6] The variables here are indexed with positive real numbers t not like in (Gnedenko [17]) where indices are natural. However, it doesn't impose any extra restrictions since it is shown in (Kurowicka [32]) that the theorems about domains of attraction are valid also in the case of real index.

where y is a point such that $R(y) = 1$ and $R(y - \epsilon) < 1$ for all $\epsilon > 0$, we get

$$\lim_{t\to\infty} t\,(1 - R(a_t x + y)) = \overline{V}_1(x) \tag{3.23}$$

where

$$\overline{V}_1(x) \quad = \quad \begin{cases} 0, & x < 0 \\ x^\alpha & x \geq 0. \end{cases}$$

We also get $a_t \to 0^+$. The condition (3.23) for $x > 0$ can be written in the following form

$$\lim_{t\to\infty} t\,(1 - R(a_t x + y)) \left[\sum_{i=1}^{a} q_i \frac{1 - R^{(i)}(a_t x + y)}{1 - R(a_t + y)} \right]$$

Hence for $x > 0$ we get

$$\lim_{t\to\infty} t\,(1 - R(a_t x + y)) \left[\sum_{i=1}^{a} q_i \frac{1 - R^{(i)}(a_t x + y)}{1 - R(a_t + y)} \right] = d(x)\overline{V}_1(x).$$

By (b) and monotonicity of $R^{(i)}$ follows that $R^{(i)}(y) = 1$ for $i = 1, 2, \ldots, a$. Hence

$$\lim_{t\to\infty} \ t \left(\sum_{i=1}^{a} q_i F^{(i)}(a_t x + y) \right) = d(x)\overline{V}_1(x) = \overline{V'}_1(x)$$

which by Lemma 3.1 means that $\overline{\mathcal{R}}_1$ is a limit reliability function of non-homogeneous series system with function d defined above. □

Theorem 3.2 *If*

(a) $R \in \{R^{(1)}, R^{(2)}, \ldots, R^{(a)}\}$,

(b) $\lim_{r\to-\infty} \frac{1-R(r)}{1-R(rx)} = x^\alpha, \ \ x > 0$

(c) $\lim_{r\to\infty} \frac{1-R^{(i)}(rx)}{1-R(rx)} \leq 1$ *for* $i = 1, 2, \ldots, a, \ x \in (-\infty, \infty)$,
there exists function

$$d(x) \quad = \quad \lim_{r\to\infty} \sum_{i=1}^{a} q_i \frac{1 - R^{(i)}(rx)}{1 - R(rx)}, \ x \in (-\infty, \infty)$$

then $\overline{\mathcal{R}'}_2$ is a limit reliability function of the non-homogeneous series system.

Proof. Let (a), (b) and (c) be satisfied. It is shown in the proof of theorem about domain of attraction of limit distribution for series systems (Gnedenko [17]) that for normalising functions a_t and b_t defined as follows:

$$a_t = \inf\{x | R(-x(1+0)) \leq 1 - \frac{1}{t} \leq R(-x(1-0))\}, x > 0$$

and

$$b_t = 0,$$

we get

$$\lim_{t\to\infty} t\,(1 - R(a_t x)) = \overline{V}_2(x) \qquad (3.24)$$

where

$$\overline{V}_2(x) \quad = \quad \begin{cases} (-x)^{-\alpha}, & x < 0 \\ \infty & x \geq 0. \end{cases}$$

We get $a_t \to \infty$. Hence by (c) and since by (b) and (c) $d(x) \neq 0$ for all $x \in (-\infty, \infty)$

$$\lim_{t\to\infty} t\,(1 - R(a_t x)) \left[\sum_{i=1}^{a} q_i \frac{1 - R^{(i)}(a_t x)}{1 - R(a_t x)} \right] = d(x)\overline{V}_2(x).$$

Finally we obtain

$$\lim_{t\to\infty} \; t \left(\sum_{i=1}^{a} q_i F^{(i)}(a_t x) \right) = d(x)\overline{V}_2(x) = \overline{V'}_2(x)$$

which by Lemma 3.1 means that $\overline{\mathcal{R}}_2$ is a limit reliability function of non-homogeneous series system with function d defined above. □

Theorem 3.3 *If*

(a) $R \in \{R^{(1)}, R^{(2)}, \ldots, R^{(a)}\}$,

(b) $\lim_{t\to\infty} t\,(1 - R(a_t x + b_t)) = e^x$
where

$$\begin{aligned} b_t &= \inf\{x | R(x+0) \le 1 - \frac{1}{t} \le R(x-0)\}, \\ a_t &= \inf\{x | R(x(1+0)+b_t) \le 1 - \frac{e}{t} \le R(x(1-0)+b_t)\}, \end{aligned}$$

(c) $\exists_T \forall_{t>T} R(a_t x + b_t) \neq 1$ *for* $x < x_0$ *and* $R(a_t x + b_t) = 1$ *for* $x \ge x_0$
where $x_0 \in [-\infty, \infty)$,

(d) $\lim_{t\to\infty} \frac{1-R^{(i)}(a_t x+b_t)}{1-R(a_t x+b_t)} \le 1$ *for* $i = 1, 2, \ldots, a,\ x \ge x_0$,
there exists function

$$d(x) = \begin{cases} 0, & x < x_0 \\ \lim_{t\to\infty} \sum_{i=1}^{a} q_i \frac{1-R^{(i)}(a_t x+b_t)}{1-R(a_t x+b_t)}, & x \ge x_0 \end{cases}$$

then $\overline{\mathcal{R}'}_3$ *is a limit reliability function of the non-homogeneous series system.*

Proof. Let (a), (b) and (c) be satisfied. We get

$$\lim_{t\to\infty} t\,(1 - R(a_t x + b_t)) = e^x = \overline{V}_3(x). \tag{3.25}$$

Hence

$$\lim_{t\to\infty} t\,(1 - R(a_t x + b_t)) \left[\sum_{i=1}^{a} q_i \frac{1 - R^{(i)}(a_t x + b_t)}{1 - R(a_t x + b_t)} \right] = d(x)\overline{V}_3(x).$$

and

$$\lim_{t\to\infty} t \left(\sum_{i=1}^{a} q_i F^{(i)}(a_t x + b_t) \right) = d(x)\overline{V}_3(x) = \overline{V'}_3(x)$$

which by Lemma 3.1 means that $\overline{\mathcal{R}}_3$ is a limit reliability function of non-homogeneous series system with function d defined above. □

Similarly using theorems about domains of attraction the limit reliability functions for non-homogeneous parallel systems can be found. By Lemma 3.4 and Theorems 3.1, 3.2 and 3.3 the theorems which give procedure of finding reliability function of non-homogeneous parallel system can be formulated and proved.

Theorem 3.4 *If*

(a) $R \in \{R^{(1)}, R^{(2)}, \ldots, R^{(a)}\}$,

(b) $\lim_{r\to\infty} \frac{R(r)}{R(rx)} = x^{\alpha}, \;\; x > 0$

(c) $\lim_{r\to\infty} \frac{R^{(i)}(rx)}{R(rx)} \leq 1$ *for* $i = 1, 2, \ldots, a, \; x \in (-\infty, \infty)$,
there exists function

$$d(x) \;\; = \;\; \lim_{r\to\infty} \sum_{i=1}^{a} q_i \frac{R^{(i)}(rx)}{R(rx)}, \; x \in (-\infty, \infty)$$

then $\mathcal{R}'_1$ *is a limit reliability function of the non-homogeneous parallel system.*

Theorem 3.5 *If*

(a) $R \in \{R^{(1)}, R^{(2)}, \ldots, R^{(a)}\}$,

(b) *there exists* y *such that* $R(y) = 0$ *and* $R(y - \epsilon) > 0$ *for all* $\epsilon > 0$,
$\lim_{r\uparrow 0} \frac{R(rx+y)}{R(r+y)} = x^{\alpha}, \;\; x > 0$

(c) $\lim_{r\uparrow 0} \frac{R^{(i)}(rx+y)}{R(rx+y)} \leq 1$ *for* $i = 1, 2, \ldots, a, \; x < 0$,
there exists function

$$d(x) \;\; = \;\; \begin{cases} \lim_{r\downarrow 0} \sum_{i=1}^{a} q_i \frac{R^{(i)}(rx+y)}{R(rx+y)}, & x < 0 \\ 0, & x \geq 0 \end{cases}$$

then $\mathcal{R}'_2$ *is a limit reliability function of the non-homogeneous parallel system.*

Theorem 3.6 *If*

(a) $R \in \{R^{(1)}, R^{(2)}, \ldots, R^{(a)}\}$,

(b) $\lim_{t\to\infty} t\,(R(a_t x + b_t)) = e^{-x}$
where

$$b_t \;\; = \;\; \inf\{x | R(x+0) \leq \frac{1}{t} \leq R(x-0)\},$$
$$a_t \;\; = \;\; \inf\{x | R(x(1+0) + b_t) \leq \frac{1}{et} \leq R(x(1-0) + b_t)\},$$

(c) $\exists_T \forall_{t>T} R(a_t x + b_t) \neq 0$ *for* $x < x_0$ *and* $R(a_t x + b_t) = 0$ *for* $x \geq x_0$ *where* $x_0 \in (-\infty, \infty]$,

(d) $\lim_{t\to\infty} \frac{R^{(i)}(a_t x+b_t)}{R(a_t x+b_t)} \leq 1$ *for* $i = 1, 2, \ldots, a,\ x \geq x_0$, *there exists function*

$$d(x) = \begin{cases} \lim_{t\to 0} \sum_{i=1}^{a} q_i \frac{R^{(i)}(a_t x+b_t)}{R(a_t x+b_t)}, & x < x_0 \\ 0, & x \geq x_0 \end{cases}$$

then $\overline{\mathcal{R}'}_3$ *is a limit reliability function of the non-homogeneous parallel system.*

Example 1

Let us consider series system composed of 100 elements of two types, which occur in the system with the same frequency. The life time of the first type elements has the exponential distribution with parameter $\lambda = 0.03$ and the life time of second kind elements is distributed according to Rayleigh with parameter $\alpha = 0.001$. The limit reliability function of this system is $\overline{\mathcal{R}'}_1$.

Justification. We must show that the conditions of Theorem 3.1 are satisfied. Let us notice that if we choose exponential distribution than conditions (b) and (c) hold:

$$\text{for } y = 0 \text{ we get } \lim_{r\downarrow 0} \frac{1 - R(rx+y)}{1 - R(r+y)} = \lim_{r\downarrow 0} \frac{1 - \exp[-\lambda rx]}{1 - \exp[-\lambda r]} = x,\ x > 0;$$

and

$$\begin{aligned} \lim_{r\downarrow 0} \frac{1 - R^{(i)}(rx-y)}{1 - R(rx+y)} &= \lim_{r\downarrow 0} \frac{1 - \exp[-\alpha(rx)^2]}{1 - \exp[-\lambda rx]} \\ &= \lim_{r\downarrow 0} \exp[-\alpha(rx)^2 + \lambda rx] \frac{2\alpha rx}{\lambda} = 0 \leq 1 \\ \lim_{r\downarrow 0} \frac{1 - R(rx+y)}{1 - R(rx+y)} &= 1. \end{aligned}$$

From the above function d has the following form

$$d(x) = \begin{cases} 0, & x < 0 \\ 0.5, & x \geq 0. \end{cases}$$

The conditions of Theorem 3.1 are satisfied hence the function

$$\overline{\mathcal{R}'}_1(x) = \begin{cases} 1, & x < 0 \\ \exp[-0.5x], & x \geq 0. \end{cases}$$

is a limit reliability function of this system.
By (3.1) the exact reliability function of this system is given by

$$\begin{aligned} \overline{\Re'}_{100}(x) &= (\exp[-0.03x])^{50}(\exp[-0.001x^2])^{50} \\ &= e^{-1.5x}e^{-0.05x^2} \text{ for } x \geq 0. \end{aligned}$$

Since by (3.21) and (3.22) the norming functions

$$a_t = \frac{1}{\lambda t} = \frac{1}{0.03 \cdot 100} = \frac{1}{3} \text{ and } b_t = 0$$

then by (3.7) the approximate reliability function of this system is of the following form

$$\overline{\Re'}_{100}(x) \approx \overline{\mathcal{R}'}_1\left(\frac{x - b_t}{a_t}\right) = \begin{cases} 1, & x < 0 \\ \exp[-1.5x], & x \geq 0. \end{cases}$$

The difference between exact and approximate reliability functions is shown in the Table 1 and Figure 5 below.

t	$\mathcal{R}'_1$	$\Re_{100}$	$\mathcal{R}'_1 - \Re_{100}$
0.0	1.0000	1.0000	0.0000
0.5	0.4724	0.4665	0.0059
1.0	0.2231	0.2122	0.0109
1.5	0.1054	0.0942	0.0112
2.0	0.0498	0.0408	0.0090
2.5	0.0235	0.0172	0.0063
3.0	0.0111	0.0071	0.0040
3.5	0.0052	0.0028	0.0024
4.0	0.0025	0.0011	0.0014
5.0	0.0005	0.0001	0.0004

Table 3.1: The difference between exact and approximate reliability functions.

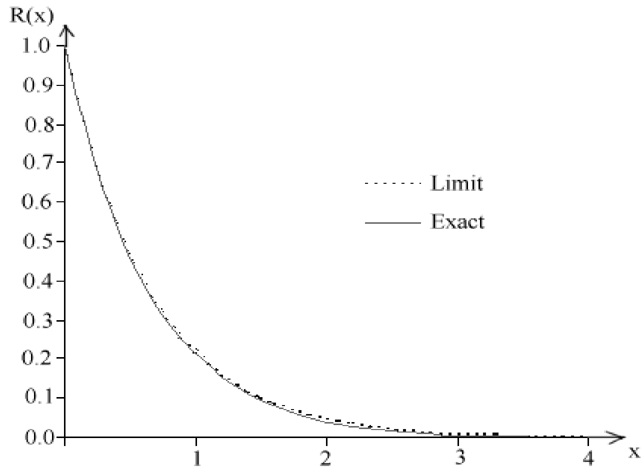


Figure 3.5. The difference between exact and approximate reliability function of that system.

3.4 Limit reliability functions for non- homogeneous series-parallel systems

In (Kolowrocki [24]) it is shown that the possible limit reliability functions of the non-homogeneous series systems are of the following form

$$\mathcal{R}'_1(x) = \begin{cases} 1, & x < 0 \\ 1 - \exp[-d(x)x^{-\alpha}], & x \geq 0 \end{cases}$$

$$\mathcal{R}'_2(x) = \begin{cases} 1 - \exp[-d(x)(-x)^{\alpha}], & x < 0 \\ 0, & x \geq 0 \end{cases}$$

$$\mathcal{R}'_3(x) = 1 - \exp[-d(x)\exp(-x)], \quad x \in (-\infty, \infty).$$

$$\mathcal{R}'_4(x) = \begin{cases} 1, & x < 0 \\ 1 - \exp[-d(x)\exp(-x^{\alpha} - \frac{s}{c})], & x \geq 0 \end{cases}$$

$$\mathcal{R}'_5(x) = \begin{cases} 1 - \exp[-d(x)\exp((-x)^{\alpha} - \frac{s}{c})], & x < 0 \\ 0, & x \geq 0 \end{cases}$$

$$\mathcal{R}'_6(x) = \begin{cases} 1 - \exp[-d(x)\exp(\beta(-x)^{\alpha} - \frac{s}{c})], & x < 0 \\ 1 - \exp[-d(x)\exp(-x^{\alpha} - \frac{s}{c})], & x \geq 0 \end{cases}$$

$$\mathcal{R}'_7(x) = \begin{cases} 1, & x < x_1 \\ 1 - \exp[-d(x)\exp(-\frac{s}{c})], & x_1 \le x < x_2 \\ 0, & x \ge x_2,\ x_1 < x_2. \end{cases}$$

$$\mathcal{R}'_8(x) = \begin{cases} 1 - \prod_{i=1}^{a}[1 - d_i(x)\exp[-(-x)^{-\alpha}]]^{q_i k}, & x < 0 \\ 0, & x \ge 0 \end{cases}$$

$$\mathcal{R}'_9(x) = \begin{cases} 1, & x < 0 \\ 1 - \prod_{i=1}^{a}[1 - d_i(x)\exp[-x^{\alpha}]]^{q_i k}, & x \ge 0 \end{cases}$$

$$\mathcal{R}'_{10}(x) = 1 - \prod_{i=1}^{a}[1 - d_i(x)\exp[-\exp x]]^{q_i k}, \quad x \in (-\infty, \infty).$$

As in the Section 3.3, where theorems about domains of attraction were used to determine the limit reliability function for the non-homogeneous series and parallel systems, we show now how to find limit reliability function of non-homogeneous series-parallel system. The theorems about domains of attraction will be used which can be found in the chapter " Domains of attraction of limit reliability functions" of this thesis or in (Kurowicka [32]), (Kurowicka [33]) and (Kurowicka [34]).

Theorem 3.7 *If*

(a) $R \in \{R^{(i)}, i = 1, 2, \ldots, a\}$ *where* $R^{(i)}$ *is given by (3.4),*

(b) $R \in D_{\mathcal{R}_j}, j = 1, 2, \ldots, 7$ *with* a_t *and* b_t *specified in respective theorem,*

(c) $\exists_T \forall_{t>T} R(a_t x + b_t) \neq 0$ *for* $x < x_0$ *and* $R(a_t x + b_t) = 0$ *for* $x \ge x_0$ *where* $x_0 \in (-\infty, \infty]$,

(d) $\lim_{t\to\infty} \frac{R^{(i)}(a_t x + b_t)}{R(a_t x + b_t)} \le 1$ *for* $i = 1, 2, \ldots, a,\ x \ge x_0$, *there exists function*

$$d(x) = \begin{cases} \lim_{t\to\infty} \sum_{i=1}^{a} q_i \left(\frac{R^{(i)}(a_t x + b_t)}{R(a_t x + b_t)}\right)^{l_t}, & x < x_0 \\ 0, & x \ge x_0 \end{cases}$$

then $\mathcal{R}'_j, j = 1, 2, \ldots, 7$ *is a limit reliability function of the non-homogeneous series-parallel system with function* d *defined above.*

Proof. Since (a) and (b) hold then by theorem about domain of attraction of $\mathcal{R}_j = 1 - \exp[V_j]$ we get

$$\lim_{t\to\infty} k_t \left(R(a_t x + b_t)\right)^{l_t} = V_j(x) \text{ for } x \in C_{V_j}. \tag{3.26}$$

Since (c) and (d) are satisfied then (3.26), for $x < x_0$, can be written in the following form

$$\lim_{t\to\infty} k_t \left(R(a_t x + b_t)\right)^{l_t} \sum_{i=1}^{a} q_i \left(\frac{R^{(i)}(a_t x + b_t)}{R(a_t x + b_t)}\right)^{l_t} = d(x)V_j(x).$$

For $x \geq x_0$, by (c) and (d), $V_j(x)$ and $d(x)$ are both equal zero hence

$$\lim_{t\to\infty} k_t \sum_{i=1}^{a} q_i \left(R^{(i)}(a_t x + b_t)\right)^{l_t} = d(x)V_j(x) = V'_j(x),\ x \in C_{V'}.$$

which by Lemma 3.2 means that $\mathcal{R}'_j, j = 1, 2, \ldots, 7$ is a limit reliability function of that system. □

Theorem 3.8 *If*

(a) $R \in \{R^{(i)}, i = 1, 2, \ldots, a\}$ *where* $R^{(i)}$ *is given by (3.4),*

(b) $R \in D_{\mathcal{R}_j}, j = 8, 9, 10$ *with* a_t *and* b_t *specified in respective theorem,*

(c) $\exists_T \forall_{t>T} R(a_t x + b_t) \neq 0$ *for* $x < x_0$ *and* $R(a_t x + b_t) = 0$ *for* $x \geq x_0$ *where* $x_0 \in (-\infty, \infty]$,

(d) $\lim_{t\to\infty} \frac{R^{(i)}(a_t x + b_t)}{R(a_t x + b_t)} \leq 1$ *for* $i = 1, 2, \ldots, a,\ x \geq x_0$, *there exists function*

$$d_i(x) = \begin{cases} \lim_{t\to\infty} \left(\frac{R^{(i)}(a_t x + b_t)}{R(a_t x + b_t)}\right)^{l_t}, & x < x_0 \\ 0, & x \geq x_0 \end{cases}$$

then $\mathcal{R}'_j, j = 8, 9, 10$ *is a limit reliability function of the non-homogeneous series-parallel system with function* d *defined above.*

Proof. Analogous way like in the proof of Theorem 3.7 using Lemma 3.3 and theorems about domains of attraction of $\mathcal{R}_j, j = 8, 9, 10$ we can show that if the conditions (a)-(d) are satisfied then $\mathcal{R}'_j, j = 8, 9, 10$ is a limit reliability function of the non-homogeneous series-parallel system. □

Example 2 Let us consider a water supply system composed of 3 lines with 100 segment pipes. The first line is composed of 80 segment pipes with exponential reliability functions with $\lambda_1 = 0.025$ and 20 segment pipes with

Erlang distribution with parameters $k = 2$, $\lambda_2 = 0.065$. The other 2 lines are composed of 10 segment pipe with Rayleigh distribution with parameter $\alpha_1 = 0.001$ and 90 segment pipes with Weibull with shape parameter $\beta = \frac{1}{3}$ and .... with parameter $\alpha_2 = 0.6$.

