Computation of implied dividend based on option market data

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Contents

Chapter 1

Introduction

Options are popularly traded in today's financial market. They are often connected to some item, such as a listed stock, an exchange index, futures contracts, or real estate. In this thesis, the stock option is discussed. There are two basic types of options, the European and American. A European option is an option contract that can only be exercised on the expiration date. Futures contracts (i.e., options on commodities) are generally European style options. An American option is an option contract that can be exercised at any time between the date of purchase and the expiration date. Most exchange-traded options are American-Style. Stock options are typically American style.

The famous Black-Scholes model is a fast and effective way to calculate the option price. An analytical solution for European options exist, however, for the stock options which are American style, a numerical approach is necessary. In real markets, many companies pay dividends to the stock holders not to the option holders. Whereas, the classical Black-Scholes model cannot deal with the dividend payment, so we use Wilmott's model which is an improvement of the Black-Scholes model to include the discrete dividend.

Sometimes, the announcement of the amount of dividend payment and the ex-dividend date cannot be obtained by the investors. At this time, the dividend is *implied*. To calculate the implied dividend as well as the implied volatility, two calibration methods are applied with the fixed risk-free interest rate. The data set is collected from the ING Group from Jan 2005 till Jun 2006.

In this thesis, the following issues are discussed. In chapter 2, the definitions and properties of European options are discussed; the Black-Scholes equation is derived and some simple dividend payment models are introduced. In chapter 3, the properties of American options are discussed. In chapter 4, some finite difference approximation methods for computing both of the European and American styles option prices are introduced. In chapter 5, we illustrate how the option pricing parameters needed in the B-S model influence the option price. In chapter 6, two basic optimization approaches for computing implied parameters are introduced and the results of the thesis are presented.

Chapter 2

Black-Scholes Analysis with European Options

2.1 Black-Scholes model without dividend

With *European Options*: the holder of the option has the right, not the obligation to buy (call) or to sell (put), at a fixed date (*expiry date*) for a fixed price (*exercise price*) an asset (*share, goods, derivative*).

The value V (*premium*) of an option (C *for a call,* P *for a put*) will depend on the following parameters:

- present value of the asset S
- time t till expiry T
- volatility of asset
- the risk-free interest rate r
- the exercising price E

The Black-Scholes model is valid under the following assumptions:

- the asset price S follows a lognormal random walk
- the risk free interest rate r and the volatility σ of the assets are assumed constant for the entire life time of the option
- no transaction costs for portfolio-hedging are included
- no dividends are paid on the asset during the option contract

- there are no arbitrage possibilities
- trading with the underlying asset can be done continuously
- short selling is permitted, and the asset can be divided arbitrarily

The asset price S is assumed to follow a lognormal random walk, a simple mathematical formula for S is

$$\frac{dS}{S} = \mu dt + \sigma dX \tag{2.1}$$

Where μ is known as the drift, usually μ is constant and it represents the average rate of growth of the asset price. σ is defined as the *volatility*, which measures the standard deviation of the return. Here both μ and σ are assumed to be constant. dX is a Wiener process, a normal distribution with mean 0 and variance dt, which describes the randomness of the asset price.

If $f(S,t)$ is a smooth function in both S and t, discarding the stochastic moment of S and t, expand $f(S,t)$ by the Taylor series up to second order terms, given

$$df = \frac{\partial f}{\partial S}dS + \frac{\partial f}{\partial t}dt + \frac{1}{2}(\frac{\partial^2 f}{\partial S^2}dS^2 + 2\frac{\partial^2 f}{\partial S \partial t}dSdt + \frac{\partial^2 f}{\partial t^2}dt^2) + \ldots \tag{2.2}$$

dS is given by (2.1), so

$$dS^2 = \sigma^2 S^2 dX^2 + 2\sigma\mu S^2 dtdX + \mu^2 S^2 dt^2 \tag{2.3}$$

where $dX^2 = dt$, and the other terms are of lower order. So that

$$df = \sigma S \frac{df}{dS}dX + (\mu S \frac{df}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2}dS^2 + \frac{\partial f}{\partial t})dt \tag{2.4}$$

Eq. (2.4) is called Ito's lemma.