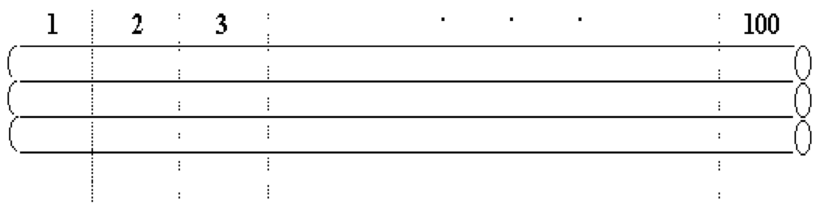


Figure 3.6. The water supply system.

Assuming that failure of the lines are independent we may consider this supply system as a non-homogeneous, series-parallel system.

We have

$$k_t = 3,\ l_t = 100,\ a = 2,\ q_1 = \frac{2}{3},\ q_2 = \frac{1}{3}$$

and

$$e_1 = 2,\ p_{11} = 0.8,\ p_{12} = 0.2.$$

Hence

$$\begin{aligned} R^{(1)}(x) &= \exp[-0.025 \cdot 0.8x - 0.0065 \cdot 0.2x](1 + 0.1x)^{0.2} \\ &= \exp[-0.033x](1 + 0.1x)^{0.2}. \end{aligned}$$

Since

$$e_2 = 2,\ p_{21} = 0.9,\ p_{22} = 0.1$$

then

$$\begin{aligned} R^{(2)}(x) &= \exp[-0.6 \cdot 0.9x^{\frac{1}{3}} - 0.001 \cdot 0.1x^2] \\ &= \exp[-0.54x^{\frac{1}{3}} - 0.0001x^2]. \end{aligned}$$

We will show that the function $R^{(1)}$ satisfies conditions of Theorem 3.8:

- The condition (b) is satisfied since we can show

$$\lim_{r\downarrow 0}\frac{1-R^{(1)}(rx)}{1-R^{(1)}(r)} = \lim_{r\downarrow 0}\frac{1-\exp[-0.033rx](1+0.1rx)^{0.2}}{1-\exp[-0.033r](1+0.1r)^{0.2}} = x \text{ for } x>0,$$

and $a_t = \frac{1}{0.02l_t},\ b_t = 0.$

- The condition (c) holds

$$\lim_{t\to\infty}\frac{R^{(1)}(a_t x+b_t)}{R^{(1)}(a_t x+b_t)} = 1,$$

$$\lim_{t\to\infty}\frac{R^{(2)}(a_t x+b_t)}{R^{(1)}(a_t x+b_t)} = \lim_{t\to\infty}\frac{\exp\left[-0.54\left(\frac{x}{0.02l_t}\right)^{\frac{1}{3}}-0.0001\left(\frac{x}{0.02l_t}\right)^{2}\right]}{\exp\left[-0.033\frac{x}{0.02l_t}\right](1+0.1\frac{x}{0.02l_t})^{0.2}} \leq 1.$$

The functions $d_i,\ i=1,2$ have the form

$$d_1(x) = \lim_{t\to\infty}\left(\frac{R^{(1)}(a_t x+b_t)}{R^{(1)}(a_t x+b_t)}\right)^{l_t} = 1,$$

$$d_2(x) = \lim_{t\to\infty}\frac{R^{(2)}(a_t x+b_t)}{R^{(1)}(a_t x+b_t)} = \lim_{t\to\infty}\left(\frac{R^{(2)}(\frac{x}{0.02l_t})}{R^{(1)}\left(\frac{x}{0.02l_t}\right)}\right)^{l_t}$$

$$= \lim_{t\to\infty}\left(\frac{\exp\left[-0.54\left(\frac{x}{0.02l_t}\right)^{\frac{1}{3}}-0.0001\left(\frac{x}{0.02l_t}\right)^{2}+0.033\left(\frac{x}{0.02l_t}\right)\right]}{(1+0.1\frac{x}{0.02l_t})^{0.2}}\right)^{l_t}$$

$$= 0.$$

From the above and by Theorem 3.8, we get

$$\mathcal{R}'_9(x) = \begin{cases} 1, & x<0 \\ 1-[1-\exp[-x]]^2 & x\geq 0 \end{cases}$$

is the limit reliability function of this system.

Hence the approximate reliability function is given by

$$\Re_{3,100}(x) \approx \mathcal{R}'_9(\frac{x-b_t}{a_t}) = 1-[1-\exp[-2x]]^2,\ \ x\geq 0.$$

By (3.3) and (3.4) the exact reliability function of the water supply system is as follows

$$\begin{aligned} \Re_{3,100}(x) &= 1 - \prod_{i=1}^{2} [1 - (R^{(i)}(x))^{100}]^{q_i 3} \\ &= 1 - [1 - e^{-3.3x}(1 + 0.065x)^{0.2}]^2 [1 - e^{-54x^{\frac{1}{3}} - 0.01x^2}]. \end{aligned}$$

The difference between exact and approximate reliability functions of that system is shown in Table 2 and Figure 6 below.

t	$\mathcal{R}'_9$	$\Re_{3,100}$	$\mathcal{R}'_9 - \Re_{3,100}$
0.0	1.0000	1.0000	0.0000
0.2	0.8913	0.8906	0.0007
0.4	0.6968	0.6935	0.0033
0.6	0.5117	0.5055	0.0062
0.8	0.3630	0.3547	0.0083
1.0	0.2523	0.2430	**0.0093**
1.2	0.1732	0.1639	**0.0093**
1.4	0.1179	0.1093	0.0086
1.6	0.0799	0.0723	0.0076
2.0	0.0363	0.0311	0.0048
3.0	0.0049	0.0035	0.0014
4.0	0.0007	0.0004	0.0003
5.0	0.0001	0.0000	0.0001
6.0	0.0000	0.0000	0.0000

Table 3.2: The difference between exact and approximate reliability functions.

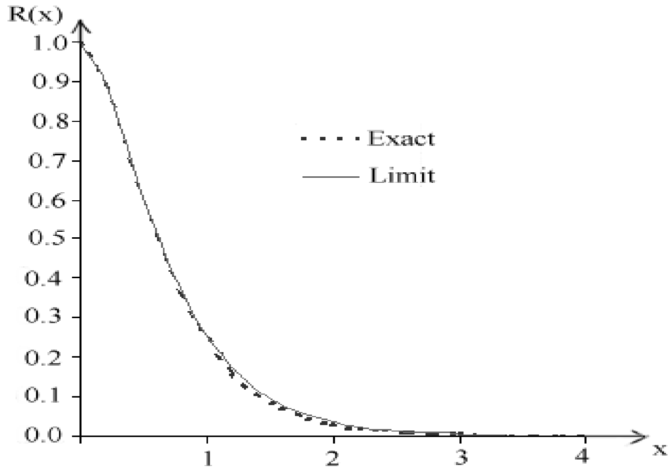


Figure 3.7. The difference between exact and approximate reliability function of the system.

3.5 Conclusions

It was shown how to find the limit reliability functions for non-homogeneous series, parallel and series-parallel systems. Because of duality (see Lemma 3.4) the obtained results can be transformed to parallel-series systems. We have seen, in the presented examples, that exact and approximate reliability functions are very close.

Part II

Chapter 4

A parametrization of positive definite matrices in terms of partial correlation vines

Dorota Kurowicka, Roger Cooke

Abstract: We present a parametrization of the class PD(n) of positive definite $n \times n$ matrices using regular vines and partial correlations. Using a bijection from $(-1,1)^{\binom{n}{2}} \to C(n)$ (C(n)-class of correlation matrices) with a clear probabilistic interpretation (Bedford and Cooke [3]), we suggest a new approach to various problems involving positive definiteness.

Keywords: correlation, tree dependence, positive definite matrix, matrix completion.

4.1 Introduction

Positive (semi) definiteness is an important property of square matrices. There are algorithms for testing positive definiteness such as the Choleski decomposition or algorithms based on finding eigenvalues of a matrix. We propose to study positive definiteness using partial correlations (Yule and Kendall [54]) in conjunction with a new structure which we call a regular vine (Cooke [6], Bedford and Cooke [3]). A symmetric real matrix with elements in the interval (-1,1) and with "1"'s on the main diagonal is called a *proto correlation matrix*. For a given $n \times n$ proto correlation matrix the canonical regular vine is considered. A vine is a set of trees such that the edges of the tree T_i are nodes of the tree T_{i+1} and all trees have the maximum number of edges. A vine is regular if two edges of T_i are joined by an edge of T_{i+1} only if these edges share a common node in T_i. A regular vine is called canonical if each tree T_i has a unique node of degree $n-i$. Partial correlations can be assigned to the edges of the canonical vine such that conditioning and conditioned sets of the vine and partial correlations are equal (see Section 4.2). In general we have $\binom{n}{2}$ edges in a regular vine on n elements. All assignments of partial correlations from the interval (-1,1), are consistent (see Theorem 4.2) and in this way the bijection from $(-1,1)^{\binom{n}{2}}$ to $C(n)$ is constructed. This relationship can be used to specify dependence in high dimensional distributions (Kurowicka and Cooke [35]) but also to decide whether a proto correlation matrix is positive definite. This algorithm can also be used to transform a non-positive definite matrix into a positive definite matrix. With the new algorithm these alterations have a clear probabilistic interpretation. This approach can be useful where a high dimensional correlation matrix should be specified (e.g. dependent Monte Carlo simulations). In complex problems many entries in the correlation matrix may be unspecified, and this partially specified matrix must be extended to a positive definite matrix. We present preliminary results for the matrix completion problem using canonical vine partial correlation specifications. In particular, we present procedures for deciding whether a partially specified matrix can be extended to a positive definite matrix for certain non-chordal graphs (Laurent [38] and [39], Fiedler [13], Barrett, Johnson and Loewy [52], Barrett, Johnson and Tarazaga [53]).

This chapter is organized as follows. In the Section 4.2 we introduce vines and present definitions and theorems showing the relationship between vines and positive definite matrices. Section 4.3 contains an algorithm for testing positive definiteness of a matrix using the canonical vine. The relationship between the new algorithm and the known matrix theory results is also shown. In Section 4.4 repairing violation of positive definiteness is discussed and finally in Section 4.5 the algorithm solving the completion problem for few special cases of the proto correlation matrix is presented.

4.2 Vines

Definition 4.1 (Tree) *$T = (N, E)$ is a tree with nodes N and edges E if it is connected graph with no cycle. That is, there does not exist a sequence $a_1, \ldots, a_k$ of elements of N such that*

$$\{a_1, a_2\} \in E, \ldots, \{a_{k-1}, a_k\} \in E, \{a_k, a_1\} \in E.$$

Definition 4.2 (Regular vine) *V is a regular vine on n elements if*

1. $V = (T_1, \ldots, T_{n-1})$

2. *T_1 is a tree with nodes $N_1 = \{1, \ldots, n\}$, and edges E_1; $\#E_1 = n - 1$; for $i = 2, \ldots, n-1$ T_i is a tree with nodes $N_i = E_{i-1}$, and edges E_i; $\#E_i = i - 1$.*

3. (proximity) *for $i = 2, \ldots, n-1$, $\{a, b\} \in E_i, \#a \triangle b = 2$ where $\triangle$ denotes the symmetric difference. In other words, if a and b are nodes of T_i connected by an edge, where $a = \{a_1, a_2\}, b = \{b_1, b_2\}$, then exactly one of the a_i equals one of the b_i.*

Definition 4.3 (Constraint set)

1. *For $j \in E_i, i \leq n - 1$*
$U_j(k) = \{e \mid \exists \; e_{i-(k-1)} \in e_{i-(k-2)} \in \ldots \in j, \;\; e \in e_{i-(k-1)}\}$
is called the k-fold union[1] *of j; $k = 1, \ldots, i$*
$U_j^* = U_j(i)$ is the complete union of j, that is, set of $\{1, \ldots, n\}$ reachable from j by the membership relation.
$U_j(1) = \{j_1, j_2\} = j$.
By definition we write $U_j(0) = \{j\}$.

[1] 1-fold union of the set is the set of elements i.e. set itself, 2-fold union e_2 is the set of elements of elements of e, etc.

2. *For* $i = 1, \ldots, n-1, e_i \in E_i$, *if* $e_i = \{j, k\}$ *then the conditioning set associated with* e_i *is*

$$D_{e_i} = U_j^* \cap U_k^*$$

and the conditioned sets associated with e_i *are*

$$C_{e_i,j} = U_j^* \setminus D_{e_i}; C_{e_i,k} = U_k^* \setminus D_{e_i}.$$

3. *The constraint set for* V *is*

$$CV = \{D_{e_i}, C_{e_i,j}, C_{e_i,k} \,|e_i \in E_i; e_i = \{j,k\}, i = 1, \ldots, n-1\}.$$

Note that for $e \in E_1$, the conditioning set is empty.

For $e_i \in E_i, i \leq n-1, e_i = \{j,k\}$ we have $U_{e_i}^* = U_j^* \cup U_k^*$.

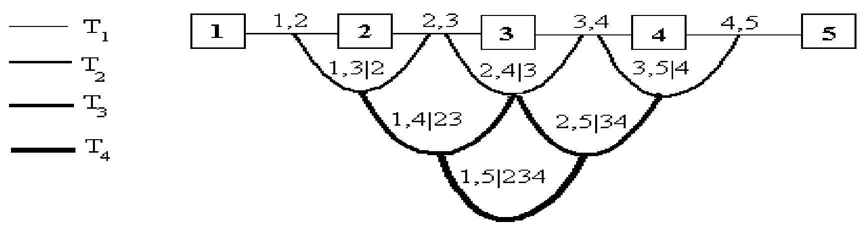


Figure 4.1. A regular vine on 5 elements showing conditioned and conditioning sets.

Definition 4.4 (Canonical vine) *A regular vine is called a canonical if each tree has a unique node of degree* $n - i$.

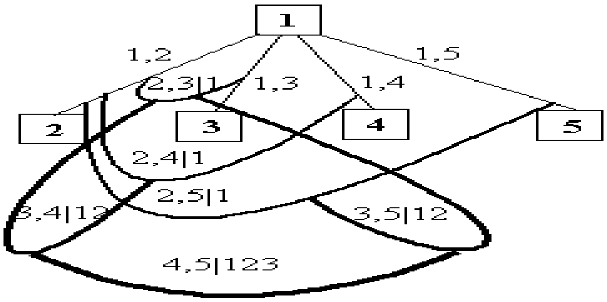


Figure 4.2. A canonical vine on 5 elements.

For regular vines the structure of the constraint set is particularly simple, as shown by the following lemmata (Cooke [6]).

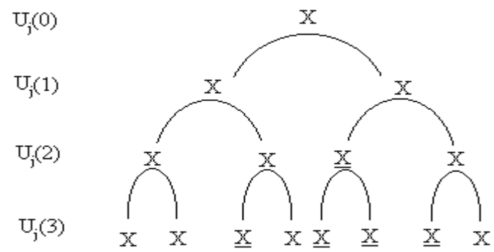
Lemma 4.1 *Let V be a regular vine on n elements, and let $j \in E_i$. Then*

$$\#U_j(k) = k + 1; k = 0, 1, \ldots, i.$$

Proof. The statement clearly holds for $k = 0, k = 1$. By the proximity property it follows immediately that it holds for $k = 2$. We claim that in general

$$\#U_j(k) = 2\#U_j(k-1) - \#U_j(k-2); k = 2, 3, \ldots.$$

To see this, we represent the $U_j(k)$ as a complete binary tree whose nodes are in set of nodes of V. The repeated nodes are underscored, and children of underscored nodes are underscored. Because of proximity, nodes with a common parent must have a common child. Letting X denote an arbitrary node, we have:



Evidently, the number of newly underscored nodes on echelon k (i.e. nodes which are not children of an underscored node) is equal to the number of non-underscored nodes in echelon $k-2$. Hence, the number of non-underscored nodes in echelon k is

$$2\#U_j(k-1) - \#U_j(k-2). \quad \Box$$

Lemma 4.2 *If V is a regular vine on n elements then for all $i = 1, \dots n-1$, and all $e_i \in E_i$, the conditioned sets associated with e_i are singletons, $\#U^*_{e_i} = i+1$, and $\#D_{e_i} = i-1$.*

Proof. Let $e_i \in E_i$ and $e_i = \{j,k\}$. By Lemma 4.1 $\#U^*_{e_i} = i+1$. Let $D = U^*_j \cap U^*_k$ and $C = U^*_j \triangle U^*_k$. It suffices to show that $\#C = 2$. We get

$$i + 1 = \#D + \#C \tag{4.1}$$

and

$$2i = \#C + 2\#D. \tag{4.2}$$

When we divide (4.2) by 2 and subtract from (4.1) then

$$\#C = 2.$$

Hence $\#(U^*_j \setminus D) = 1$, $\#(U^*_k \setminus D) = 1$ and $\#D = i-1$. $\Box$

Lemma 4.3 *Let V be a regular vine, and suppose for $j, k \in E_i, U^*_j = U^*_k$, then $j = k$.*

Proof. There is a largest x such that $U_j(x) \neq U_k(x)$ and $U_j(x+1) = U_k(x+1)$. Since $\#U_j(x+1) = x+2$, there can be at most $x+1$ edges between these elements. $\#U_j(x) = \#U_k(x) = x+1$, but this implies that $U_j(x) = U_k(x)$, since otherwise T_j would not be a tree. □

Lemma 4.4 *If the conditioned sets of nodes i, j in a regular vine are equal, then $i = j$.*

Proof. Suppose i and j have the same conditioned sets. By Lemma 4.2 the conditioned sets are singletons, say $\{a, b\}, a \in N, b \in N$. Let D_i respectively D_j be the conditioning sets of nodes i and j. Then in the tree T_1 there is a path from a to b through the nodes in D_i, and also a path from a to b through the nodes in D_j. If $D_i \neq D_j$, then there must be a cycle in the edges E_1, but this is impossible since T_1 is a tree. It follows that $D_i = D_j$, and from Lemma 4.3 it follows that $i = j$. □

Definition 4.5 (Partial correlation) *Let $X_i, \ldots X_n$ be random variables, and let i, j, k be distinct indices, and let C be a (possibly empty) set of indices disjunct from $\{i, j, k\}$. The partial correlation of X_i and X_j given $\{X_k, \cup\{X_h | h \in C\}\}$ is*

$$\rho_{ij;kC} = \frac{\rho_{ij;C} - \rho_{ik;C}\rho_{jk;C}}{\sqrt{1-\rho^2_{ik;C}}\sqrt{1-\rho^2_{jk;C}}}; \quad \rho^2_{ik;C} < 1, \; \rho^2_{jk;C} < 1. \tag{4.3}$$

where $\rho_{ij} = \rho(X_i, X_j)$.

If $\rho^2_{ik;C} = 1$ or $\rho^2_{jk;C} = 1$, then $\rho_{ij;kC}$ is not defined.

In general, all partial correlations can be computed from the correlations by iterating the above equation. If $X_1, \ldots, X_n$ follow a joint normal distribution with variance covariance matrix of full rank, then partial correlations correspond to conditional correlations. In the other cases partial correlations can be interpreted in terms of partial regression coefficient (Yule and Kendall [54]). The relationship between partial correlations and conditional correlations is studied in (Kurowicka and Cooke [35]).

The edges in a regular vine may be associated with a set of partial correlations in the following way:

for $i = 1, \ldots, n-1$, with $e \in E_i, e = \{j, k\}$ we associate

$$\rho_{C_{e,j}C_{e,k};D_e}$$

Definition 4.6 (Partial correlation specification) *A partial correlation specification for a regular vine is an assignment of values in* $(-1, 1)$ *to each edge of the vine.*

From Lemma 4.2 it follows the the sets $C_{e,j}$ and $C_{e,k}$ are singletons, and that their intersection with D_e is empty. For tree T_1, the conditioning sets D_e are empty and the partial correlations are just the ordinary correlations. The order of a partial correlation is the cardinality of the conditioning set. Hence this association involves $(n-1)$ partial correlations of order zero, $(n-2)$ of order one, ... and one of order $(n-2)$. In total there are

$$\sum_{j=1}^{n-1} j = \binom{n}{2}$$

edges in a regular vine and the same number of partial correlations associated with the edges of a regular vine. Since the conditioned sets of each edge must be distinct, it follows that each pair of indices appears once as conditioned variables in a regular vine.

The following theorem shows that the correlations are uniquely determined by the partial correlations on a regular vine.

Theorem 4.1 *Let* $X_1, \ldots, X_n$ *and* $Y_1, \ldots Y_n$ *be random variables satisfying the same partial correlation vine specification. Then for* $i \neq j$

$$\rho(X_i, X_j) = \rho(Y_i, Y_j).$$

Proof. It suffices to show that the the correlations $\rho_{ij} = \rho(X_i, X_j)$ can be calculated from the partial correlations specified by the vine. Proof is by induction on the number of elements n. The basic case $(n = 2)$ is trivial. Assume the theorem holds for $i = 2, \ldots n-1$. For a regular vine over n elements the tree T_{n-1} has one edge, say $e = \{j, k\}$. By Lemma 4.2, $\#D_e = n-2$. Re-indexing the variables $X_1, \ldots X_n$ if necessary, we may assume that

$$\begin{aligned}
C_{e,j} &= U_j^* \setminus D_e = X_1, \\
C_{e,k} &= U_k^* \setminus D_e = X_n, \\
U_j^* &= \{1, \ldots, n-1\} \\
U_k^* &= \{2, \ldots, n\} \\
D_e &= \{2, \ldots, n-1\}.
\end{aligned}$$

The correlations over U_j^* and U_k^* are determined by the induction step. It remains to determine the correlation ρ_{1n}. The left hand side of

$$\rho_{1n;2\ldots n-1} = \frac{\rho_{1n;3\ldots n-1} - \rho_{12;3\ldots n-1}\rho_{2n;3\ldots n-1}}{\sqrt{1-\rho^2_{12;3\ldots n-1}}\sqrt{1-\rho^2_{2n;3\ldots n-1}}} \tag{4.4}$$

is determined by the vine specification. The terms

$$\rho_{12;3\ldots n-1},\ \rho_{2n;3\ldots n-1}$$

are determined by the induction hypothese. It follows that we can solve the above equation for $\rho_{1n;3\ldots n-1}$, and write

$$\rho_{1n;3\ldots n-1} = \frac{\rho_{1n;4\ldots n-1} - \rho_{13;4\ldots n-1}\rho_{3n;4\ldots n-1}}{\sqrt{1-\rho^2_{13;4\ldots n-1}}\sqrt{1-\rho^2_{3n;4\ldots n-1}}}$$

Proceeding in this manner, we eventually find

$$\rho_{1n;n-1} = \frac{\rho_{1n} - \rho_{1n-1}\rho_{nn-1}}{\sqrt{1-\rho^2_{1n-1}}\sqrt{1-\rho^2_{nn-1}}}$$

This equation may now be solved for ρ_{1n}. □

The following lemma shows that $\rho_{1n;2\ldots n-1}$ can be chosen arbitrarily in (4.4) and the resulting system could be solved for ρ_{1n}. This idea is the basic for the proof of Theorem 4.2 below.

Lemma 4.5 *If $z, x, y \in (-1,1)$, then also $w \in (-1,1)$, where*

$$w = z\sqrt{(1-x^2)(1-y^2)} + xy.$$

Proof. We substitute $x = \cos\alpha, y = \cos\beta$, and use

$$\begin{aligned} 1-\cos^2\alpha &= \sin^2\alpha; \\ \cos\alpha\cos\beta &= \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}; \\ \sin\alpha\sin\beta &= \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}; \end{aligned}$$

and find

$$z\frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2} + \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2} = w.$$

Write this as

$$z\frac{a+b}{2}+\frac{a-b}{2}=w,$$

where $a,b\in(-1,1)$. As the left hand side is linear in a,b, and z, its extreme values must occur when $z=1$ or $z=-1$. It is easy to check that in these cases, $w\in(-1,1)$. $\square$

In the next lemma we will see that it is always possible to find a single unknown variable on the right hand side of equation (4.4) such that the left hand side will lie in the interval $(-1,1)$.

Lemma 4.6 *Let* $w,y\in(-1,1)$, $x\in(-1,1)$ *and*

$$z \quad = \quad \frac{w-xy}{\sqrt{(1-x^2)(1-y^2)}} \tag{4.5}$$

then

$$z\in(-1,1) \quad \Leftrightarrow \quad x\in I_z,\;\; I_z\neq\emptyset,\; I_z=(\underline{x},\overline{x})\cap(-1,1)$$

where

$$\begin{aligned}\underline{x} &= yw-\sqrt{(1-y^2)(1-w^2)}\\ \overline{x} &= yw+\sqrt{(1-y^2)(1-w^2)}.\end{aligned}$$

Proof. It suffices to find the solution of the following inequality

$$(w-xy)^2<(1-x^2)(1-y^2)$$

which is equivalent to

$$x^2-2wyx+w^2+y^2-1<0. \tag{4.6}$$

We get

$$\begin{aligned}\Delta &= 4w^2y^2-4w^2-4y^2+4\\ &= 4(1-y^2)(1-w^2).\end{aligned}$$

Since $y,w\in(-1,1)$ then $\Delta>0$ and inequality (4.6) has always solution

$$I_x=\left(yw-\sqrt{(1-y^2)(1-w^2)};yw+\sqrt{(1-y^2)(1-w^2)}\right)$$

Since $w, y \in (-1,1)$ then $wy \in (-1,1)$. We want also $x \in (-1,1)$ so we get

$$I = I_x \cap (-1,1) \tag{4.7}$$

which is always non-empty. □

Remark 4.1 *Similar considerations hold if we change the role of w and x in (4.5). Then we obtain that we can always find w such that $z \in (-1,1)$. This solution belongs to the following non-empty interval*

$$I = \left(xy - \sqrt{(1-x^2)(1-y^2)}; xy + \sqrt{(1-x^2)(1-y^2)}\right) \cap (-1,1).$$

Lemma 4.7 *Let $w \in (-1,1)$ and z given by (4.5).*

$$z \in (-1,1) \Leftrightarrow (x,y) \in A(w)$$

where

$$A(w) = \{(x,y) : \frac{(y-wx)^2}{1-w^2} + x^2 < 1\}. \tag{4.8}$$

Proof. It suffices to find points $(x,y) \in (-1,1)^2$ such that

$$(w-xy)^2 < (1-x^2)(1-y^2)$$

therefore we test when the function

$$\begin{aligned} g(x,y) &= (w-xy)^2 - (1-x^2)(1-y^2) \\ &= x^2 + y^2 - 2wxy + w^2 - 1 \\ &= x^2(1-w^2) + (y-xw)^2 - (1-w^2) \end{aligned}$$

is less than zero.

It is easy to notice that $g(x,y)$ is less then zero if $(x,y) \in A(w)$. □

Remark 4.2 *Note that point (0,0) always belongs to $A(w)$.*

In (Bedford and Cooke [3]) the following is proved:

Theorem 4.2 *For any regular vine on n elements there is a one to one correspondence between the set of $n \times n$ positive definite correlation matrices and the set of partial correlation specifications for the vine.*

4.3 Positive definiteness

If A is an $n \times n$ symmetric matrix with positive numbers on the main diagonal, then we may transform A to a matrix A^* by setting

$$a^*_{ij} = \frac{a_{ij}}{\sqrt{a_{ii}a_{jj}}}. \qquad (4.9)$$

A^* has "1"-s on the main diagonal. Since it is known that A^* is positive definite if and only if all principle submatrices are positive definite then we can restrict our further considerations to the matrices which after transformation (4.9) have all $a^*_{ij} \in (-1,1)$ where $i \neq j$. The matrix with all elements from the interval (-1,1) and with "1"-s on the main diagonal is called proto correlation matrix.

Then we get that

$$A^* = DAD$$

where

$$d_{ij} = \begin{cases} \frac{1}{\sqrt{a_{ii}}} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.3 *A is positive definite ($A \succ 0$) if and only if A^* is positive definite.*

Proof. It is easy to see that D is invertible

$$D^{-1} = [c_{ij}]$$

where

$$c_{ij} = \begin{cases} \sqrt{a_{ii}} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Let $A \succ 0$ and $y \in R^n - \{0\}$ arbitrary vector. We want to show $yA^*y^T > 0$. Since $A^* = DAD$ then

$$yA^*y^T = yDADy^T.$$

Let $x = yD$ then $x \in R^n - \{0\}$. It is easy to see that $x^T = Dy^T$. Hence we get

$$yA^*y^T = xAx^T > 0.$$

Analogously if we assume that $A^* \succ 0$ then taking an arbitrary vector $y \in R^n - \{0\}$ and using the following relationship between A and A^*

$$A = D^{-1} A^* D^{-1}$$

we can show $yAy^T > 0$. $\square$

In order to check positive definiteness of the matrix A^* we will use the partial correlation specification for the canonical vine. If all partial correlations from the partial correlation specification on the vine are in the interval $(-1, 1)$ then A^* is positive definite.

We illustrate this algorithm for 5×5 proto correlation matrix given by

$$\begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} \\ \rho_{21} & 1 & \rho_{23} & \rho_{24} & \rho_{25} \\ \rho_{31} & \rho_{32} & 1 & \rho_{34} & \rho_{35} \\ \rho_{41} & \rho_{42} & \rho_{43} & 1 & \rho_{45} \\ \rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & 1 \end{bmatrix}$$

For this matrix we will consider the canonical vine on 5 variables as follows

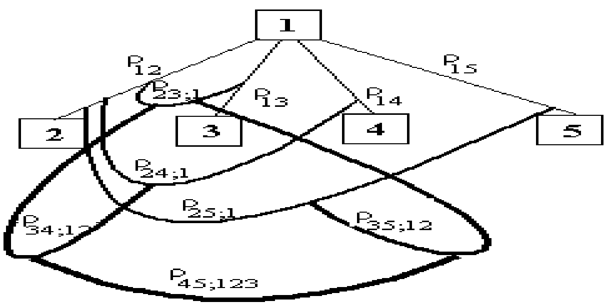


Figure 4.3. Partial correlation specification for the canonical vine on 5 variables.

In the first tree we have to read correlations from the matrix A^*. For the second tree we will use formula (4.4) and calculate the following correlations:

$$\rho_{23;1},\ \rho_{24;1},\ \rho_{25;1}.$$

To calculate correlations $\rho_{34;12}, \rho_{35;12}$ we will also have to calculate $\rho_{34;1}$ and $\rho_{35;1}$. Similarly, to calculate $\rho_{45;123}$ we will need $\rho_{45;12}$ and $\rho_{45;1}$.

In general we must calculate using formula (4.4)

$$\binom{n-1}{2} \quad \text{partial correlations of the first order}$$
$$\binom{n-2}{2} \quad \text{partial correlations of the second order}$$
$$\vdots$$
$$\binom{n-(n-2)}{2} \quad \text{partial correlations of the (n-2) order.}$$

Hence in order to verify positive definiteness of the matrix A^* we have to calculate

$$\sum_{k=1}^{n-2} \binom{n-k}{2} = \frac{(n-2)(n-1)n}{6} < \frac{n^3}{6}$$

partial correlations using formula (4.4).

Example 1

Let us consider the matrix

$$A = \begin{bmatrix} 25 & 12 & -7 & 0.5 & 18 \\ 12 & 9 & -1.8 & 1.2 & 6 \\ -7 & -1.8 & 4 & 0.4 & -6.4 \\ 0.5 & 1.2 & 0.4 & 1 & -0.4 \\ 18 & 6 & -64 & -0.4 & 16 \end{bmatrix}$$

and transform A to proto correlation matrix using formula (4.9). Then we get

$$A^* = \begin{bmatrix} 1 & 0.8 & -0.7 & 0.1 & 0.9 \\ 0.8 & 1 & -0.3 & 0.4 & 0.5 \\ -0.7 & -0.3 & 1 & 0.2 & -0.8 \\ 0.1 & 0.4 & 0.2 & 1 & -0.8 \\ 0.9 & 0.5 & -0.8 & -0.8 & 1 \end{bmatrix}.$$

Since

$$\begin{bmatrix} \rho_{23;1}, & \rho_{24;1}, & \rho_{25;1} \end{bmatrix} = \begin{bmatrix} 0.6068, & 0.5360, & -0.8412 \end{bmatrix}$$

$$\begin{bmatrix} \rho_{34;12}, & \rho_{35;12} \end{bmatrix} = \begin{bmatrix} 0.0816, & -0.0830 \end{bmatrix}$$
$$\begin{bmatrix} \rho_{45;123} \end{bmatrix} = \begin{bmatrix} 0.0351 \end{bmatrix}$$

are all between $(-1, 1)$, it follows that, A^* and A are positive definite.

We now show the relationship between above procedure of testing positive definiteness using canonical vine correlations specification and the known matrix theory results.

Theorem 4.4 (Schur complement)
Suppose that symmetric matrix M is partitioned as

$$M = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$$

where X, Z are square.

$$M \succ 0 \Leftrightarrow X \succ 0 \text{ and } Z - Y^T X^{-1} Y \succ 0.$$

Let A be an $n \times n$ proto correlation matrix and A is partitioned in the following way

$$A = \begin{bmatrix} X_k & Y_k \\ Y_k^T & Z_k \end{bmatrix}$$

where X is $k \times k, 1 \leq k \leq n-2$ and Z $n-k \times n-k$ is a matrix.

We introduce the following notation:

$A_{;12\ldots k}$: matrix of the k-th order partial correlations with conditioned set $\{12\ldots k\}$.[2]

If M is a square matrix with positive elements on the main diagonal then let M^* denote the matrix M transformed to proto correlation matrix using transformation (4.9).

Theorem 4.5

$$A \succ 0 \quad \Leftrightarrow \quad \forall_{1 \leq k \leq n-2} \left(Z_k - Y_k^T X_k^{-1} Y_k \right)^* = A_{;12\ldots k}.$$

[2]Note that $A_{;12\ldots n-2} = \begin{bmatrix} 1 & \rho_{n,n-1;12\ldots n-2} \\ & 1 \end{bmatrix}$.

Proof. The proof is by iteration with respect to k. For $k = 1$ we get

$$A \quad = \quad \begin{bmatrix} X_1 & Y_1 \\ Y_1^T & Z_1 \end{bmatrix}$$

where

$$X_1 = [1]\,, \quad Y_1 = [\rho_{12}\ \rho_{13}\ \ldots\ \rho_{1n}]\,, \quad Z_1 = \begin{bmatrix} 1 & \rho_{23} & \rho_{24} & \ldots & \rho_{2n} \\ & 1 & \rho_{34} & \ldots & \rho_{3n} \\ & \ldots & \ldots & \ldots & \ldots \\ & & & 1 & \rho_{n-1,n} \\ & & & & 1 \end{bmatrix}.$$

Since certainly $X_1 \succ 0$, then by Theorem 4.4

$$A \succ 0 \text{ if and only if } Z_1 - Y_1^T X_1^{-1} Y_1 \succ 0.$$

We get

$$\widetilde{A_{;1}} = Z_1 - Y_1^T X_1^{-1} Y_1 = \begin{bmatrix} 1-\rho_{12}^2 & \rho_{23} - \rho_{12}\rho_{13} & \ldots & \rho_{2n} - \rho_{12}\rho_{1n} \\ & 1-\rho_{13}^2 & \ldots & \rho_{3n} - \rho_{13}\rho_{1n} \\ & \ldots & \ldots & \ldots \\ & & & 1-\rho_{1n}^2 \end{bmatrix}$$

Since A is the proto correlation matrix then $\rho_{ij} \in (-1, 1)$ where $i, j = 1, 2, \ldots, n$ and $i \neq j$. Hence all elements on the main diagonal of $\widetilde{A_{;1}}$ are positive so the transformation (4.9) can be applied. After transformation $\widetilde{A_{;1}}$ will be of the form

$$\widetilde{A_{;1}}^{\,*} = \begin{bmatrix} 1 & \frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{(1-\rho_{12}^2)(1-\rho_{13}^2)}} & \ldots & \frac{\rho_{2n}-\rho_{12}\rho_{1n}}{\sqrt{(1-\rho_{12}^2)(1-\rho_{1n}^2)}} \\ & 1 & \ldots & \frac{\rho_{3n}-\rho_{13}\rho_{1n}}{\sqrt{(1-\rho_{13}^2)(1-\rho_{1n}^2)}} \\ & \ldots & \ldots & \ldots \\ & & 1 & \frac{\rho_{n-1,n}-\rho_{1n-1}\rho_{1n}}{\sqrt{(1-\rho_{1n-1}^2)(1-\rho_{1n}^2)}} \\ & & & 1 \end{bmatrix}.$$

By formula (4.4) we see that $\widetilde{A_{;1}}^{\,*} = A_{;1}$. Hence and by Theorem 4.3 $A \succ 0$ if and only if $A_{;1} \succ 0$.

[3] Note that this is equivalent to checking if all correlations in the second tree in canonical partial correlation vine are in (-1,1).

If any element of the first row of $A;1$ is not in (-1,1) then A and $A_{;1}$ are not positive definite.[3] If all elements of the first row of $A_{;1}$ are in (-1,1) we can apply the above steps with $A_{;1}$ in the role of A. Let $A_{;1}(i,j)$ denote the (i,j) element of $A_{;1}$. The result of applying Theorem 4.4 and formula (4.9) to $A_{;1}$ yields a matrix with off-diagonal elements (i,j), $i,j = 2,3,\dots n$

$$\frac{A_{;1}(i,j) - A_{;1}(2,i)A_{;1}(2,j)}{\sqrt{(1-A^2_{;1}(2,i))(1-A^2_{;1}(2,j))}}.$$

By the recursive formula (4.4), this is equal to

$$\rho_{ij;12}$$

so $A_{;12}$ is the matrix of second order partials. The proof is completed by iteration.□

Corollary 4.1

$$A \succ 0 \Leftrightarrow A_{;1} \succ 0 \Leftrightarrow \dots \Leftrightarrow A_{;12\dots n-2} \succ 0 \Leftrightarrow \rho_{n,n-1;12\dots n-2} \in (-1,1).$$

4.4 Repairing violations of positive definiteness

Partial correlations can be used to alter a non-positive definite matrix A so as to obtain a positive definite matrix B. If the matrix is not positive definite then there exists at least one element in the partial correlation specification of the canonical vine which is not in the interval $(-1,1)$. We will change the value of that element and recalculate partial correlations on the vine using the following algorithm:

for $1 \le s \le n-2$, $j = s+2, s+3, \dots, n$

$$\rho_{s+1,j;12\dots s} \notin (-1,1) \quad \Rightarrow \quad \rho_{s+1,j;12\dots s} := V(\rho_{s+1,j;12\dots s})$$

where $V(\rho_{s+1,j;12\dots s}) \in (-1,1)$ is the altered value of $\rho_{s+1,j;12\dots s}$.

Recalculate partial correlations of lower order as follows:

$$\begin{aligned} V(\rho_{s+1,j;1\dots t-1}) = {} & V(\rho_{s+1,j;1\dots t})\sqrt{(1-\rho^2_{t,s+1;1\dots t-1})(1-\rho^2_{s+1,j;1\dots t-1})} \\ & + \rho_{t,s+1;1\dots t-1}\rho_{s+1,j;1\dots t-1}, \end{aligned} \quad (4.10)$$

where $t = s, s-1, \dots, 1$.