We suppose the option price $V(S,t)$ satisfies Ito's lemma defined by (2.4), so

$$dV(S,t) = \sigma S \frac{\partial V}{\partial S}dX + (\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t})dt \tag{2.5}$$

To eliminate the stochastic part dX, we construct a portfolio consisting one option and $-\Delta$ assets. The value of this portfolio is

$$\Pi = V - \Delta S \tag{2.6}$$

The change in portfolio $d\Pi$ reads:

$$d\Pi = dV - \Delta dS \tag{2.7}$$

Here Δ is constant for dt, and Π then follows the random walk

$$d\Pi = \sigma S(\frac{\partial V}{\partial S} - \Delta)dX + (\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu\Delta S)dt \quad (2.8)$$

Choosing $\Delta = \frac{\partial V}{\partial S}$ the randomness can be eliminated to some extent, so that a deterministic portfolio is obtained. Due to the absence of arbitrage, the return of the portfolio in dt must equal that of the risk-free bank account:

$$r\Pi dt = d\Pi \quad (2.9)$$

This leads to the famous Black-Scholes equation!

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \quad (2.10)$$

Remarks:

- A derivative satisfying the above assumptions and which only depends on the present value S and t can be described by the Black-Scholes equation
- The value of an option is independent of the drift parameter μ
- The (linear) Black-Scholes operator

$$L_{BS} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} - r \quad (2.11)$$

  has a financial interpretation as the difference between the hedged portfolio and the return of a bank deposit.

2.2 Boundary and Initial conditions for European option

The Black-Scholes equation needs a final condition and boundary conditions to derive the unique solution of the partial differential equation. For the European call, one can define a vanilla option value by $C(S,t)$, with exercise price E and expiry date T.

The final condition at $t = T$, the value of call option is known to be the payoff

$$C(S,T) = max(S - E, 0) \quad (2.12)$$

When $S = 0$, then $dS = 0$, so that the asset price doesn't change during dt, and at expiry the payoff is zero. We have

$$C(0,t) = 0 \tag{2.13}$$

When $S \to \infty$, the option will be exercised and the magnitude of exercise price is less important. So the option value becomes the asset value at this time:

$$C(S,t) \sim S - Ee^{-r(T-t)} \quad \text{as} \quad S \to \infty \tag{2.14}$$

For a European call option, it is not possible to exercise early, (2.10) and (2.12)-(2.14) can be solved to give the Black-Scholes analytical solution of the call option.

For a put option, with value $P(S,t)$, the final condition is the payoff

$$P(S,T) = max(E - S, 0) \tag{2.15}$$

Similar to the European call case, as $S = 0$, the final payoff of European put is certainly $E.P(0,t)$ is the present value of E received at T. Assuming the risk-free interest rate is r, then

$$P(0,t) = Ee^{-r(T-t)} \tag{2.16}$$

As $S \sim \infty$, the option is unlikely to be exercised, then

$$P(S,t) \to 0 \quad \text{as} \quad S \to \infty \tag{2.17}$$

2.3 Derive the Black-Scholes Formula

A European call $C(S,t)$ price given by the Black-Scholes equation can be written as:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{2.18}$$

with

$$C(0,t) = 0, \quad C(S,t) - Ee^{-r(T-t)} \sim S \quad \text{at} \quad S \to \infty$$

and

$$C(S,T) = max(S - E, 0)$$

In order to transform (2.18) to a diffusion equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{2.19}$$

we need to set

$$S = Ee^x, \quad t = T - \tau/\frac{1}{2}\sigma^2, \quad C = Ev(x,\tau) \tag{2.20}$$

Then we get

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv \tag{2.21}$$

where $k = r/\frac{1}{2}\sigma^2$, and the initial condition for v is

$$v(x,0) = max(e^x - 1, 0)$$

Then let

$$v(x,\tau) = e^{\alpha x + \beta\tau}u(x,\tau) \tag{2.22}$$

The two unknowns α and β needs to be solved, so put (2.22) to (2.21), and differentiate it, then

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 v}{\partial x^2} + (k-1)(\alpha u + \frac{\partial u}{\partial x}) - ku$$