Theorem 4.6 *The following hold:*

(a) *all recalculated partial correlations are in the interval* $(-1, 1)$*,*

(b) *changing the value of the partial correlation on the vine leads to changing only one correlation in the matrix and doesn't effect correlations which were already changed.*

(c) *there is a linear relationship between altered value of partial correlation and correlation with the same conditioned set in the proto correlation matrix,*

(d) *this method always produce positive definite matrix.*

Proof.

(a) This condition follows directly from Lemma 4.5.

(b) The condition (b) is a result of observation that changing the value of the correlation $\rho_{s+1,j;12\ldots s}$ in the above algorithm leads to recalculate correlations of the lower order but only with the same indices before ";", that is, $s+1, j$.

(c) Since $\rho_{s+1,j;12\ldots t-1}$ is linear in $\rho_{s+1,j;12\ldots t}$ for all $t = s, s-1, \ldots, 1$ the linear relationship between $\rho_{s+1,j}$ and $\rho_{s+1,j;12\ldots s}$ follows by substitution.

(d) Applying the above algorithm whenever a partial correlation outside the interval $(-1, 1)$ is found, we eventually obtain that all partial correlations in partial correlation specification on the vine are in $(-1, 1)$, that is, the altered matrix is positive definite. □

Let us consider following example:
Example 2
Let

$$A = \begin{bmatrix} 1 & -0.6 & -0.8 & 0.5 & 0.9 \\ -0.6 & 1 & 0.6 & -0.4 & -0.4 \\ -0.8 & 0.6 & 1 & 0.1 & -0.5 \\ 0.5 & -0.4 & 0.1 & 1 & 0.7 \\ 0.9 & -0.4 & -0.5 & 0.7 & 1 \end{bmatrix}.$$

We get $\rho_{34;12} = 1.0420$ hence A is not positive definite.

Since $\rho_{34;12} > 1$ then we will change its value to $V(\rho_{34;12}) = 0.9$ and recalculate lower order correlations

$$V(\rho_{34;1}) = V(\rho_{34;12})\sqrt{(1-\rho_{23;1}^2)(1-\rho_{24;1}^2)} + \rho_{23;1}\rho_{24;1}$$

and

$$V(\rho_{34}) = V(\rho_{34;1})\sqrt{(1-\rho_{13}^2)(1-\rho_{14}^2)} + \rho_{13}\rho_{14}$$

This way we will get for our example $V(\rho_{34;1}) = 0.9623$ and finally the new value in the proto correlation matrix $V(\rho_{34}) = 0.0293$. Next we will apply the same algorithm to verify if this altered matrix become positive definite. We obtained that matrix

$$B = \begin{bmatrix} 1 & -0.6 & -0.8 & 0.5 & 0.9 \\ -0.6 & 1 & 0.6 & -0.4 & -0.4 \\ -0.8 & 0.6 & 1 & 0.0293 & -0.5 \\ 0.5 & -0.4 & 0.0293 & 1 & 0.7 \\ 0.9 & -0.4 & -0.5 & 0.7 & 1 \end{bmatrix}$$

is positive definite.

4.5 Completion problem

In this section we apply the canonical vine to completion problem. First, however, we quote the known results of completion problem which can be found in [38], [39], [13], [52], [53].

We define the set of correlation matrices $\mathcal{E}_{n\times n}$ as follows:

$$\mathcal{E}_{n\times n} = \{X = (x_{ij}) \text{ symmetric } n\times n | X \succeq 0, x_{ii} = 1 \text{ for all } i = 1,2,\ldots,n\}.$$

Let $G = (N, E)$ be a graph where $N = \{1, 2, \ldots, n\}$. G is simple i.e. has no loops or parallel edges. We define set $\mathcal{E}$ as a projection of $\mathcal{E}_{n\times n}$ on the subspace R^E indexed by the edge set of G

$$\mathcal{E}(G) = \left\{x \in R^E | \exists A = (a_{ij}) \in \mathcal{E}_{n\times n} \text{ such that } a_{ij} = x_{ij} \text{ for all } ij \in E\right\}.$$

The sets $\mathcal{E}_{n\times n}$ and $\mathcal{E}(G)$ are called *elliptopes*.

Let $G = (N, E)$ be a graph. Given a subset $U \subseteq N$, $G(U)$ denotes the subgraph of G *induced* by U, with node set U and with edge set $\{uv \in E | u, v \in U\}$. One says that U is a *clique* in G when $G(U)$ is a complete graph.

Suppose X has diagonal entries 1, and let $x = (x_{ij})_{ij \in E} \in R^E$ denote the vector whose components are specified entries of X. Let G denote the graph with edge set E.

Definition 4.7

$$X \text{ is completable if } x \in \mathcal{E}(G).$$

Clique condition

$$x \in \mathcal{E}(G) \Rightarrow$$

$$\text{For every clique } K \text{ in } G, \text{ the projection } x_K \text{ of } x \text{ on the edge set of } K \text{ belongs to } \mathcal{E}(K). \tag{4.11}$$

Since every vector $x \in \mathcal{E}(G)$ has all entries in the interval [-1,1], we can find

$$a_e = \frac{\arccos x_e}{\pi} \in [0,1] \text{ for every } e \in E.$$

Cycle condition

$$x \in \mathcal{E}(G) \Rightarrow a = (a_e)_{e \in E} \text{ satisfies condition}$$

$$\sum_{e \in F} a_e - \sum_{e \in C \setminus F} a_e \leq |F| - 1 \quad \text{for } C \text{ a circuit in } G, F \subseteq C \text{ with } |F| \text{ odd.} \tag{4.12}$$

The condition (4.11) and (4.12) are not sufficient in general. There are graphs, however, for which (4.11) is sufficient. These graphs are called *chordal* (graph G is said to be *chordal* if every circuit of G with length ≥ 4 has a chord; a *chord* of the circuit C is an edge joining two nonconsecutive nodes of C). The condition (4.12) is sufficient for the circuits and *series-parallel* graphs i.e. graphs with no K_4- *minors*[4]. These two conditions taken together suffice for describing eliptope $\mathcal{E}(G)$ for the graphs called *cycle completable* i.e. chordal graphs, series-parallel graphs and their *clique sums* (where *clique sum* of graphs $G_1 = (N_1, E_1)$ and $G_2 = (N_2, E_2)$ is a graph $G = (N_1 \cup N_2, E_1 \cup E_2)$ such that the set $K = N_1 \cap N_2$ induces a clique (possibly empty) in both G_1 and G_2 and there is no edge between a node of $N_1 \setminus K$ and node of $N_2 \setminus K$).

Now we present the solution of the completion problem for the special cases of graphs using partial correlation specification of a canonical vine. We shall see that verifying the relevant conditions for completability simultaneously gives the set of completions.

Case 1

We have the following proto correlation matrix which needs to be completed

$$A \quad = \quad \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} & \cdots & \cdots & \cdots & \rho_{1n} \\ & 1 & \rho_{23} & \rho_{24} & \cdots & \cdots & \cdots & \rho_{2n} \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 1 & \rho_{k,k+1} & \rho_{k,k+2} & \cdots & \rho_{k,n} \\ & & & & 1 & \Box & \Box & \Box \\ & & & & & 1 & \Box & \Box \\ & & & & & & 1 & \Box \\ & & & & & & & 1 \end{bmatrix}$$

where $\Box$ represents an unspecified entry in this matrix.

Since all correlations from the rows 1 to k are given then we can calculate all partial correlations in canonical vine specifications up to $(k-1)$-th order.

Assigning the remaining partial correlations of order k to $n-2$ in the canonical vine the value 0, we can specify all empty cells recalculating partial correlations using the algorithm 4.10. In this case the matrix A can be completed if and only if all partial correlations of order less then k are in the interval $(-1,1)$. Hence we must evaluate formula (4.4) $\sum_{j=k+1}^{n-1}\left[\binom{j}{2}-\binom{n-k}{2}\right]$ times.

Remark 4.3 *The graph corresponding to above matrix is chordal. Since we have $n-k$ maximal cliques of size $k+1$ there, then by (4.11), the $n-k$ principal submatrices of size $k+1$ should be checked for positive definiteness.*

Case 2

[4] A graph H is said to be *minor* of the graph G if H can be obtained from G by repeatedly deleting and/or contracting edges and deleting isolated nodes. *Deleting* an edge e in graph G means discarding it from the edge set of G. *Contracting* edge $e=uv$ means identifying both end nodes of e and discarding multiple edges and loops if some are created during the identification of u and v.

$$A = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} & \dots & \dots & \dots & \rho_{1n} \\ & 1 & \rho_{23} & \rho_{24} & \dots & \dots & \dots & \rho_{2n} \\ & & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & 1 & \rho_{k,k+1} & \rho_{k,k+2} & \dots & \rho_{k,n} \\ & & & & 1 & \rho_{k+1,k+2} & \square & \square \\ & & & & & 1 & \square & \square \\ & & & & & & 1 & \square \\ & & & & & & & 1 \end{bmatrix}$$

We want to fill the $k+1$-th row and reduce this case to the previous one.

As in the Case 1 we can calculate all partials of order less then k but also $\rho_{k+1,k+2;12\dots k}$ can be calculated.

We want to choose $\rho_{k+1,s;12\dots k}$, where $s = k+3,\dots,n$ in such a way that $\rho_{k+2,s;12\dots k+1}$ will be in the interval $(-1,1)$.

Since $\rho_{k+2,s;12\dots k}$ and $\rho_{k+1,s;12\dots k}$ cannot be computed and

$$\rho_{k+2,s;12\dots k+1} = \frac{\rho_{k+2,s;12\dots k} - \rho_{k+1,k+2;12\dots k}\rho_{k+1,s;12\dots k}}{\sqrt{(1-\rho^2_{k+1,k+2;12\dots k})(1-\rho^2_{k+1,s;12\dots k})}}$$

then it is enough to assign them the value 0 and recalculate correlations using (4.10). This way we will fill all empty cells in the rows $k+1$ and $k+2$ and reduce to Case 1.

Case 3

(a)

$$A = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} & \dots & \dots & \dots & \dots & \rho_{1n} \\ & 1 & \rho_{23} & \rho_{24} & \dots & \dots & \dots & \dots & \rho_{2n} \\ & & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & 1 & \rho_{k,k+1} & \rho_{k,k+2} & \rho_{k,k+3} & \dots & \rho_{k,n} \\ & & & & 1 & \rho_{k+1,k+2} & \square & \dots & \dots \\ & & & & & 1 & \rho_{k+2,k+3} & \dots & \dots \\ & & & & & & 1 & \square & \square \\ & & & & & & & 1 & \square \\ & & & & & & & & 1 \end{bmatrix}$$

In this case we can calculate all partials of order less then k and $\rho_{k+1,k+2;1\dots k}$ and $\rho_{k+2,k+3;1\dots k}$. To find the value of $\rho_{k+2,k+3;1\dots k+1}$ we will use Lemma 4.6. We choose a value of $\rho_{k+1,k+3;1\dots k}$ which belong to the non-empty interval given by

(4.7). Similar solutions are obtained when the empty cell in row $k+1$ occupies any other position except position $(k+1, k+2)$.

(b)

$$A = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} & \cdots & \cdots & \cdots & \cdots & \cdots & \rho_{1n} \\ & 1 & \rho_{23} & \rho_{24} & \cdots & \cdots & \cdots & \cdots & \cdots & \rho_{2n} \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 1 & \rho_{k,k+1} & \rho_{k,k+2} & \rho_{k,k+3} & \rho_{k,k+4} & \cdots & \rho_{k,n} \\ & & & & 1 & \square & \square & \rho_{k+1,k+4} & \rho_{k+1,k+5} & \cdots \\ & & & & & 1 & \rho_{k+2,k+3} & \rho_{k+2,k+4} & \rho_{k+2,k+5} & \cdots \\ & & & & & & 1 & \square & \square & \square \\ & & & & & & & 1 & \square & \square \\ & & & & & & & & 1 & \square \\ & & & & & & & & & 1 \end{bmatrix}$$

The difference with Case a is that the partial correlation $\rho_{k+1,k+2;1\ldots k}$, which cannot be calculate now, appears in every correlations of order $k+1$. In this case that in addition to all partial correlations of order less then k correlations $\rho_{k+2,k+3;1\ldots k}$, $\rho_{k+2,k+4;1\ldots k}$ and $\rho_{k+2,k+5;1\ldots k}$ can be calculated. We want to find $\rho_{k+2,k+4;1\ldots k+1}$, $\rho_{k+2,k+4;1\ldots k+1}$ and $\rho_{k+2,k+4;1\ldots k}$. We find using Lemma 4.6 value of $\rho_{k+1,k+2;1\ldots k}$ which belongs to the intersection $I_{k+1,k+2;1\ldots k} = I_{k+2,k+4;1\ldots k+1} \cap I_{k+2,k+5;1\ldots k+1}$ [5] of two intervals such that correlations $\rho_{k+2,k+4;1\ldots k+1}$ and $\rho_{k+2,k+5;1\ldots k+1}$ are in $(-1,1)$. Next given this value of $\rho_{k+1,k+2;1\ldots k}$, the value of $\rho_{k+1,k+3;1\ldots k}$ can be computed. In this case the matrix can be completed if all correlations which can be computed are in $(-1,1)$ and if the interval $I_{k+1,k+2;1\ldots k}$ is not empty.

Case 4

This case is an example of non-chordal graph with one circuit with length 4.

$$A = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} & \cdots & \cdots & \cdots & \cdots & \cdots & \rho_{1n} \\ & 1 & \rho_{23} & \rho_{24} & \cdots & \cdots & \cdots & \cdots & \cdots & \rho_{2n} \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 1 & \rho_{k,k+1} & \rho_{k,k+2} & \rho_{k,k+3} & \rho_{k,k+4} & \cdots & \rho_{k,n} \\ & & & & 1 & \rho_{k+1,k+2} & \square & \rho_{k+1,k+4} & \cdots & \cdots \\ & & & & & 1 & \rho_{k+2,k+3} & \square & \cdots & \cdots \\ & & & & & & 1 & \rho_{k+3,k+4} & \cdots & \cdots \\ & & & & & & & 1 & \square & \square \\ & & & & & & & & 1 & \square \\ & & & & & & & & & 1 \end{bmatrix}$$

[5] We write $I_{k+1,k+2;1\ldots k}$ instead of $I_{\rho_{k+1,k+2;1\ldots k}}$.

In this case we can calculate all partials of order less than k and $\rho_{k+1,k+2;12\ldots k}$, $\rho_{k+1,k+4;12\ldots k}$, $\rho_{k+2,k+3;12\ldots k}$, $\rho_{k+3,k+4;12\ldots k}$.

Using Lemma 4.6 we choose $\rho_{k+1,k+3;12\ldots k}$ belonging to the intersection

$$I_{k+1,k+3;1\ldots k} = I_{k+2,k+3;1\ldots k+1} \cap I_{k+3,k+4;1\ldots k+1}$$

of two intervals such that $\rho_{k+2,k+3;12\ldots k+1}$ and $\rho_{k+3,k+4;12\ldots k+1}$ are in $(-1,1)$ if this intersection is non-empty. We next can find possible solutions for $\rho_{k+3,k+4;12\ldots k+1}$ such that $\rho_{k+3,k+4;12\ldots k+2}$ is in $(-1,1)$. Recalculating correlations using formula (4.10) we can fill empty cells in the circuit. In this case the matrix can be completed if all correlations which can be calculated are in $(-1,1)$ and if $I_{k+1,k+3;1\ldots k}$ is not empty.

Remark 4.4 *Let us notice that the procedure of finding correlation* $\rho_{k+1,k+3;12\ldots k}$ *which belong to the interval* $I_{k+1,k+3;1\ldots k}$ *allows us to choose a chord in this circuit. In this way, this case can be reduced to the previous cases where chordal graphs were considered.*

Case 5

In this case we show the general solution of the completion problem for the circuit of length n ($n \geq 4$). The following matrix corresponds to the circuit of length n

$$A = \begin{bmatrix} 1 & \rho_{12} & \square & \square & \ldots & \square & \rho_{1n} \\ & 1 & \rho_{23} & \square & \ldots & \square & \square \\ & & \ldots & \ldots & \ldots & \ldots & \ldots \\ & & & 1 & \rho_{n-3,n-2} & \square & \square \\ & & & & 1 & \rho_{n-2,n-1} & \square \\ & & & & & 1 & \rho_{n-1,n} \\ & & & & & 1 & \end{bmatrix}$$

We have to choose correlations $\rho_{13},\ \rho_{14},\ldots,\rho_{1n-1}$ such that by Lemma 4.6 and Lemma 4.7 the following system is satisfied

$$\left\{ \begin{array}{ccc} \rho_{13} & \in & I_{23;1} \\ (\rho_{13},\rho_{14}) & \in & A(\rho_{34}) \\ (\rho_{14},\rho_{15}) & \in & A(\rho_{45}) \\ \ldots & \ldots & \ldots \\ (\rho_{1,n-2},\rho_{1,n-1}) & \in & A(\rho_{n-2,n-1}) \\ \rho_{1,n-1} & \in & I_{n-1,n;1} \end{array} \right. \tag{4.13}$$

If we can solve (4.13) then this matrix can be completed. We complete our matrix with the algorithm presented below:

$$\begin{array}{rcl}
\rho_{34;12} \in (-1,1) & \Leftrightarrow & \rho_{24;1} \in I_{34;12} \Rightarrow \rho_{24} \text{ can be recalculated with (??)} \\
\rho_{35;12} \in (-1,1) & \Leftrightarrow & \rho_{25;1} \in I_{35;12} \Rightarrow \rho_{25} \text{ can be recalculated} \\
\dots & & \\
\rho_{n-1,n;12} \in (-1,1) & \Leftrightarrow & \rho_{2n;1} \in I_{n-1n;12} \Rightarrow \rho_{2n} \text{ can be recalculated} \\
\rho_{45;123} \in (-1,1) & \Leftrightarrow & \rho_{35;12} \in I_{45;123} \Rightarrow \rho_{35;1} \text{ and next } \rho_{35} \text{ can be recalculated} \\
\dots & & \\
\rho_{n-1,n;123} \in (-1,1) & \Leftrightarrow & \rho_{3n;12} \in I_{n-1,n;123} \Rightarrow \rho_{3n;1} \text{ and next } \rho_{3n} \text{ can be recalculated} \\
\dots & & \\
\rho_{n-1,n;12\dots n-2} \in (-1,1) & \Leftrightarrow & \rho_{n-2n;12\dots n-3} \in I_{n-1,n;12\dots n-2} \Rightarrow \rho_{n-2,n;12\dots n-4} \\
 & & \text{and next } \rho_{n-2,n;12\dots n-4}, \dots, \rho_{n-2,n} \text{ can be recalculated}
\end{array}$$

Case 6

In this case we consider wheel on n ($n \geq 4$) elements (a *wheel* on n elements is a graph composed of a circuit C on $n-1$ nodes together with an additional node adjacent to all nodes of C).

The following matrix corresponds to the wheel of length n

$$A = \begin{bmatrix}
1 & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} & \dots & \dots & \rho_{1n} \\
 & 1 & \rho_{23} & \Box & \Box & \dots & \Box & \rho_{2n} \\
 & & 1 & \rho_{34} & \Box & \dots & \Box & \Box \\
 & & & \dots & \dots & \dots & \dots & \dots \\
 & & & & 1 & \rho_{n-3,n-2} & \Box & \Box \\
 & & & & & 1 & \rho_{n-2,n-1} & \Box \\
 & & & & & & 1 & \rho_{n-1,n} \\
 & & & & & & & 1
\end{bmatrix}.$$

This case can be reduced to the Case 5 by applying Theorem 4.4. If correlations $\rho_{k,k+1;1}$ for $k = 2, 3, \dots, n-1$ and $\rho_{2n;1}$ are in (-1,1) then the following matrix needs to be completed

$$\begin{bmatrix} 1 & \rho_{23;1} & \square & \square & \ldots & \square & \rho_{2n;1} \\ & 1 & \rho_{34;1} & \square & \ldots & \square & \square \\ & & \ldots & \ldots & \ldots & \ldots & \ldots \\ & & & & 1 & \rho_{n-2,n-1;1} & \square \\ & & & & & 1 & \rho_{n-1,n;1} \\ & & & & & & 1 \end{bmatrix}.$$

Remark 4.5 *Note that it is shown in [39] that the wheel on n ($n \geq 4$) elements is not cycle completable.*

Example 4

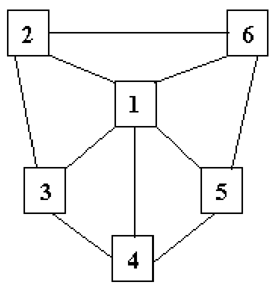


Figure 4.4 A wheel on 6 elements.

The following matrix correspond to the wheel on 6 elements

$$\begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} & \rho_{16} \\ & 1 & \rho_{23} & \square & \square & \rho_{26} \\ & & 1 & \rho_{23} & \square & \square \\ & & & 1 & \rho_{34} & \square \\ & & & & 1 & \rho_{45} \\ & & & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.8 & 0.1 & -0.3 & 0.5 & -0.4 \\ & 1 & 0 & \square & \square & -0.1 \\ & & 1 & -0.6 & \square & \square \\ & & & 1 & -0.7 & \square \\ & & & & 1 & 0.2 \\ & & & & & 1 \end{bmatrix}.$$

We can calculate

$$[\rho_{23;1},\ \rho_{26;1},\ \rho_{34;1},\ \rho_{45;1},\ \rho_{56;1}] \;=\; [-0.1340,\ 0.4001,\ -0.6005,\ -0.6658,\ 0.5040].$$

We must choose correlations $\rho_{24;1}$ and $\rho_{25;1}$ such that the system similar to (4.13) is satisfied. Since $\rho_{24;1} \in I_{34;12} = (-0.7119, 0.8729)$ and $\rho_{25;1} \in I_{56;12} = (-0.5899, 0.9932)$ then we can choose $\rho_{24;1} = \rho_{25;1} = 0$. Hence $\rho_{34;12} = -0.6060, \rho_{45;12} = -0.6658$ and $\rho_{56;12} = 0.5499$. Now we can find $\rho_{35;12} \in I_{45;123} = (-0.1900, 0.9970)$. Let us choose $\rho_{35;12} = 0$ then $\rho_{45;123} = -0.8370$ and $\rho_{36;12} \in I_{56;123} = (-0.8352, 0.8352)$. Hence we can also take $\rho_{36;12} = 0$ then $\rho_{56;123} = 0.5499$. Finally we get $\rho_{46;123} \in I_{56;1234} = (-0.9173, -0.0032)$. We take for instance $\rho_{46;123} = -0.5$ and now we can recalculate all correlations using algorithm 4.10.

$$
\begin{aligned}
\rho_{24;1} = 0 &\Rightarrow \rho_{24} = \rho_{12}\rho_{14} = -0.24 \\
\rho_{25;1} = 0 &\Rightarrow \rho_{25} = \rho_{12}\rho_{15} = 0.4 \\
\rho_{35;12} = 0 &\Rightarrow \rho_{35;1} = \rho_{23;1}\rho_{25;1} = 0 \\
&\Rightarrow \rho_{35} = \rho_{13}\rho_{15} = 0.05 \\
\rho_{36;12} = 0 &\Rightarrow \rho_{36;1} = \rho_{23;1}\rho_{26;1} = -0.0536 \\
&\Rightarrow \rho_{36} = \rho_{36;1}\sqrt{(1-\rho_{13}^2)(1-\rho_{16}^2)} + \rho_{13}\rho_{16} = -0.0889 \\
\rho_{46;123} = -0.5 &\Rightarrow \rho_{46;12} = -0.3977 \Rightarrow \rho_{46;1} = -0.3645 \Rightarrow \rho_{46} = -0.1987.
\end{aligned}
$$

We obtain that the matrix

$$
\begin{bmatrix}
1 & 0.8 & 0.1 & -0.3 & 0.5 & -0.4 \\
 & 1 & 0 & -0.24 & 0.4 & -0.1 \\
 & & 1 & -0.6 & 0.05 & -0.0889 \\
 & & & 1 & -0.7 & -0.1987 \\
 & & & & 1 & 0.2 \\
 & & & & & 1
\end{bmatrix}
$$

is positive definite.

General solution strategy

The following matrix is given (assumed to be symmetric)

$$
\begin{bmatrix}
1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1n-1} & \rho_{1n} \\
\rho_{21} & 1 & \rho_{23} & \cdots & \rho_{2n-1} & \rho_{2n} \\
\rho_{31} & \rho_{32} & 1 & \cdots & \rho_{3n-1} & \rho_{3n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\rho_{n-1,1} & \rho_{n-1,2} & \rho_{n-1,3} & \cdots & 1 & \rho_{n-1n} \\
\rho_{n1} & \rho_{n2} & \rho_{n3} & \cdots & \rho_{n,n-1} & 1
\end{bmatrix}.
$$

Some of the entries ρ_{ij} are not specified. First we must order the rows and columns such that we obtain maximal bottom right triangle in this matrix completely unspecified. The advantage of this procedure is to reduce our completion problem to those considered in Cases 1,2,3. Note that it is not always possible (see Case 5, 6). Next we calculate all partial correlations which can be computed. Let ρ_{1j}, $j \in C_1$ where $C_1 = \{j \in \{2, 3, \ldots, n\} | \rho_{1j}$ unspecified $\}$. To find these unspecified correlations we must solve a system such that all correlations of the first order $\rho_{kj;1}$ where $k \neq j$, $k \in \{2, 3, \ldots, n\}$ and ρ_{kj} specified, are in the interval (-1,1). If it is possible we choose solutions of this system with correlation's values zero if not we calculate them using Lemma 4.6 or/ and Lemma 4.7. If this system has solutions then we can fill all empty cells in the first row and the values of $\rho_{kj;1}$ are known. Next we repeat the same operation for the unspecified elements in the second row $\rho_{2j}, j \in C_2$. We must solve a system such that all correlations of the second order $\rho_{kj;12}$ where $k \neq j$ and $k \in \{3, \ldots, n\}$ such that ρ_{kj} is specified, are in the interval (-1,1). If this system has solutions we obtain the following partial correlations of the first order $\rho_{2j;1}$. Recalculating correlations using algorithm (4.10) we fill all empty cells in the second row and the values of $\rho_{kj;12}$ will be known etc.

4.6 Conclusions

We have explained the use of partial correlation specifications on a canonical vine in various problems regarding positive definiteness of the proto correlation matrices. When applicable, these algorithms possess a clear probablistic interpretations. We note, however, that they cannot apply to problems involving positive semidefiniteness. Indeed, the denominators in (4.4) must be non-zero and this implies that all partial correlations must be greater then -1 and less then 1. The speed of these algorithms appears to be comparable to that of previous algorithms.

Chapter 5

Conditional and partial correlations for graphical uncertainty models

Dorota Kurowicka, Roger Cooke

Abstract: We study the relationship between partial correlation and constant conditional correlation with particular attention to copulae used in high dimensional graphical models. Sufficient and, in some cases, necessary conditions for equality are obtained. Numerical results show that the difference between partial and conditional correlation is small for the minimum information copula with given product moment correlation. When approximate equality holds, regular vines enable us to specify a correlation structure without algebraic constraints (e.g. positive definiteness) and to translate this structure into an on-the-fly sampling algorithm.

Keywords: Partial correlation, conditional correlation, conditional independence, Markov tree, copula, entropy, information, reliability model.

5.1 Introduction

Mathematical models such as reliability diagrams, fault trees, accelerated life testing models, etc, rely on parameters whose values cannot always be perfectly measured. Nowadays, even elementary texts in risk and reliability prescribe uncertainty analysis for such models and present elementary methods (see eg Andrews and Moss [1]). Elementary methods inevitably assume that the uncertainties over different parameters are independent. This is often unrealistic. Methods for uncertainty analysis with dependence are currently an active research topic. This article develops tools for representing dependence in high dimensional distributions, such as those arising in uncertainty analysis of large fault trees.

The Markov tree method for specifying dependence in high dimensional distributions permits on the fly sampling and has attractive theoretical features (see Section 5.3). However, it is limited by the fact that only a 'treefull' of constraints can be specified. Another popular approach (Iman and Conover [21]) abandons on the fly sampling. A large sample matrix is held in memory and transformed to realize a given (rank) correlation matrix. For large problems, many cells of the correlation matrix will typically be unspecified, and this approach encounters the so called matrix completion problem (Laurent [38]): can a partially specified matrix be extended to a positive (semi) definite matrix? If an extension is possible, which extensions should be used? Furthermore, for large problems, holding a sample matrix in memory imposes unwelcome tradeoffs between speed and accuracy. Vines promise to combine the advantages of both approaches while avoiding the matrix completion problem. The key element is this: when conditional rank correlation is held constant, the partial correlation and mean conditional product moment correlation are approximately equal.

We first discuss partial and conditional correlation, and the graphical models in which these are used. We then study conditions under which these two correlations are identical. After introducing the Fréchet, the diagonal band and the minimum information copulae, we present numerical results.

5.2 Partial and conditional correlations

For variables X_1 and X_2 with zero mean and standard deviations σ_1 and σ_2, let b_{12} be the number which minimizes

$$E(X_1 - b_{12}X_2)^2.$$

The product moment or Pearson correlation $\rho(X_1, X_2)$ between X_1 and X_2 is defined as

$$\rho(X_1, X_2) = \mathrm{sgn}(b_{12})(b_{12}b_{21})^{\frac{1}{2}}.$$

It is easy to show that $\rho(X_1, X_2) = \frac{\mathrm{Cov}(X_1,X_2)}{\sigma_1\sigma_2}$. Consider variables X_i with zero mean and standard deviations $\sigma_i, i = 1, \ldots, n$.

Let the numbers $b_{12;3,\ldots,n}, \ldots, b_{1n;3,\ldots,n-1}$ minimize

$$E(X_1 - b_{12;3,\ldots,n}X_2 - \ldots - b_{1n;3,\ldots,n-1}X_n)^2;$$

then the partial correlations are defined as (Yule and Kendall [54]):

$$\rho_{12;3,\ldots,n} \quad = \quad \mathrm{sgn}(b_{12;3,\ldots,n})\,(b_{12;3,\ldots,n}b_{21;3,\ldots,n})^{\frac{1}{2}}\text{, etc.}$$

Partial correlations can be computed from correlations with the following recursive formula:

$$\rho_{12;3,\ldots,n} \quad = \quad \frac{\rho_{12;3,\ldots,n-1} - \rho_{1n;3,\ldots,n-1} \cdot \rho_{2n;3,\ldots,n-1}}{\sqrt{1-\rho^2_{1n;3,\ldots,n-1}}\sqrt{1-\rho^2_{2n;3,\ldots,n-1}}}. \tag{5.1}$$

The conditional correlation of Z and Y given X;

$$\rho_{YZ|X} \quad = \quad \rho(Y|X, Z|X)$$

is the product moment correlation computed with the conditional distribution given X. In general this depends on the value of X, but it may be constant. Letting F_X, F_Y denote the cumulative distribution functions of X and Y; the rank correlation between X and Y is:

$$r(X, Y) \quad = \quad \rho(F_X(X), F_Y(Y)).$$

For the joint normal distribution, partial and conditional correlations coincide.

We define the mean absolute difference between partial and conditional correlation or conditional rank correlation as

$$\begin{aligned}\Delta(YZ|X) \quad &= \quad E(|\rho_{YZ;X} - \rho_{YZ|X}|),\\ \Delta_r(YZ|X) \quad &= \quad E(|\rho_{YZ;X} - r_{YZ|X}|).\end{aligned}$$

If Y and Z are independent conditional on X, then of course $r_{YZ|X} = \rho_{YZ|X} = 0$ and we write

$$\Delta(YZ|X) = \Delta.$$

We shall see in Section 5.4 that Δ may be quite large, though a sharp upper bound is not known at present.

5.3 Trees, Vines and Copulae

Trees and vines are graphical modelling tools for specifying dependence structures in high dimensional distributions. We restrict attention to variables with a uniform distribution on $[0,1]$ and present the main concepts informally. A tree on N variables specifies at most $N-1$ edges between the variables. Each edge may be associated with a copula, that is a distribution on $[0,1]^2$ with uniform marginals. Popular copulae in this context are the diagonal band (Cooke and Waij [7]) and the minimum information copulae (Meeuwissen and Bedford [42]); these copulae are continuous and can realize any correlation value in $[-1,1]$ (for the other copulae see Dall'Aglio, Kotz and Salinetti [14] and Nelsen [46]). Given any tree on N variables with copulae on the edges, a joint distribution can always be constructed satisfying the tree-copulae specification. Moreover, it can be shown (Cooke [6]) that there is a unique minimum information joint distribution satisfying the tree-copulae specification and under this distribution the tree becomes a Markov tree. Distributions specified in this way can be sampled on the fly. The tree-copulae method of specifying a joint distribution is limited by the fact that there can be at most $N-1$ edges on the tree.

A vine on N variables is a nested set of trees, where the edges of tree j are the nodes of tree $j+1$; $j=1,\ldots,N-2$, and each tree has the maximum number of edges. A regular vine on N variables is a vine in which two edges in tree j are joined by an edge in tree $j+1$ only if these edges share a common node, $j=1,\ldots,N-2$. There are $(N-1)+(N-2)+\ldots+1=\frac{N(N-1)}{2}$ edges in a regular vine on N variables. Each edge in a regular vine may be associated with a constant conditional rank[1] correlation (for $j=1$ the conditions are vacuous) and, using the diagonal band or minimum information copulae, a unique joint distribution satisfying the vine-copulae specification with minimum information can be constructed and sampled on the fly (Cooke [6]). Moreover, the (constant conditional) rank correlations may be chosen arbitrarily in the interval $[-1,1]$. Figure 4.1 shows a regular vine on 5 variables. The four nested trees are distinguished by the line style of the edges; tree 1 has solid lines, tree 2 has dashed lines, etc.The conditional rank correlations associated with each edge are determined as follows: the variables reachable from a given edge are called the constraint set of that edge. When two edges are joined by an edge of the next tree, the intersection of the respective constraint sets are the conditioning variables, and the symmetric difference of the constraint sets are the conditioned variables. The regularity condition insures that the symmetric difference of the constraint sets always contains two variables. Note that each pair of variables occurs once as conditioned variables.

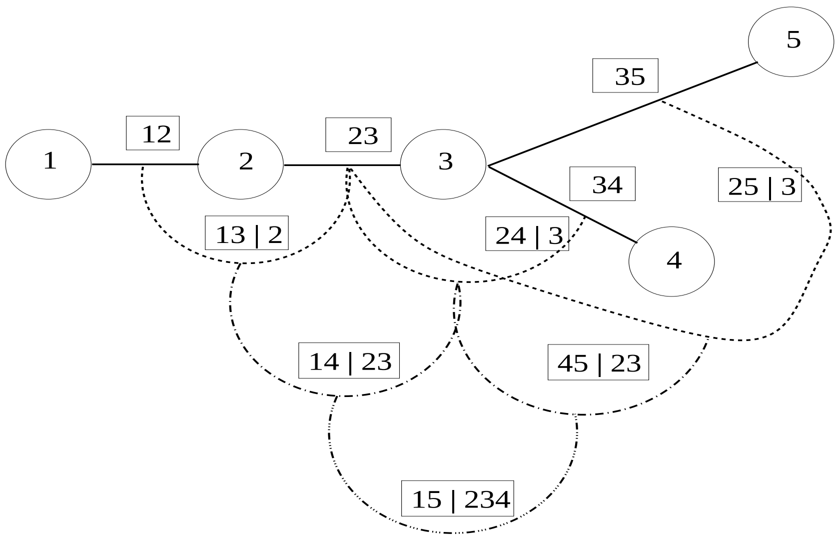


Figure 5.1. A regular vine on 5 variables

The edges of a regular vine may also be associated with partial correlations, with values chosen arbitrarily in the interval $(-1,1)$. Using the recursive formulae (5.1) it can be shown that each such partial correlation regular vine uniquely determines the correlation matrix, and every full rank correlation matrix can be obtained in this way (Bedford and Cooke [3]). In other words, a regular vine provides a bijective mapping from $(-1,1)^{N(N-1)/2}$ into the set of positive definite matrices with 1's on the diagonal. One verifies that ρ_{ij} can be computed from the sub-vine generated by the constraint set of the edge whose conditioned set is $\{i,j\}$ using recursive the formulae (5.1).We can determine numerically the mean conditional product moment correlation for a given constant conditional rank correlation. If this mean product moment correlation were (approximately) equal to the partial correlations, then the recursive formulae (5.1) could be applied to (approximately) compute the entire correlation matrix of the joint distribution constructed from the regular vine-copula specification. Alternatively, an arbitrary correlation matrix could be used to compute the partial correlations on a regular vine, and these in turn used to determine

[1] Conditional rank correlations are implemented in the sampling algorithms; however, as we know the conditional copula distributions and the relation between rank and mean product moment correlations for these distributions, we could just as well associate mean conditional product moment correlations.

the constant conditional rank correlations, and to (approximately) sample the distribution on the fly. The degree to which partial correlations and mean conditional product moment correlations agree is a property of the copulae used, and the correlation values themselves.

5.4 Conditions for $\Delta = 0$

The following example shows that Δ may be large.

Proposition 5.1 *If*

(a) *X is distributed uniformly on an interval $[0,1]$,*

(b) *Y, Z are independent given X,*

(c) *$Y|X$ and $Z|X$ are distributed uniformly on $[0, X^k], k > 0$,*

then

$$\Delta = \frac{3k^2(k-1)^2}{4(k^4+4k^2+3k+1)}, \tag{5.2}$$

as $k \to \infty$ this converge to $\frac{3}{4}$.

Proof. We get

$E(Y) = E(Z) = E(E(Y|X)) = E(\frac{X^k}{2}) = \frac{1}{2(k+1)}$,
$E(Y^2) = E(Z^2) = E(E^2(Y|X)) = E(\frac{X^{2k}}{3}) = \frac{1}{3(2k+1)}$,
$\mathrm{Var}(Y) = \mathrm{Var}(Z) = \frac{1}{3(2k+1)} - (\frac{1}{2(k+1)})^2$,
$E(XY) = E(XZ) = E(E(XY|X)) = E(X(E(Y|X)) = E(\frac{X^{k+1}}{2}) = \frac{1}{2(k+2)}$,
$\mathrm{Cov}(X,Y) = \mathrm{Cov}(X,Z) = E(XY) - E(X)E(Y) = \frac{1}{2(k+2)} - \frac{1}{2}\frac{1}{2(k+1)}$,
$E(YZ) = E(E(YZ|X)) = E(E(Y|X)E(Z|X)) = E(\frac{X^{2k}}{4}) = \frac{1}{4(2k+1)}$,
$\mathrm{Cov}(Y,Z) = E(YZ) - E(Y)(E(Z) = \frac{1}{4(2k+1)} - \frac{1}{4(k+1)^2}$.

From the above calculations we obtain

$$\rho_{YZ} = \frac{\mathrm{Cov}(Y,Z)}{\sigma_Y \sigma_Z} = \frac{3k^2}{4k^2+2k+1}$$

and

$$\rho_{XY}\rho_{XZ} = \frac{\mathrm{Cov}^2(X,Y)}{\mathrm{Var}X\mathrm{Var}Y} = \frac{9k^2(2k+1)}{(k+1)^2(4k^2+2k+1)},$$

so that

$$\Delta = \frac{\rho_{YZ} - \rho_{XY}\rho_{XZ}}{\sqrt{1-\rho_{XY}^2}\sqrt{1-\rho_{XZ}^2}} \quad = \quad \frac{3k^2(k-1)^2}{4(k^4+2k^2+k+1)} \to \frac{3}{4} \text{ as } k \to \infty. \ \square$$

Acknowledgement
The case $k = 2$ in the Proposition 5.1 was proposed by P. Groeneboom.

$Y\|X$ and $Z\|X$	Δ
$[0, X]$	0.0000
$[0, X^2]$	0.0769
$[0, X^3]$	0.2126
$[0, X^4]$	0.3243
$[0, X^5]$	0.4049
$[0, X^{10}]$	0.5824
$[0, X^{100}]$	0.7348
$[0, X^{1000000}]$	0.7500

Table 5.1: Numerical results for Proposition 5.1.

Table 6.1 shows some numerical results. We note that unconditional distributions of Y and Z are not uniform.