To eliminate the terms of u and $\partial u/\partial x$, we have

$$\beta = \alpha^2 + (k-1)\alpha - k, \quad 0 = 2\alpha + (k-1)$$

then α and β are calculated as:

$$\alpha = -\frac{1}{2}(k-1), \quad \beta = -\frac{1}{4}(k+1)^2$$

After these transformations, we finally substitute (2.18) into (2.19), where

$$v = e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau}u(x,\tau)$$

with the initial condition

$$u(x,0) = u_0(x) = max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0)$$

so that we obtain the diffusion equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad -\infty < x < \infty, \quad \tau > 0$$

The solution of the diffusion equation is

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u_0(S)e^{-(x-s)^2/4\tau}ds \tag{2.23}$$

To evaluate the integral in (2.23), make the transformation $x' = (s-x)/\sqrt{2\tau}$, we get

$$\begin{aligned} u(x,\tau) &= \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} u_0(x'\sqrt{2\tau}+x)e^{-\frac{1}{2}x'^2}dx' \\ &= \frac{1}{2\sqrt{\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k+1)(x+x'\sqrt{2\tau})}e^{-\frac{1}{2}x'^2}dx' \\ &\quad -\frac{1}{2\sqrt{\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k-1)(x+x'\sqrt{2\tau})}e^{-\frac{1}{2}x'^2}dx' \\ &= I_1 - I_2 \end{aligned}$$

To evaluate I_1 by completing the square in the component to get a standard integral:

$$\begin{aligned} I_1 &= \frac{1}{2\sqrt{\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k+1)(x+x'\sqrt{2\tau})}e^{-\frac{1}{2}x'^2}dx' \\ &= \frac{e^{\frac{1}{2}(k+1)x}}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{4}(k+1)^2\tau}e^{-\frac{1}{2}(x'-\frac{1}{2}(k+1)\sqrt{2\tau})^2}dx' \\ &= \frac{e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}-\frac{1}{2}(k+1)\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\eta^2}d\eta \\ &= e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}N(d_1), \end{aligned}$$

where

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}$$

and

$$N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} e^{-\frac{1}{2}s^2}ds$$

The computation of I_2 is similar to the approach of I_1 except replacing $(k+1)$ by $(k-1)$. Recall that

$$v(x,\tau) = e^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau}u(x,\tau) \tag{2.24}$$

and $x = log(S/E)$, $\tau = \frac{1}{2}\sigma^2(T-t)$ so for a European call option the solution of the BS equation reads

$$C_E(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2) \tag{2.25}$$

With $N(x)$, the cumulative normal distribution function, and

$$d_1 = \frac{log(S/E) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{2.26}$$

$$d_2 = \frac{log(S/E) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{2.27}$$

2.4 Put-Call Parity

The Put-Call parity is a relationship, first identified by Stoll (1969), that exists between the prices of European put and call options that both have the same underlying, strike price and expiration date. The relationship is derived using arbitrage arguments. Consider two portfolios consisting of:

1.The call option and an amount of cash equal to the present value of the strike price.

2.The put option and the underlying.

Comparing the expiration value for these two portfolios, with E representing the common strike price, we find the following:

A portfolio comprising a call option and an amount x of cash equal to the present value of the option's strike price has the same expiration value as a portfolio comprising the corresponding put option and the underlying. For European options, early exercise is not possible. If the expiration values of the two portfolios are the same, then their present values must also be the same. This equivalence is the so-called Put-Call parity.

If the two portfolios are going to have the same value at expiration, then they must have the same value today. Otherwise, an investor can make an arbitrage profit by purchasing the less expensive portfolio, selling the more expensive one and holding the long-short position to expiration. Accordingly, we have the price equality:

$$c + Ee^{-r(T-t)} = p + S \tag{2.28}$$

Where c is the price of European call option, p is the price of European put option, S is the asset price and r is the risk-free interest rate given the option's lifetime T and the time right now t.

Note that, the Put-Call parity applies only to European options, since a possibility of early exercise can cause a divergence in the present values of the two portfolios.

The Put-Call parity offers a simple test of option pricing models. Any option pricing model that produces put and call prices that do not satisfy put-call parity must be rejected as unsound. Such a model will suggest trading opportunities where none exist.