Theorem 5.1 *Let*

(a) X, Y, Z *have mean 0,*

(b) Y *and* Z *be independent given* X*,*

(c) $E(Y|X) = AX,\ E(Z|X) = BX,$

then

$$\Delta = 0. \tag{5.3}$$

Proof. Since $\Delta = 0$ is equivalent to

$$\frac{\text{Cov}(Y,Z)}{\sigma_Y \sigma_Z} \quad = \quad \frac{\text{Cov}(Y,X)}{\sigma_Y \sigma_X}\frac{\text{Cov}(Z,X)}{\sigma_Z \sigma_X}$$

it suffices to show that

$$\sigma_X^2 \text{Cov}(Y,Z) \quad = \quad \text{Cov}(Y,X)\text{Cov}(Z,X) \tag{5.4}$$

We get

$$E(ZY) = E(E(YZ|X)) = E(E(Y|X)(E(Z|X)) = AB\sigma_X^2,$$
$$E(XY) = E(XE(Y|X)) = A\sigma_X^2,$$
$$E(XZ) = B\sigma_X^2.$$

Hence $\Delta = 0$ holds. □

Theorem 5.2 *Suppose* $\Delta = 0$ *and*

(a) X, Y, Z *have mean* 0*,*

(b) Y *and* Z *are independent given* X*,*

(c) $E(Y|X) = bE(Z|X)$*,*

then

$$E(Y|X) = AX.$$

Proof. If $\Delta = 0$ then (5.4) is satisfied as well. From (b) and (c) we get

$$\sigma_X^2 E([E(Y|X)]^2) \quad = \quad [E(XE(Y|X))]^2.$$

Applying Cauchy-Schwarz inequality in the case of equality we obtain

$$E(Y|X) \quad = \quad AX. \ \square$$

5.5 Copulae

Definition 5.1 *A copula* C *is a distribution on the unit square with uniform marginals.*

Definition 5.2 *Random variables* X *and* Y *are joined by copula* C *if and only if their joint distribution can be written*

$$F_{XY}(x, y) = C(F_X^{-1}(x), F_Y^{-1}(x)).$$

In the following we transform the unit square to $[-\frac{1}{2}, \frac{1}{2}]^2$ to simplify the calculations.

The simplest copulae are the Fréchet copula, where all mass is spread uniformly on the main diagonal. We get

$$\begin{aligned} M(s,t) &= \min(s,t), \\ W(s,t) &= \max(s+t-\frac{1}{2}, -\frac{1}{2}). \end{aligned}$$

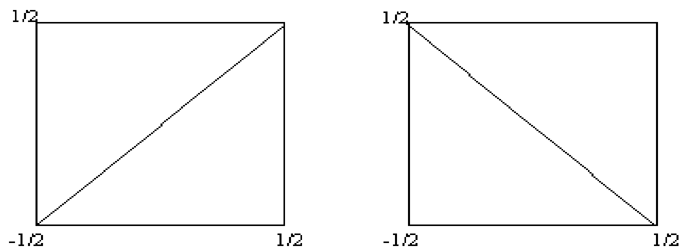


Figure 5.2. Illustration of the diagonals where mass of M (left) and W (right) is concentrated.

Let $\Phi(A)$ denote the class of mixtures of the Fréchet copulae with parameter $A \in [0,1]$ on the unit square $[-\frac{1}{2}, \frac{1}{2}]^2$, then whole mass is concentrated on the diagonal and antidiagonal which depends on parameter A. We get

$$C_A(s,t) = AM(s,t) + (1-A)W(s,t).$$

It is easy to see that this mixture of the Fréchet copulas has linear regression.

For the variables X, Y, Z joined by the mixtures of the Fréchet copulas the assumptions of Theorem 5.2 are fulfilled so in this case $\Delta = 0$.

The *diagonal band* copula with the density on the unit square $[-\frac{1}{2}, \frac{1}{2}]^2$ is given below. For the positive correlation the mass is concentrated on the diagonal band with vertical bandwidth $\beta = 1 - \alpha$. Mass is distributed uniformly on the inscribed rectangle and is uniform but is "twice as thick" in the triangular corners. We can easily verify that the mass on the rectangle is equal to $\frac{1}{2\beta}$ and on the triangles $\frac{1}{\beta}$. For negative correlation the band is drawn between the other corners.

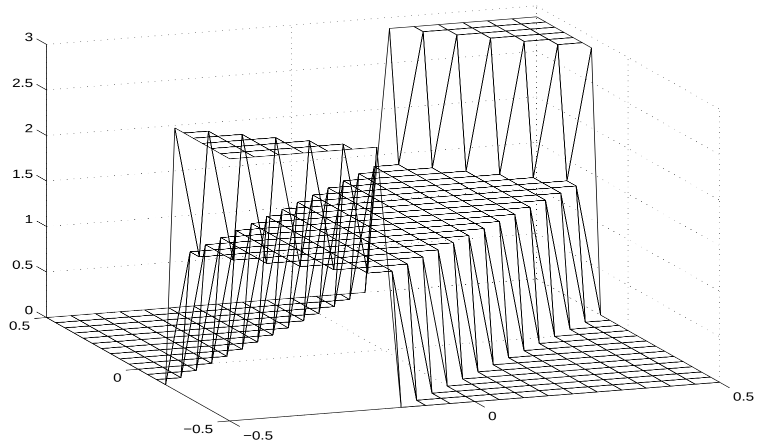


Figure 5.3. The diagonal band distribution with correlation 0.8.

The regression for the diagonal band distribution are given by the following formulae:

$0 \leq \alpha \leq 0.5$

$$E(Y|X,\alpha) = \begin{cases} \frac{1}{2(1-\alpha)}[X^2 + X + (\alpha - 0.5)^2], & -0.5 \leq X \leq -0.5+\alpha \\ \frac{\alpha}{1-\alpha}X, & -0.5+\alpha \leq X \leq 0.5-\alpha \\ -\frac{1}{2(1-\alpha)}[X^2 - X + (\alpha - 0.5)^2], & 0.5-\alpha \leq X \leq 0.5 \end{cases}$$

$0.5 \leq \alpha \leq 1$

$$E(Y|X,\alpha) = \begin{cases} \frac{1}{2(1-\alpha)}[X^2 + X + (\alpha - 0.5)^2], & -0.5 \leq X \leq 0.5-\alpha \\ X, & 0.5-\alpha \leq X \leq -0.5+\alpha \\ -\frac{1}{2(1-\alpha)}[X^2 - X + (\alpha - 0.5)^2], & -0.5+\alpha \leq X \leq 0.5 \end{cases}$$

For negative α we can find the regression as follows

$$E(Y|X=x,\alpha) = E(Y|X=-x,-\alpha).$$

The correlation coefficient can be calculated from the formula (Cooke and Waij [7])

$$\rho = \operatorname{sgn}(\alpha)((1-|\alpha|)^3 - 2(1-|\alpha|)^2 + 1).$$

The *minimum information* copula is the distribution with the joint density function $g(x,y)$ with minimal relative information with respect to the uniform density given uniform marginals and a given correlation. The density $g(x, y)$ has functional form (Meeuwissen and Bedford [42])

$$g(x, y) = \kappa(x)\kappa(y)e^{\theta xy}$$

for (x, y) in unit square $[-\frac{1}{2}, \frac{1}{2}]^2$. Function $\kappa(x)$ is even around $x = 0$.

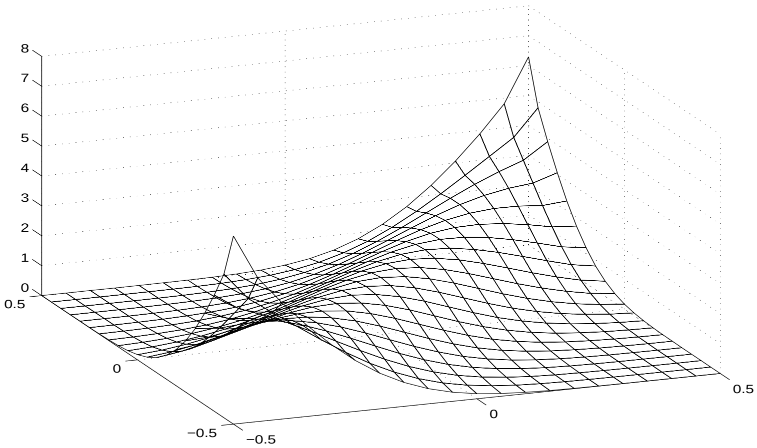


Figure 5.4. The minimum information distribution with correlation 0.8.

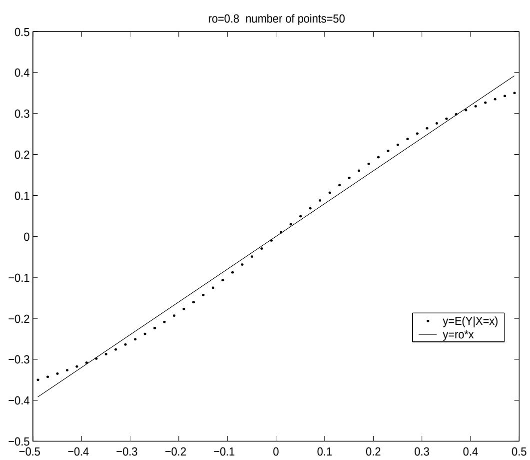


Figure 5.5. The conditional expectation for minimum information distribution with correlation 0.8.

5.6 Computing Δ

Let us consider variables X, Y uniform on $[-\frac{1}{2}, \frac{1}{2}]$. We may write

$$E(Y|X) \quad = \quad kX + \epsilon(X).$$

We want to find the coefficient k which minimizes the square error given by

$$E\left((E(Y|X) - kX)^2\right).$$

Setting the derivative with respect to k equal to 0:

$$\frac{d}{dx} E\left((E(Y|X) - kX)^2\right) \quad = \quad 0$$

implies

$$E\left(X(E(Y|X) - kX)\right) \quad = \quad 0,$$

which is equivalente to

$$E(XE(Y|X)) \quad = \quad E(kX^2).$$

Hence

$$\mathrm{Cov}(X, Y) \quad = \quad k\sigma_X^2$$

and finally since $\sigma_X = \sigma_Y$

$$k = \frac{\mathrm{Cov}(X, Y)}{\sigma_X^2} = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y} = \rho.$$

Thus, the best approximation of the regression is the line with coefficient equal to ρ.

The mean square difference with linear regression is equal to the variance of the conditional expectation minus a correction term $\rho^2\sigma_X^2$.

Proposition 5.2

$$E\left((E(Y|X) - \rho X)^2\right) = \mathit{Var}(E(Y|X)) - \rho^2\sigma_X^2.$$

Proof.

$$\begin{aligned} E\left((E(Y|X) - \rho X)^2\right) &= E\left(E(Y|X)^2\right) - 2\rho\mathrm{Cov}(X, Y) + \rho^2\sigma_X^2 \\ &= E\left(E(Y|X)^2\right) - \rho^2\sigma_X^2 \\ &= \mathrm{Var}(E(Y|X)) - \rho^2\sigma_X^2. \quad \Box \end{aligned}$$

Remark 5.1

$$CR = \frac{Var(E(Y|X))}{\sigma_Y^2} \text{ is called correlation ratio.}$$

Theorem 5.3 *Let us consider variables X, Y, Z uniform on $[-\frac{1}{2}, \frac{1}{2}]$ and such that Y and Z are independent given X then*

$$\Delta = \frac{\rho_{YZ} - \rho_{XY}\rho_{XZ}}{\sqrt{1-\rho_{XY}^2}\sqrt{1-\rho_{XZ}^2}} = \frac{\frac{E\left(\epsilon_Y(X)\epsilon_Z(X)\right)}{\sigma_Y\sigma_Z}}{\sqrt{1-\rho_{XY}^2}\sqrt{1-\rho_{XZ}^2}},$$

where

$$\begin{aligned} \epsilon_Y(X) &= E(Y|X) - \rho_{XY}X \\ \epsilon_Z(X) &= E(Z|X) - \rho_{ZY}X. \end{aligned}$$

Proof. We get

$$\begin{aligned} &E(YZ) = E(E(YZ|X)) = E(E(Y|X)E(Z|X)) = \\ &\quad \rho_{XY}\rho_{XZ}\sigma_X^2 + \rho_{XY}E\left(X\epsilon_Z(X)\right) + \rho_{XZ}E\left(X\epsilon_Y(X)\right) + E\left(\epsilon_Y(X)\epsilon_Z(X)\right). \end{aligned}$$

Since

$$\begin{aligned} E\left(X\epsilon_Z(X)\right) &= E(X\epsilon_Y(X)) \\ &= E(XE(Y|X) - \rho_{XY}X^2) \\ &= \mathrm{Cov}(X,Y) - \rho_{XY}\sigma_X^2 = \mathrm{Cov}(X,Y) - \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y}\sigma_X^2 = 0 \end{aligned}$$

then we get

$$E(YZ) = \rho_{XY}\rho_{XZ}\sigma_X^2 + E(\epsilon_Y(X)\epsilon_Z(X)).$$

We can easily calculate that

$$E(XY) = \rho_{XY}\sigma_X^2 \text{ and } E(XZ) = \rho_{XZ}\sigma_X^2.$$

Hence we get

$$\begin{aligned} |\rho_{YZ} - \rho_{XY}\rho_{XZ}| &= \left|\frac{E(YZ)}{\sigma_Y\sigma_Z} - \frac{E(XY)E(XZ)}{\sigma_Y\sigma_Z\sigma_X^2}\right| \\ &= \frac{1}{\sigma_Y\sigma_Z}\left|E(\epsilon_Y(X)\epsilon_Z(X))\right| \end{aligned}$$

which concludes the proof. □

Remark 5.2 *In the case when $E(Y|X) = E(Z|X)$ we get*

$$\Delta = \frac{|\rho_{YZ} - \rho^2|}{1-\rho^2} \quad = \quad \frac{Var(E(Y|X))}{\sigma_X^2(1-\rho^2)} - \frac{\rho^2}{1-\rho^2} = \frac{CR-\rho^2}{1-\rho^2} \leq 1. \qquad (5.5)$$

Next we examine the difference between conditional and partial correlation when we assume that conditional correlation is constant.

Theorem 5.4 *Let*

(a) *X, Y, Z be uniform on $[-\frac{1}{2}, \frac{1}{2}]$,*

(b) *$E(Y|X) = E(Z|X) = AX$,*

(c) *$\sigma_{Y|X} = \sigma_{Z|X}$,*

(d) *$\rho_{YZ|X} = r$,*

then

$$\rho_{YZ;X} = r.$$

Proof. It is easy to see that

$$\rho_{XZ} = \rho_{XY} = A.$$

Since

$$r = \rho_{YZ|X} = \frac{\mathrm{Cov}(Y|X, Z|X)}{\sigma_{Y|X}\sigma_{Z|X}} \quad = \quad \frac{E(YZ|X) - A^2X^2}{\mathrm{Var}(Y|X)}$$

then

$$E(YZ|X) \quad = \quad r\mathrm{Var}(Y|X) + A^2X^2.$$

From the above we get

$$\rho_{YZ} = \frac{E(E(YZ|X))}{\sigma_X^2} \quad = \quad \frac{E(r\mathrm{Var}(Y|X) + A^2X^2)}{\sigma_X^2}.$$

Since

$$E(\mathrm{Var}(Y|X)) = \mathrm{Var}(Y) - \mathrm{Var}(E(Y|X)) \quad = \quad \sigma_X^2 - A^2\sigma_X^2 = \sigma_X^2(1-A^2)$$

then

$$\rho_{YZ} = \frac{r\sigma_X^2(1-A^2) + A^2\sigma_X^2}{\sigma_X^2} = r(1-A^2) + A^2.$$

Finally we can calculate the partial correlation Y, Z given X

$$\rho_{YZ;X} = \frac{\rho_{YZ} - \rho_{XY}\rho_{XZ}}{\sqrt{1-\rho_{XY}^2}\sqrt{1-\rho_{XZ}^2}} = \frac{r(1-A^2) + A^2 - A^2}{1-A^2} = r. \quad \square$$

Theorem 5.5 *Let*

(a) X, Y, Z *be uniform on* $[-\frac{1}{2}, \frac{1}{2}]$,

(b) Y, X *and* Z, X *be joined by the mixture of the Fréchet copulae with parameters* B_Y, B_Z *respectively,*

(c) $\rho_{YZ|X} = r$,

then

$$\rho_{YZ;X} = r.$$

Proof. We get

$$\begin{aligned} E(Y|X) &= X(2A_Y - 1), \\ E(Z|X) &= X(2A_Z - 1), \\ E((Y|X)^2) &= X^2(1-A_Y) + X^2A_Y = X^2, \end{aligned}$$

Let $2B_Y - 1 = A_Y$ and $2B_Z - 1 = A_Z$ then

$$\mathrm{Var}(Y|X) = X^2 - X^2A_Y^2 = X^2(1-A_Y^2).$$

From the above we obtain

$$\sigma_{Y|X} = X\sqrt{1-A_Y^2} \text{ and } \sigma_{Z|X} = X\sqrt{1-A_Z^2}.$$

We also get

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sigma_Y\sigma_X} = \frac{E(xE(Y|X))}{\sigma_Y\sigma_X} = \frac{A_Y\sigma_X^2}{\sigma_X^2} = A_Y.$$

Analogously we obtain

$$\rho_{XZ} = A_Z.$$

From the above calculations and by (c)

$$r = \rho_{YZ|X} = \frac{\mathrm{Cov}(Y|X, Z|X)}{\sigma_{Y|X}\sigma_{Z|X}} = \frac{E(YZ|X) - A_Y A_Z X^2}{X^2\sqrt{1-A_Y^2}\sqrt{1-A_Z^2}}.$$

Hence we calculate that

$$\begin{aligned} E(YZ|X) &= r\sqrt{1-A_Y^2}\sqrt{1-A_Z^2}X^2 + A_Y A_Z X^2 \\ &= X^2(r\sqrt{1-A_Y^2}\sqrt{1-A_Z^2} + A_Y A_Z) \end{aligned}$$

and find

$$\begin{aligned} \rho_{YZ} &= \frac{E(E(YZ|X))}{\sigma_Y \sigma_Z} \\ &= \frac{E(X^2(r\sqrt{1-A_Y^2}\sqrt{1-A_Z^2} + A_Y A_Z))}{\sigma_X^2} \\ &= \frac{\sigma_X^2(r\sqrt{1-A_Y^2}\sqrt{1-A_Z^2} + A_Y A_Z)}{\sigma_X^2} \\ &= r\sqrt{1-A_Y^2}\sqrt{1-A_Z^2} + A_Y A_Z. \end{aligned}$$

Hence the partial correlation Y, Z given X is as follows

$$\rho_{YZ;X} = \frac{\rho_{YZ} - \rho_{XY}\rho_{XZ}}{\sqrt{1-\rho_{XY}^2}\sqrt{1-\rho_{XZ}^2}} = \frac{r\sqrt{1-A_Y^2}\sqrt{1-A_Z^2} + A_Y A_Z - A_Y A_Z}{\sqrt{1-A_Y^2}\sqrt{1-A_Z^2}} = r$$

which concludes the proof. □

5.7 Numerical results

We calculate Δ for several values of $\rho = \rho_{XY} = \rho_{XZ}$ using (5.5). The results are prepared in Matlab 5.3 and presented in Table 5.2. A discrete version of the minimum information distribution was obtained in Matlab 5.3 as a solution of the optimization problem. Table 5.2 contains the results for discrete minimum information distribution where the unit interval was divided uniformly into 50 equal segments. If we take a better approximation of this distribution we find that Δ becomes smaller.

The above results show that in the case of conditional independence for diagonal band and minimum information distributions the correlation between conditionally independent variables is almost equal to the product of correlation between them and the variable on which we conditionalize. In all cases Δ is lower for the minimum information copula.

ρ	Δ	
	DIAGONAL BAND	MINIMUM INFORMATION
0.1	7.08594e-6	2.0455e-6
0.2	1.54056e-4	1.0002e-5
0.3	8.64019e-4	4.0192e-5
0.4	2.82514e-3	1.6511e-4
0.5	6.84896e-3	6.1690e-4
0.6	1.34098e-2	2.0390e-3
0.7	2.16325e-2	5.8727e-3
0.8	2.92942e-2	1.3867e-2
0.9	3.27922e-2	2.2426e-2

Table 5.2: The comparison of Δ for diagonal band and minimum information distributions for $\rho = \rho_{XY} = \rho_{XZ}$.

Suppose in Table 5.3 we fix ρ_{XY}, ρ_{XZ} and the conditional rank correlation and sample using the minimum information copula. Table 5.3 compares partial correlation, and the mean conditional product moment correlation for some illustrative cases. We see that the difference between them can be in order of 4%, and thus is larger than in Table 5.2 where Y and Z are conditionally independent given X.

Stipulated			Computed				
ρ_{XY}	ρ_{XZ}	$r_{YZ\|X}$	ρ_{YZ}	$\rho_{YZ;X}$	$E\rho_{YZ\|X}$	Δ_r	Δ
0.1	0.7	0.0	0.0702	0.0002	-1.347e-14	2.5e-4	2.5e-4
0.9	0.9	0.9	0.9795	0.8921	0.9004	0.0079	0.0083
-0.9	0.9	-0.9	-0.9739	-0.8626	-0.8469	0.0374	0.0157
-0.9	0.9	0.9	-0.6631	0.7729	0.8098	0.1271	0.0369
-0.5	0.4	0.7	0.3267	0.6635	0.6525	0.0365	0.0110
0.3	0.9	-0.2	0.1906	-0.1909	-0.1896	0.0091	0.0013
-0.1	-0.3	-0.8	-0.7173	-0.7873	-0.7719	0.0127	0.0154
0.8	0.8	0.8	0.9143	0.7619	0.7540	0.0381	0.0079

Table 5.3: The results of the simulations for minimum information distribution.

In Table 5.2 the conditional rank correlation and the conditional product moment correlation are equal and equal to zero. In Table 5.3 the conditional rank correlation is constant, and not equal to the (non constant) conditional product moment correlation. We believe that improved numerical routines will give better approximations as the stipulated correlations become more extreme.

5.8 Conclusions

We have seen that mean conditional product moment correlation under constant conditional rank correlation with the minimum information copula provides a good approximation to the partial correlation, particularly if the stipulated correlation values are less than 0.9 in absolute value. As explained in Section 5.3, this means that we can (approximately) specify a correlation structure by giving the partial correlation values on a regular vine. The advantage of this is that these values are algebraically independent; they need satisfy no condition like positive definiteness, and the matrix completion problem does not arise. Alternatively, we can start with an arbitrary correlation matrix, and compute the partial correlations on a regular vine. Setting these equal to mean conditional product moment correlations under constant conditional rank correlations, we can retrieve the conditional rank correlations and thus, combined with the minimum information copula, determine a sampling routine. This sampling routine works on the fly: We draw one sample vector at a time, we need not retain large numbers of sample vectors in memory.

Chapter 6

Elliptical copulae

Dorota Kurowicka, Jolanta Misiewicz, Roger Cooke

Abstract: In this chapter we construct a copula, that is, a distribution with uniform marginals. This copula is continuous and can realize any correlation value in $(-1, 1)$. It has linear regression and has the properties that partial correlation is equal to constant conditional correlation . This later property is important in Monte Carlo simulations. The new copula can be used in graphical models specifying dependence in high dimensional distributions such as Markov trees and vines.

Keywords: correlation, conditional correlation, conditional independence, partial correlations, tree dependence, copulae, vines

6.1 Introduction

In modelling high dimensional distributions the problems encountered include

(a) Determining whether a partially specified matrix can be extended to a correlation matrix;

(b) Finding a convenient way of representing correlation matrices;

(c) Choosing an unique joint distribution to realize a correlation matrix.

(a) is so called matrix completion which is receiving attention at the moment (Laurent [38]). To tackle these problems the graphical models called vines were introduced by (Cooke [6]). A vine is a set of trees such that the edges of the tree T_i are nodes of the tree T_{i+1} and all trees have the maximum number of edges. A vine is regular if two edges of T_i are joined by an edge of T_{i+1} only if these edges share a common node in T_i. Partial correlations, defined in (Yule and Kendall [54]), can be assigned to the edges of the regular vine such that conditioning and conditioned sets of the vine and partial correlations are equal (for the details we refer readers to Bedford and Cooke [3]). There are $\binom{n}{2}$ edges in the regular vine and there is a bijection from $(-1,1)^{\binom{n}{2}}$ to the set of full rank correlation matrices ([3]). Using regular vines with partial correlations we thus determine the entire correlation matrix in convenient way (b). Using regular vines with conditional correlations we can determine a convenient sampling routines (c). In general, however, partial and conditional correlations are not equal. For popular copulas such as the diagonal band (Cooke and Waij [7]) and the minimum information copulae with given correlation (Meeuwissen and Bedford [42]), when conditional rank correlation is held constant, the partial correlation and mean conditional product moment correlation are approximately equal (Kurowicka and Cooke [35]). This approximation, however, deteriorates as the correlations become more extreme. For the well known Fréchet copulae the partial and constant conditional correlations are equal but these copulae are not very useful from the application point of view ([35]). In (Kurowicka and Cooke [36]) it is shown how regular vines can be applied to the completion problem (a). For other copulae and their properties we refer to (Dall'Agilo, Kotz and Salinetti [14]) and (Nelsen [46]). In this article we present the new copula for which partial and constant conditional correlations are equal. In constructing this new copula the properties of elliptically contoured and rotationally invariant random vectors were used (see Harding [5] and Misiewicz [43]). These copula present a striking companion with copulae previously used in Monte Carlo simulation codes (Unicorn and PREP/SPOP [23])

This chapter is organized as follows. In Section 6.2 the uniform distribution on the sphere and its properties is presented. In Section 6.3 the copula is given. The properties of this function are shown. In Section 6.4 after introducing definitions of the partial and conditional correlations the equality of partial and constant conditional correlations for the new copula is proven. Section 6.5 contains conclusions.

6.2 Uniform distribution on the sphere and its properties

Let $X = (X_1, X_2, X_3)$ have the uniform distribution on the sphere with the radius r, $S_2(r) \subseteq R^3$ where sphere in R^n is defined as

$$S^{n-1}(r) = \{x \in R^n | \sum_{k=1}^{n} x_k^2 = r^2\}.$$

We can see that for every $t \in [-r, r]$

$$\begin{aligned} P(X_1 < t) = P(X_2 < t) = P(X_3 < t) &= \frac{2\pi}{4\pi r^2} \int_{-r}^{t} \sqrt{r^2 - x^2} \sqrt{1 + \left[\frac{d}{dx}\sqrt{r^2 - x^2}\right]^2} dx \\ &= \frac{1}{2} + \frac{t}{2r}, \end{aligned}$$

which means that each of the variables X_1, X_2 and X_3 has a uniform distribution on the interval $[-r, r]$.

Consider now a linear operator $A : R^3 \to R^3$ represented by the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Since the random vector X is rotationally invariant, the random vector

$$W = (W_1, W_2, W_3) = AX^T$$

is elliptically contoured.

[1] A random vector $X = (X_1, X_2, \ldots, X_n)$ is *elliptically contoured* if it is pseudo isotropic with a function $c : R^n \to [0, \infty)$ defined by an inner product on R^n; i.e there exists a symmetric positive definite $n \times n$ matrix Σ such that

$$c(\xi)^2 = <\xi, \Sigma\xi>, \forall \xi \in R^n.$$

If $\Sigma = I$ then the vector X is called *rotationally invariant.*

Harding proved (see [5]) that every elliptically contoured[1] random vector on $R^n, n \geq 2$ has the linear regression property if it has second moment. This means in particular that for every $j \neq k, j, k \in \{1,2,3\}$ there exists a_{jk} such that

$$E(W_j|W_k) \quad = \quad a_{jk}W_k.$$

The numbers a_{jk} can be calculated directly:

$$W_j \quad = \quad a_{j1}X_1 + a_{j2}X_2 + a_{j3}X_3.$$

Since $E(X_k) = 0$, $E(X_jX_k) = 0$ for $k \neq j$ and $\mathrm{Var}(X_k) = \frac{1}{2r}\int_{-r}^{r} x^2 dx = \frac{1}{3}r^2$ thus W_1, W_2, W_3 have expectations 0 and

$$\mathrm{Var}(W_j) \quad = \quad E\left((a_{j1}X_1 + a_{j2}X_2 + a_{j3}X_3)^2\right) = \frac{1}{3}r^2\left(a_{j1}^2 + a_{j2}^2 + a_{j3}^2\right).$$

According to Harding's result the conditional expectation $E(W_j|W_k)$ coincides with the orthogonal projection of vector W_j onto W_k. Then we can calculate for $k \neq j$

$$\begin{aligned} E(W_jW_k) &= E\left((a_{j1}X_1 + a_{j2}X_2 + a_{j3}X_3)(a_{k1}X_1 + a_{k2}X_2 + a_{k3}X_3)\right) \\ &= \frac{1}{3}r^2(a_{j1}a_{k1} + a_{j2}a_{k2} + a_{j3}a_{k3}). \end{aligned}$$

Finally we get

$$E(W_j|W_k) \quad = \quad \frac{E(W_jW_k)}{\mathrm{Var}W_k}W_k = \frac{a_{j1}a_{k1} + a_{j2}a_{k2} + a_{j3}a_{k3}}{a_{k1}^2 + a_{k2}^2 + a_{k3}^2}W_k. \qquad (6.1)$$

Notice now that random variables W_1, W_2, W_3 have uniform distributions, which with appropriate choice of A will be uniform distributions on $[-r, r]$. We use a very helpful property of rotationally invariant random vectors, namely:

if $Y \in R^n$ is rotationally invariant and $a \in R^n$ then the distribution of $a_1Y_1 + \ldots + a_nY_n$ is the same as the distribution of $\|a\|_2Y_1$, where $\|a\|_2$ is the Euclidean norm of the vector a.

Now we can write the following:

$$\begin{aligned} P(W_k < t) &= P(a_{k1}X_1 + a_{k2}X_2 + a_{k3}X_3 < t) \\ &= P\left(X_1 < \frac{t}{\sqrt{a_{k1}^2 + a_{k2}^2 + a_{k3}^2}}\right) = \frac{1}{2} + \frac{t}{2r\sqrt{a_{k1}^2 + a_{k2}^2 + a_{k3}^2}} \end{aligned}$$

for $t \in [-r\sqrt{a_{k1}^2 + a_{k2}^2 + a_{k3}^2}, r\sqrt{a_{k1}^2 + a_{k2}^2 + a_{k3}^2}]$.

It is enough to assume that for all $k = 1, 2, 3$

$$a_{k1}^2 + a_{k2}^2 + a_{k3}^2 \quad = \quad 1 \tag{6.2}$$

to have uniform distribution on $[-r, r]$.

6.3 The elliptical copulae

Taking the projection of the uniform distribution on the sphere $S^2(\frac{1}{2})$ on a plane (X, Y) we can construct copulae.

The area of surface given in functional form $z = g(x, y)$ above area $D \subseteq R^2$ can be calculated as:

$$\int\int_D \sqrt{1 + \left(\frac{d}{dx}g(x,y)\right)^2 + \left(\frac{d}{dy}g(x,y)\right)^2} dxdy. \tag{6.3}$$

Using (6.3) for the sphere with radius $\frac{1}{2}$, hence for function $g(x, y) = 2\sqrt{\frac{1}{4} - x^2 - y^2}$, and dividing by the whole area of surface of $S^2(\frac{1}{2})$, which is equal to π, we obtain

$$\int\int_{x^2+y^2<\frac{1}{4}} \frac{1}{\pi\sqrt{\frac{1}{4} - x^2 - y^2}} dxdy = 1.$$

Hence function

$$f(x, y) \quad = \quad \begin{cases} \frac{1}{\pi\sqrt{\frac{1}{4}-x^2-y^2}} & (x, y) \in B \\ 0 & (x, y) \notin B \end{cases} \tag{6.4}$$

where $B = \{(x, y) | x^2 + y^2 < \frac{1}{4}\}$ is a density function in R^2.

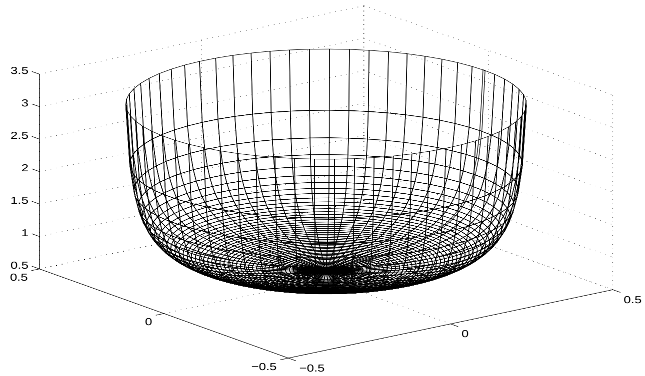


Figure 5.1. The density function f.

We can easily check that the function f has uniform marginals.

For $x \in [-\frac{1}{2}, \frac{1}{2}]$

$$f_X(x) = \int_{-\sqrt{\frac{1}{4}-x^2}}^{\sqrt{\frac{1}{4}-x^2}} \frac{1}{\pi\sqrt{\frac{1}{4}-x^2-y^2}}\, dy = \frac{1}{\pi} \arcsin \frac{y}{\sqrt{\frac{1}{4}-x^2}} \Bigg|_{-\sqrt{\frac{1}{4}-x^2}}^{\sqrt{\frac{1}{4}-x^2}} = 1.$$

To construct family of copulae which can represent all correlations $\rho \in (-1, 1)$ we consider linear transformation represented by a matrix

$$A = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.5}$$

where

$$\varphi \in (-\frac{\pi}{4}, \frac{\pi}{4}).$$

This transformation satisfies condition (6.2).

Let $(x', y', z') \in S^2(\frac{1}{2})$. Applying transformation (6.5) we get points (x, y, z) from ellipsoid

$$\begin{aligned} x &= \cos(\varphi)x' + \sin(\varphi)y' \\ y &= \sin(\varphi)x' + \cos(\varphi)y' \\ z &= z'. \end{aligned}$$

We now find the equation of this ellipsoid. Since

$$x'^2 + y'^2 + z'^2 = \frac{1}{4}$$

and

$$\begin{aligned} x' &= \frac{\cos(\varphi)x - \sin(\varphi)y}{\cos(2\varphi)} \\ y' &= \frac{-\sin(\varphi)x + \cos(\varphi)y}{\cos(2\varphi)} \\ z' &= z. \end{aligned}$$

then the ellipsoid is given by

$$x^2 + y^2 - 2\sin(2\varphi)xy + (\cos^2(2\varphi))z^2 \quad = \quad \frac{\cos^2(2\varphi)}{4}.$$

This can be also written as

$$x^2 + \left(\frac{y - \sin(2\varphi)x}{\cos(2\varphi)}\right)^2 + z^2 \quad = \quad \frac{1}{4}.$$

For all points from ellipse

$$x^2 + \left(\frac{y - \sin(2\varphi)x}{\cos(2\varphi)}\right)^2 < \frac{1}{4}$$

density function is given by following formula

$$f_\varphi(x, y) \quad = \quad \frac{1}{\pi\cos(2\varphi)} \frac{1}{\sqrt{\frac{1}{4} - x^2 - \left(\frac{y - 2\sin(2\varphi)x}{\cos(2\varphi)}\right)}}. \tag{6.6}$$

The distribution with density function given by formula (6.6) has uniform marginals so this is a copula. This copula depends on parameter φ. We will write C_φ. For two variables joined by copula C_φ on $[-\frac{1}{2}, \frac{1}{2}]^2$ the following holds:

Proposition 6.1 *If X, Y are joined by the copula C_φ, then*

$$\rho_{XY} \quad = \quad \sin(2\varphi). \tag{6.7}$$

Proof. We get

$$\rho_{XY} = \frac{E(XY)}{\sigma_X \sigma_Y} = \frac{E(XE(Y|X))}{\sigma_X^2}.$$

By (6.1)

$$\begin{aligned} E(Y|X) &= 2\cos(\varphi)\sin(\varphi)X \\ &= \sin(2\varphi)X \end{aligned}$$

hence

$$\rho_{XY} = \frac{\sin(2\varphi)E(X^2)}{\sigma_X^2} = \sin(2\varphi)$$

which concludes the proof. □

We can see that the function f given by (6.4) and presented on the Figure 5.1 is a density function of the copula C_0. Correlation between variables X and Y joined by the copula C_0 is equal to 0.

It is more convenient to start with the assumption that elliptical copula depends on correlation $\rho \in (-1,1)$. The parameter φ can be recovered as follows

$$\varphi = \frac{\arcsin(\rho)}{2}.$$

We will consider from now on the copula C with given correlation ρ and write C_ρ.

The density function of the elliptical copulae with given correlation $\rho \in (-1,1)$ is

$$f_\rho(x,y) = \begin{cases} \frac{1}{\pi\sqrt{1-\rho^2}} \frac{1}{\sqrt{\frac{1}{4} - \frac{x^2+y^2-2\rho xy}{1-\rho^2}}} & (x,y) \in B \\ 0 & (x,y) \notin B \end{cases}$$

where

$$B = \{(x,y)|\; x^2 + \left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)^2 < \frac{1}{4}\}$$

The figures below show graphs of density function of the copula C with correlation $\rho = 0.8$ and projection of this density on the plane.

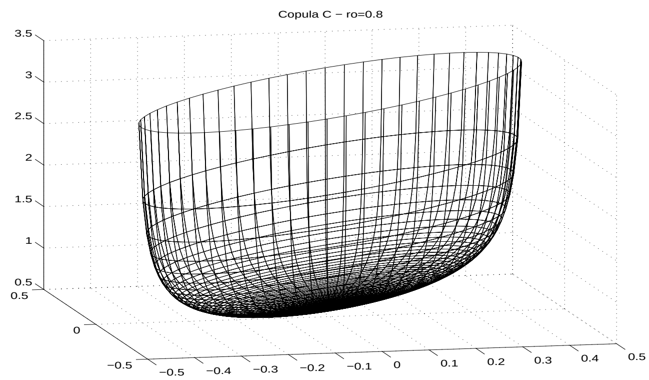


Figure 5.2. A density function of the copula C with correlation $\rho = 0.8$.

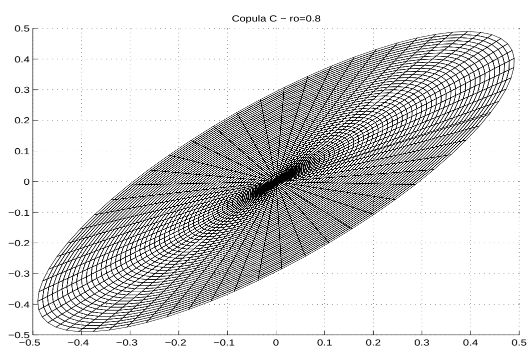


Figure 5.3. Projecting of density function of the copula C with correlation $\rho = 0.8$ on the plane.

For comparison we present below graphs of the density functions for diagonal band and minimum information copulae with correlation 0.8.

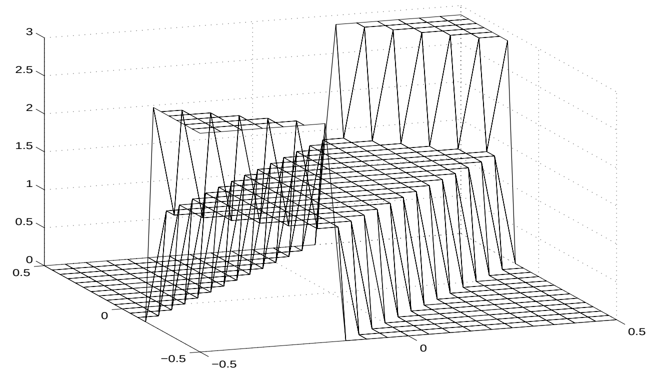


Figure 5.4. The diagonal band distribution with correlation 0.8.

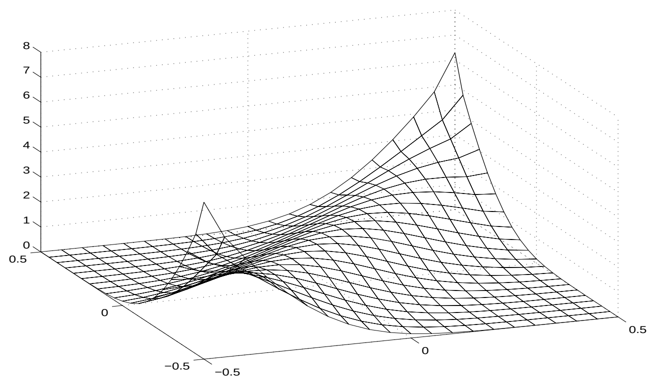


Figure 5.5. The minimum information distribution with correlation 0.8.

We show now some properties of the copula C_ρ.