2.5 Some option pricing models with discrete dividend

During the life time of the option we have a single dividend payment D at t_d. Due to the absence of arbitrage:

$$S(t_d^+) = S(t_d^-) - D \tag{2.29}$$

where t_d^+ and t_d^- are the instants immediately before and after the ex-dividend date. The value V of the option must be smooth as a function of time over the time of payment

$$V(S(t_d^+), t_d^+) = V(S(t_d^-), t_d^-) \tag{2.30}$$

We distinguish the following approaches to include a discrete dividend.

a. Wilmott's method: This method is derived directly from (5.7) and (5.8), which follows the following steps:
-solve BS differential equation backwards from expiry T to t_d^+
-incorporate jump condition (5.7) and (5.8) to find value for t_d^-
-solve BS differential equation backwards with this value as final condition from t_d^- to t

$$C_d(S,t) = C(S,t,E), t_d^+ \leq t \leq T$$
$$C_d(S,t_d^-) = C_d(S-D,t_d^+) = C(S-D,t_d^+,E)$$

$C(S-D,t,E)$ is still a solution for all $t \leq t_d^-$.

b. Back to Basics's method [1]:
In the "Back to Basics" article, the extreme situation has been considered for paying dividend. For instance, the company want to pay S (liquidator) or 0 (survivor) for the dividend. Actually, we usually assume S, so today's price of a European call option can be calculated by the following integral:

$$C_E(S,0,D,t_d) = e^{-rt_d} \int_D^\infty C_{BS}(S-D,t_d)\phi(S_0,S,t_d)dS \tag{2.31}$$

c. Volatility adjusted model
Based on the solution of BS equation, Merton (1973) used $S - e^{-rt_d}D$ instead of S, which means the asset price is replaced by the difference of asset price minus the discounted dividend D:

$$C_E(S,0,D,t_d) = (S - e^{-rt_d}D)N(d_1) - Ee^{-rT}N(d_2) \tag{2.32}$$

Because there is some jump in S, the volatility of the asset price will change after the dividend payment. To keep the volatility constant, there are some methods to adjust it. From an adjusted S, one can derive the adjusted σ directly:

$$\sigma(S,t,D) = \begin{cases} \sigma_c \frac{S-e^{-rt_d}D}{S} & (t \in [0,t_d]) \\ \sigma_c & (t \in [t_d,T]) \end{cases} \tag{2.33}$$

$$\sigma(S,t,D) = \begin{cases} \sigma_c & (t \in [0,t_d]) \\ \sigma_c \frac{S+e^{-rt_d}D}{S} & (t \in [t_d,T]) \end{cases} \tag{2.34}$$

In some situations, like the Japanese market, the interest rate is close to 0. The adjustment can be simplified to:

$$\sigma(S,t,D) = \begin{cases} \sigma_c \frac{S-D}{S} & (t \in [0,t_d]) \\ \sigma_c & (t \in [t_d,T]) \end{cases}$$

$$\sigma(S,t,D) = \begin{cases} \sigma_c & (t \in [0,t_d]) \\ \sigma_c \frac{S+D}{S} & (t \in [t_d,T]) \end{cases}$$

In this situation, the option price produced by Wilmott's method changes by time, which is contrary to the real world, because the option price should be independent of t_D when $r = 0$. Applying the volatility adjusted approach, (2.33) or (2.34), we can get the unchanged option price C in the option's lifetime independent of t_D.

When $r \neq 0$, each of the method performs well, whereas the option price given by (2.34) is a bit higher than that by (2.33) for the volatility generated by (2.34) is a bit larger than that by (2.33). There is also a drawback in our model. For an American option, the price should be higher than its equivalent of a European option without dividend payment. But in the volatility adjusted model, the European option is more expensive than the American option when dividend payment is close to the expiry date.

Chapter 3

Black-Scholes Analysis for American Options

3.1 Basic idea of American Options

An *American Option* is an option that can be exercised anytime during its life time. The majority of exchange-traded options are American in the real market.

In fact, the American option holders have greater flexibility than the European counterpart, because of their right to exercise early. Therefore the American options have typically higher values than the European equivalents. For the American call option,

$$C(S,t) \geq max(S-E,0)$$

and for the American put option,

$$P(S,t) \geq max(E-S,0)$$

In this case, during the life of the option there will be some values of S for which it is optimal for the holders to exercise the American option. Unlike the European option, one may be interested in determining the value of $S_f(t)$, for which it is optimal to exercise.