Theorem 6.1 *If* X, Y *joined by the copula* C_ρ *then*

(a) $E(Y|X) = \rho X$,

(b) $Var(Y|X) = \frac{1}{2}(1-\rho^2)\left(\frac{1}{4} - X^2\right)$.

Proof. By (6.1) and (6.7) the copula C_ρ has linear regression with coefficient equal to correlation hence (a) holds. We verify condition (b)

$$\mathrm{Var}(Y|X) = \int_{\rho X - \sqrt{1-\rho^2}\sqrt{\frac{1}{4}-X^2}}^{\rho X + \sqrt{1-\rho^2}\sqrt{\frac{1}{4}-X^2}} (y - \rho X)^2 \frac{1}{\pi\sqrt{1-\rho^2}} \frac{1}{\sqrt{\frac{1}{4} - \frac{X^2+y^2-2\rho X y}{1-\rho^2}}} dy$$

$$
\begin{aligned}
&= \frac{1}{\pi\sqrt{1-\rho^2}} \int_{\rho X-\sqrt{1-\rho^2}\sqrt{\frac{1}{4}-X^2}}^{\rho X+\sqrt{1-\rho^2}\sqrt{\frac{1}{4}-X^2}} (y-\rho X)^2 \\
&\quad \frac{1}{\sqrt{\frac{1}{4}-X^2}} \frac{1}{\sqrt{1-\left(\frac{y-\rho X}{\sqrt{1-\rho^2}\sqrt{\frac{1}{4}-X^2}}\right)^2}} \\
&= \frac{1}{\pi}(1-\rho^2)(\frac{1}{4}-X^2)\int_{-1}^{1} t^2 \frac{1}{\sqrt{1-t^2}} dt \\
&= \frac{1}{2}(1-\rho^2)(\frac{1}{4}-X^2)
\end{aligned}
$$

which concludes the proof. □

6.4 Partial and conditional correlations

Let us consider variables X_i with zero mean and standard deviations σ_i, $i = 1, \ldots, n$. Let the numbers $b_{12;3,\ldots,n}, \ldots, b_{1n;3,\ldots,n-1}$ minimize

$$E\left((X_1 - b_{12;3,\ldots,n}X_2 - \ldots - b_{1n;3,\ldots,n-1}X_n)^2\right);$$

then the *partial correlations* are defined as (Yule and Kendall [54]):

$$\rho_{12;3,\ldots,n} \quad = \quad \operatorname{sgn}(b_{12;3,\ldots,n})\left(b_{12;3,\ldots,n}b_{21;3,\ldots,n}\right)^{\frac{1}{2}}, \text{ etc.}$$

Partial correlations can be computed from correlations with the following recursive formula:

$$\rho_{12;3,\ldots,n} \quad = \quad \frac{\rho_{12;3,\ldots,n-1} - \rho_{1n;3,\ldots,n-1}\cdot\rho_{2n;3,\ldots,n-1}}{\sqrt{1-\rho^2_{1n;3,\ldots,n-1}}\sqrt{1-\rho^2_{2n;3,\ldots,n-1}}}. \tag{6.8}$$

The *conditional correlation* of Z and Y given X

$$\rho_{YZ|X} \quad = \quad \rho(Y|X, Z|X)$$

is the product moment correlation computed with the conditional distribution given X. In general this depends on the value of X, but it may be constant.

We are interested in finding the relationship between partial $\rho_{YZ;X}$ and conditional correlations $\rho_{ZY|X}$ if variables X and Y are joined by the copula $C_{\rho_{XY}}$ and X, Z are joined by the copula $C_{\rho_{XZ}}$.

It is shown in (Kurowicka and Cooke [35]) that the linear regression property leads to equality of partial and conditional correlations in the case of conditional independence. We present now some numerical results prepared in Matlab 5.3. We assume variables X and Y are joined by the copula $C_{\rho_{XY}}$ and X, Z are joined by the copula $C_{\rho_{XZ}}$ and Y and Z conditionally independent given X.

Stipulated		Computed	
ρ_{XY}	ρ_{XZ}	ρ_{YZ}	$\rho_{YZ;X}$
0	0	-7.74e-19	-7.74e-19
0.4	0.8	0.3199	-0.0002
0.2	-0.9	-0.1799	0.0002
-0.8	0.7	-0.5598	0.0005
0.9	-0.9	-0.8097	0.0017

Table 6.1: Numerical results for conditional independence.

Theorem 6.2 *Let X, Y, Z be uniform on $[-\frac{1}{2}, \frac{1}{2}]$ and suppose*

(a) *X, Y are joined by $C_{\rho_{XY}}$,*

(b) *X, Z are joined by $C_{\rho_{XZ}}$,*

(c) $\rho_{YZ|X} = \rho$

then

$$\rho_{YZ;X} = \rho.$$

Proof. By Theorem 6.1

$$\begin{aligned} E(Y|X) &= \rho_{XY} X, \\ E(Z|X) &= \rho_{XZ} X. \end{aligned}$$

The partial correlation $\rho_{YZ;X}$ can be calculated in the following way

$$\rho_{ZY;X} = \frac{\rho_{ZY} - \rho_{XY}\rho_{XZ}}{\sqrt{(1-\rho_{XY}^2)(1-\rho_{XZ}^2)}}.$$

We also get

$$\rho = \rho_{YZ|X} = \frac{E(YZ|X) - E(Y|X)E(Z|X)}{\sigma_{Y|X}\sigma_{Z|X}} = \frac{E(YZ|X) - \rho_{XY}\rho_{XZ}X^2}{\sigma_{Y|X}\sigma_{Z|X}}.$$

Hence

$$E(YZ|X) = \rho\sigma_{Y|X}\sigma_{Z|X} + \rho_{XY}\rho_{XZ}X^2.$$

Since

$$\rho_{ZY} = \frac{E(E(YZ|X))}{\sigma_X^2}$$

then

$$\rho_{YZ;X} = \frac{\rho E(\sigma_{Y|X}\sigma_{Z|X})}{\sigma_X^2\sqrt{(1-\rho_{XY}^2)(1-\rho_{XZ}^2)}}.$$

Since by Theorem 6.1

$$\sigma_{Y|X} = \sqrt{\frac{1}{2}(1-\rho_{XY}^2)(\frac{1}{4}-X^2)}, \qquad \sigma_{Z|X} = \sqrt{\frac{1}{2}(1-\rho_{XZ}^2)(\frac{1}{4}-X^2)}$$

then

$$\begin{aligned} \rho_{YZ;X} &= \frac{\rho E\left(\sqrt{\frac{1}{2}(1-\rho_{XY}^2)(\frac{1}{4}-X^2)}\sqrt{\frac{1}{2}(1-\rho_{XY}^2)(\frac{1}{4}-X^2)}\right)}{\sigma_X^2\sqrt{(1-\rho_{XY}^2)(1-\rho_{XZ}^2)}} \\ &= \frac{r\frac{1}{2}E(\frac{1}{4}-X^2)}{\sigma_X^2} = \frac{r\frac{1}{2}(\frac{1}{4}-\frac{1}{12})}{\frac{1}{12}} = \rho. \quad \Box \end{aligned}$$

6.5 Conclusions

1. Elliptical copulae are continuous and can realize all correlation values $\rho \in (-1, 1)$.
2. This copula has linear regression and for variables joined by this copula we showed that partial and constant conditional correlations are equal.
3. Similar properties characterize normal and Fréchet distribution.
4. Combining elliptical copulae with graphical model called vines, presents attractive way of representing high dimensional distribution and can be used in direct sampling procedures.

Samenvatting

Bij het analyseren van fysische systemen stellen we ons ten doel verband te leggen tussen de input en de output van een model. We representeren een model gewoonlijk als een vector functie

$$Y = [Y_1, Y_2, \ldots, Y_m]$$

met input vector

$$X = [X_1, X_2, \ldots, X_n].$$

De waarden voor X geven ons, doorgegeven via het model, de bijbehorende waarden voor Y. Gewoonlijk zijn modellen zeer complex en de dimensies van de vectoren X en Y kunnen zeer groot zijn. Omdat we de waarden voor X vaak niet precies kunnen vaststellen, moet de input vector X opgevat worden als een random vector met een bepaalde verdeling. Voor een exacte analyse hebben we de gezamenlijke verdeling van X nodig om zo verbanden en afhankelijkheden te leggen tussen de elementen van X. Voor complexe problemen is het erg moeilijk deze gezamenlijke verdeling te vinden. De volgende benaderingen zijn dan mogelijk:

1. We kunnen het model vereenvoudigen door aan te nemen dat de componenten van X onafhankelijk zijn. De gezamelijke verdeling van de input vector is dan gelijk aan het produkt van de verdelingen van de componenten van X.

2. We kunnen het model vereenvoudigen door de afbeelding die de input vector afbeeldt op de output vector te vereenvoudigen (bijvoorbeeld door te lineariseren).

3. Grafische modellen kunnen worden gebruikt om hoog-dimensionale verdelingen te representeren (bijvoorbeeld Markov bomen en generalisaties naar influence diagrammen of nieuwe grafische modellen zoals vines).

Een bekend voorbeeld waarbij eerst onafhankelijkheid werd verondersteld en waarbij later resultaten zijn gegeneraliseerd naar afhankelijke gevallen is de extreme waarden theorie. Daar heeft men laten zien dat wanneer de componenten van X onafhankelijk en gelijk verdeeld zijn en de afbeelding het minimum (of maximum) van deze variabelen is, dat er dan drie mogelijke verdelingsklassen voor de output vector Y zijn als de dimensie van X groot is. Bovendien zijn de aantrekkingsgebieden van de mogelijke verdelingsklassen bekend, d.w.z. dat noodzakelijke en voldoende voorwaarden op de verdelingen van X zodat Y tot één van de drie verdelingsklasse behoort, bekend zijn (Gnedenko [17], Haan [9]). Deze klassieke extreme waarden theorie kan worden gegeneraliseerd door afhankelijkheden toe te staan (stationariteit of Markov afhankelijkheid) of door componenten met verschillende verdelingen toe te staan (Lindgren, Leadbetter and Rootzen [44]). Deze theorie is jarenlang intensief bestudeerd en toegepast op verscheidene problemen (zoals het testen van matriaalsterkte en analyse van golf- en vloeddata (Lindgren, Leadbetter and Rootzen [44], Castillo [4], Gumbel [18])).

Het eerste gedeelte van dit proefschrift bevat een bijdrage aan de extreme waarden theorie. De dubbel geïndexeerde, onafhankelijke en gelijkverdeelde variabelen

$$X_{11},\ X_{12},\ X_{22},\ X_{21},\ X_{13},\ X_{23},\ X_{33},\ X_{32},\ X_{31},\ldots$$

worden geplaatst in een rechthoekige matrix $[X_{ij}]$. De uitvoer variabele wordt gedefinieerd als

$$Y = \min_i \max_j X_{ij}$$

of

$$Y = \max_i \min_j X_{ij}.$$

Er worden tien mogelijke limiet survival functies (betrouwbaarheidsfunctieklassen genoemd) van Y bepaald onder lineaire normalisatie (Kolowrocki [29],[28]). De mogelijke verdelingen hangen af van de vorm van de matrix $[X_{ij}]$, d.w.z. van de relaties tussen het aantal elementen in de rijen en kolommen van deze matrix. In dit proefschrift worden de aantrekkingsgebieden van deze limiet verdelingen bepaald (hoofstuk 2). In hoofstuk 3 wordt laten zien hoe de theorie over aantrekkingsgebieden kan worden gebruikt om mogelijke limiet functies te vinden voor niet-homogene minmax modellen (waarbij X_{ij} verschillende verdelingen heeft).

Eenvoudige modellen kunnen worden geconstrueerd door middel van een lineaire transformatie van onafhankelijke variabelen met gegeven marginale verdelingen. Deze benadering is geïntroduceerd door Steffensen [49].

Een gebruikelijke manier om een hoog-dimensionale verdeling te definiëren is door iedere input variabele te transformeren naar een univariate normale verdeling en vervolgens een multivariate normale verdeling te introduceren met gegeven afhankelijkheden tussen de variabelen (Lauritzen [40], Muirhead [45]).

Grafische modellen lijken een handige manier om hoog-dimensionale verdelingen te representeren. Ze kunnen een gegeven model visueel representeren en helpen de afhankelijkheden te beschrijven. Een belangrijke eigenschap van grafische modellen is dat ze complexe structuren op een modulaire wijze kunnen beschrijven, door afhankelijkheden van aangrenzende elementen te combineren. De meest bekende methode is die van een boomstruktuur. Een boom met n variabelen specificeert hoogstens $n-1$ takken tussen de variabelen. Iedere tak kan worden geassocieerd met een copula, een verdeling op $[0,1]^2$ met uniforme marginale verdelingen. Populaire copulae zijn de diagonale band (Cooke and Waij [7]) en de minimale informatie copulae (Meeuwissen and Bedford [42]).

In het laatste hoofdstuk van dit proefschrift, hoofdstuk 6, wordt de elliptische copula geïntroduceerd en worden haar eigenschappen bestudeerd. De elliptische copula is continu en kan waarden aan nemen op het interval $(-1,1)$. Bij het construeren van deze copula worden eigenschappen van rotatie invariate random vectors met elliptische vorm gebruikt (Harding [5], Misiewicz [43]). Een dichtheidsfunctie van de elliptische copula met correlatie $\rho \in (-1,1)$ wordt gegeven door:

$$f_\rho(x,y) = \begin{cases} \frac{1}{\pi\sqrt{1-\rho^2}} \frac{1}{\sqrt{\frac{1}{4}-x^2-\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)^2}} & (x,y) \in B \\ 0 & (x,y) \notin B \end{cases}$$

waarbij

$$B = \left\{(x,y) | \, x^2 + \left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)^2 < \frac{1}{4}\right\}.$$

De elliptische copula heeft de eigenschap van lineaire regressie.

Gegeven een boom met n variabelen met copulae toegekend aan de takken, kan er altijd een gezamenlijke verdeling worden geconstrueerd die voldoet aan de specificatie volgens de boom-copulae. Bovendien kan men laten zien (Cooke [6]) dat er een unieke minimum informatie gezamenlijke verdeling bestaat die aan de boom-copulae specificatie voldoet en dat onder deze verdeling de boom een Markov boom is geworden. Verdelingen die op deze manier zijn gespecificeerd kunnen "on the fly" worden gesampled. Het specificeren van gezamenlijke verdelingen volgens de boom-copulae methode wordt beperkt door het feit dat er hoogstens $n-1$ takken aan de boom zijn.

Een generalisatie van Markov bomen zijn belief netwerken en influence diagrammen waarin acyclische grafen worden gebruikt als representatie van conditionele onafhankelijkheids relaties. Deze structuren zijn gebruikt in Bayesiaanse gevolgtrekking en beslissingsanalyse.

Een nieuwe klasse van Markov bomen werd geïntroduceerd in (Cooke [6]); Een vine met n variabelen is een verzameling bomen, waarvan de takken van de boom j de knopen van de boom $j+1$ zijn en waarbij elke boom het maximaal aantal takken heeft. Een regelmatige vine met n variabelen is een vine waarin twee takken in boom j alleen kunnen worden samengevoegd door een tak in boom $j+1$ als deze takken een knoop gemeen hebben. Het verschil tussen Markov bomen en vines is dat de conditionele onafhankelijkheid van Markov bomen wordt vervangen door conditionele afhankelijkheid met een gegeven conditionele correlatie coefficient.

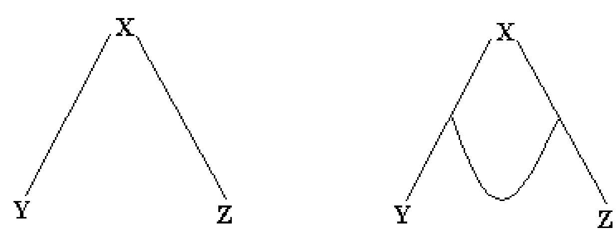


Figure 1. Een Markov boom (links) en een vine (rechts) met 3 elementen

In Figuur 1 worden voorbeelden gegeven van een Markov boom en een vine met 3 variabelen. In de Markov boom zijn de variabelen Y en Z conditioneel onafhankelijk gegeven X en in de vine zijn Y en Z niet conditioneel onafhankelijk. In hoofdstuk 5 wordt laten zien dat partiële correlatie tussen Y en Z met X groot kan zijn zelfs als Y en Z conditioneel onafhankelijk zijn gegeven X.

Aan de takken van een regelmatige vine kunnen partiële correlaties of conditionele (rang) correlaties toegekend worden. Een regelmatige vine met n elementen heeft $\binom{n}{2}$ takken en er is een bijectie van $(-1,1)^{\binom{n}{2}}$ naar de verzameling van volle rang correlatie matrices (Bedford and Cooke [3]). We kunnen dus een volle rang correlatie matrix met $\binom{n}{2}$ getallen vinden die niet aan algebraïsche beperkingen hoeven te voldoen (zoals positief definiet zijn).

Door gebruik te maken van regelmatige vines met conditionele rang correlaties kunnen we op een handige manier een hoog-dimensionale verdeling representeren en kunnen we de correlatie matrix bepalen en uit deze verdeling "on the fly" samples trekken.

In hoofdstuk 5 van dit proefschrift wordt het verband tussen partiële en conditionele correlatie bestudeerd waarbij vooral aandacht besteed wordt aan copulae die in hoog-dimensionale grafische modellen gebruikt worden. Voldoende, en in sommige gevallen noodzakelijke voorwaarden voor gelijkheid tussen partiële en conditionele correlatie worden verkregen. Numerieke resultaten laten zien dat het verschil tussen partiële en conditionele correlatie klein is wanneer de minimale informatie copula met een gegeven produkt moment correlatie wordt gebruikt. Als de gelijkheid bij benadering geldt, kunnen we m.b.v. regular vines een correlatie struktuur zonder algebraïsche beperkingen construeren (zoals positief definiet zijn).

In hoofdstuk 4 worden de technieken gebaseerd op de eigenschappen van regelmatige vines gebruikt om een aantal problemen te verhelpen met betrekking tot het positief definiet zijn van een matrix. Van een proto correlatie matrix, gedefinieerd als een symmetrische, reëele matrix met elementen in (-1,1) en met énen op de hoofddiagonaal, kunnen we bepalen of deze positief definiet is door de partiële correlaties toegekend aan de takken van een regelmatige vine te berekenen. Als we een partiële correlatie van een regelmatige vine vinden die niet in (-1,1) zit, dan is de betreffende matrix niet positief definiet. De snelheid van dit algoritme lijkt vergelijkbaar met bestaande algoritmen. Met dit algoritme kan een niet positief definiete matrix worden getransformeerd naar een positief definiete matrix door de waarden van de partiële correlaties in een regelmatige vine die niet in (-1,1) liggen te veranderen en de respectieve correlaties om te zetten naar een initiële proto correlatie matrix. Met dit nieuwe algoritme hebben deze aanpassingen een duidelijke probabilistische interpretatie. In complexe problemen zijn vaak veel elementen van de correlatie matrix niet gespecificeerd en de gedeeltelijk gespecificeerde matrix moet worden uitgebreid naar een volle, positief definite matrix. Hiervoor moet het completeringsprobleem worden opgelost (Laurent [38]).

In hoofdstuk 5 laten we zien hoe een regelmatige vine kan worden gebruikt om te bepalen of een gedeeltelijk gespecificeerde matrix kan worden uitgebreid naar een correlatie matrix. Deze benadering kan nuttig zijn als een hoog-dimensionale correlatie matrix gespecificeerd moet worden (zoals afhankelijke Monte Carlo simulaties).

Curriculum Vitae

Dorota Kurowicka was born 24th of January 1967 in Olsztyn, Poland. After completing her Secondary School in Olsztyn in 1985 she become student of Gdansk University in Gdansk, Poland. She studied mathematics and specialized in numerical methods. Her final thesis "The difference-functional inequalities" was carried out under supervision of Prof Z. Kamont and was completed in 1990. After graduation she worked at Warminsko-Mazurski University in Olsztyn in Faculty of Informatics, then she started her research at Gdynia Maritime Academy under supervision of Dr. K. Kolowrocki. Period 1999-2000 she worked as young research fellow in Department of Control risk, Optimization, Stochastic and System Theory (CROSS) of the Delft University of Technology under supervision of Prof. R.M. Cooke. Combined results of research carried out at Gdynia Maritime Academy and Delft University of Technology are presented in this Ph.D. Thesis.

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