If there is no dividend paid during the option's life time, the American call option is equal to the European call option, which means that it is not optimal to exercise the American call option prematurely if the asset is non-dividend payment. And if there is some dividend paid, it may be optimal to exercise the American call option early. See Fig.(3.1) and (3.2), the American call will never cross the payoff unlike its European equivalent, in the case of dividend. If the American call price is below the payoff function, there is

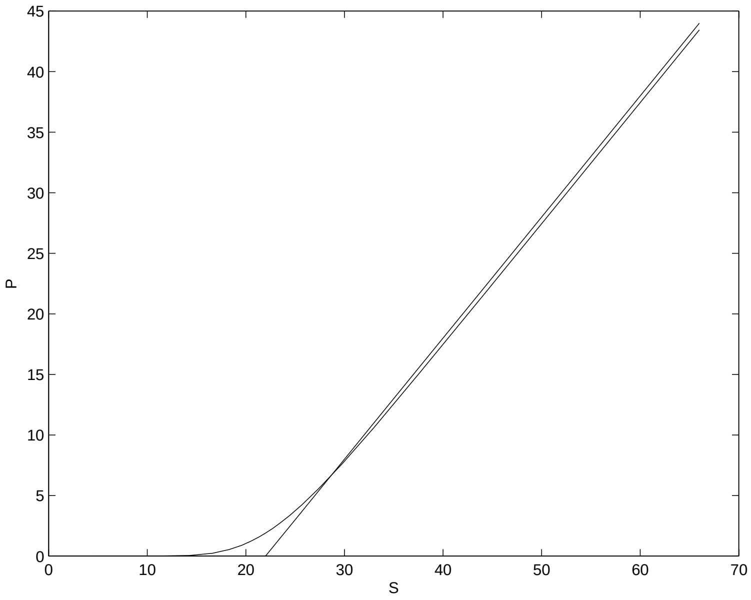

Figure 3.1: European call option with strike price $E = 22$, implied volatility $\sigma = 0.25$, risk-free interest rate $r = 0.02$, dividend payment $D = 1$ ex-dividend date $t_d = 0.5T$

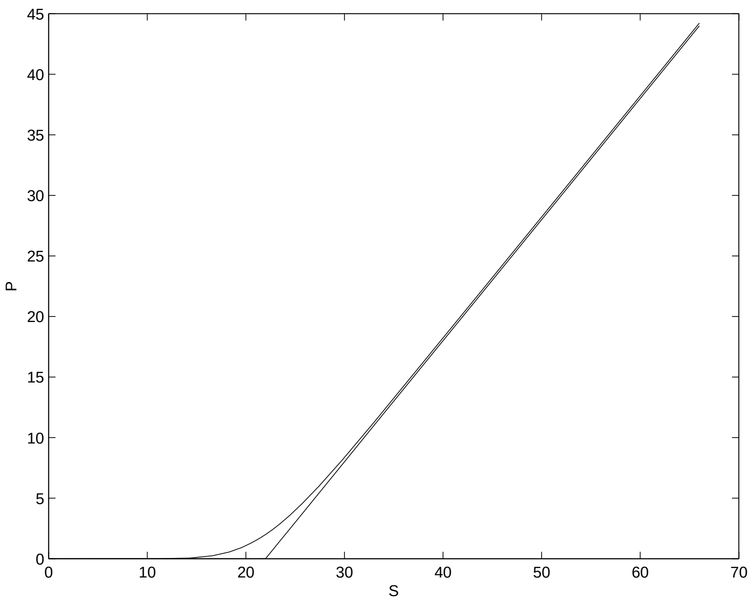

Figure 3.2: American call option with strike price $E = 22$, implied volatility $\sigma = 0.25$, risk-free interest rate $r = 0.02$, dividend payment $D = 1$ ex-dividend date $t_d = 0.5T$

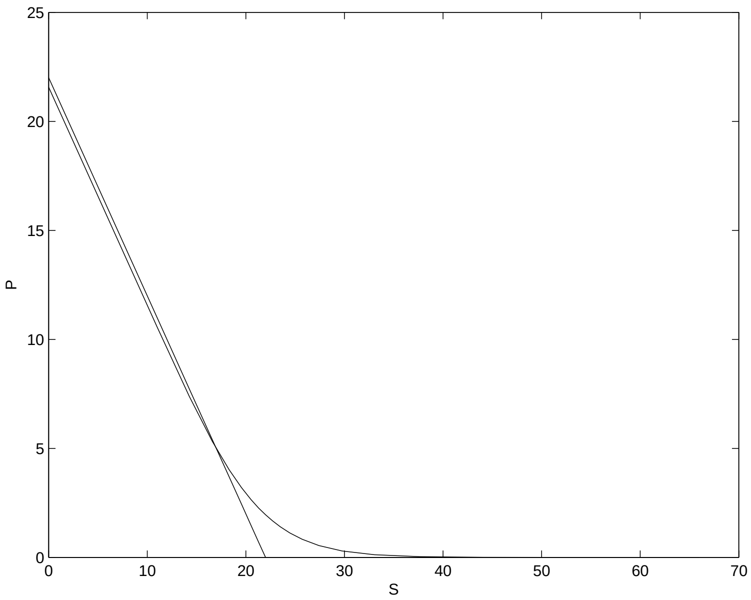

Figure 3.3: European put option with strike price $E = 22$, implied volatility $\sigma = 0.25$, risk-free interest rate $r = 0.02$, non-dividend payment

an arbitrage opportunity to make instant profit by buying the option and exercise it immediately.

In contrast, for the American put option, it is always optimal to exercise prematurely, compared to the European put option with the same parameters. Fig.(3.3) and (3.4) show that for the non-dividend American put it is even optimal to exercise early. If some dividend payment exists, the American put will be optimal to exercise right after the ex-dividend date, which will be discussed later.

3.2 American put option as free boundary problems

The valuation of American options is known as solving a free boundary problem. Typically at each time t there is a value of S, which marks the boundary between two regions: to one side one should hold the option and to the other side one should exercise it. Define the **optimal exercise price** by $S_f(t)$. Unfortunately, we don't know $S_f(t)$ analytically, so we call the unknown boundary the **free boundary**.

For the Delta-hedged portfolio Π the change in value over time dt is $d\Pi$. Arbitrage considerations show that it is impossible to have $d\Pi > r\Pi dt$ be-

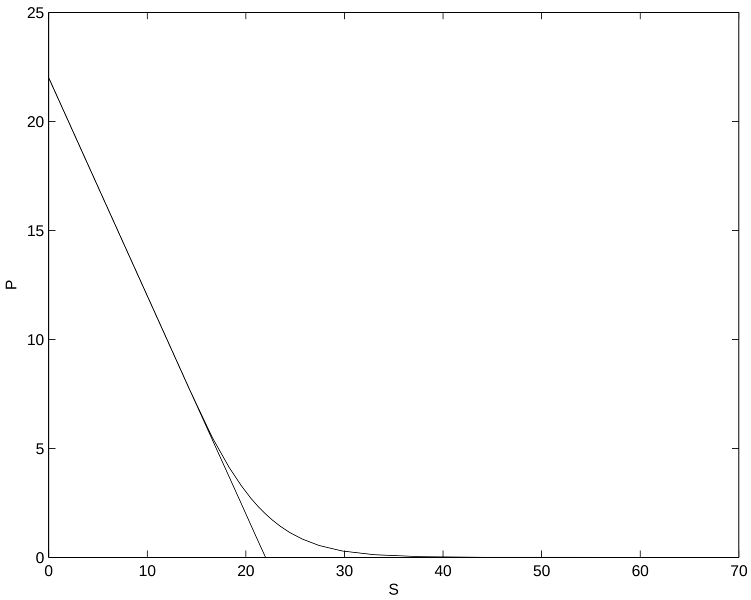

Figure 3.4: American put option with strike price $E = 22$, implied volatility $\sigma = 0.25$, risk-free interest rate $r = 0.02$, non-dividend payment

cause an investor can borrow money at the risk-free rate and invest in the risk-free portfolio Π to make a risk-free profit. However, if $d\Pi < r\Pi dt$ the equivalent strategy (short the portfolio and earn the risk-free rate on the money) is not always possible in the presence of early exercise. While in the case of a European option the return on the option in a risk-neutral world must be equal to the risk-free rate, in the case of the American option the following inequality holds:

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt \leq r\Pi dt = r(V - \frac{\partial V}{\partial S}S)dt$$

So that,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\frac{\partial V}{\partial S}S - rV \leq 0 \tag{3.1}$$

For American call option without dividend, (3.1) indicates the equation, which is the same as the European counterpart, because the optimal exercise strategy. For an American put option, it is a free boundary problem, as

$$\begin{aligned} P = E - S, &\quad \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + r\frac{\partial P}{\partial S}S - rP < 0 \\ P > E - S, &\quad \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + r\frac{\partial P}{\partial S}S - rP = 0 \end{aligned} \tag{3.2}$$

The boundary conditions at $S = S_f(t)$ are that P and its slope are continuous:

$$P(S_f(t),t) = max(E - S_f(t),0), \quad \frac{\partial P}{\partial S}(S_f(t),t) = -1 \tag{3.3}$$

The first equation of (3.3) is to determine the option price at the free boundary, and the second one is to determine the location of the free boundary.

The American option pricing usually implemented as a *Linear Complementary Problem*. Firstly, transform the original (S,t) variables to (x,τ) as (2.20), while there is an optimal exercise boundary $S = S_f(t)$, which can be rewritten as $x = x_f(\tau)$. The payoff function $max(E - S, 0)$ now reads

$$g(x,\tau) = e^{\frac{1}{2}(k+1)^2\tau} max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0) \tag{3.4}$$

Then we obtain

$$\begin{aligned} \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad \text{for} \quad x > x_f(\tau) \\ u(x,\tau) = g(x,\tau) \quad \text{for} \quad x \leq x_f(\tau) \end{aligned} \tag{3.5}$$

with the initial condition

$$u(x,0) = g(x,0) = max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0) \tag{3.6}$$

and the boundary conditions read

$$u(x^+,\tau) = 0, \quad u(-x^-,\tau) = g(-x^-,\tau) \tag{3.7}$$

where x^+ and x^- are large numbers. Then linear complementary problem for the American option transforms into

$$\begin{aligned} (\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2})(u(x,\tau) - g(x,\tau)) = 0 \\ (\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}) \geq 0, \quad (u(x,\tau) - g(x,\tau)) \geq 0 \end{aligned} \tag{3.8}$$

with the initial condition (3.6) and boundary conditions (3.7). And both $u(x,\tau)$ and $\frac{\partial u}{\partial x}(x,\tau)$ are continuous. The same transformation to the linear complementary formulation can be done for the original option pricing formulation with the Black-Scholes operator.

3.3 American options with discrete dividend

When the asset pays discrete dividends, the asset price will decrease by the same amount as the dividend right after the dividend date if there are no other factors affecting the income.

Early exercise policies for American call options are as following:

First consider an American call option on an asset which pays one dividend D at time t_d. And let $S_d^-(S_d^+)$ denote the asset price right before (after) the discrete dividend payment at time t_d^- (t_d^+). If the option is exercised at time t_d, then the option price will be $S_d - E$, otherwise, the asset price will go down to $S_d^+ = S_d^- - D$ immediately after the dividend payment. In the period of $t_d^+ \to T$, the American call will be equal to its European counterpart. At time t_d, the option price is

$$V(S_d, t_d) = max(S_d^- - E, S_d^+ - Ee^{-r(T-t_d^+)})$$

which compares the option price of being optimal to exercise the option right before t_d and the option price of being optimal to hold the option right after t_d. The situation that

$$S_d^- - E \leq S_d^+ - Ee^{-r(T-t_d)} = (S_d^- - D) - Ee^{-r(T-t_d)} \tag{3.9}$$

is not optimal to exercise the option the American call because the call is worth more when it is held than exercised for all S_d^-. Here

$$D \leq E(1 - e^{-r(T-t_d)}) \tag{3.10}$$

When $D > E(1-e^{-r(T-t_d)})$, it may be optimal to exercise the American call if the asset price S_d^- is larger than some S_f. Given the continuity property of the option price across the ex-dividend date,

$$C_{BS}(S_d^+, T - t_d^-) = C_{BS}(S_d^- - D, T - t_d^+) \tag{3.11}$$

where $C_{BS}(S_d^- - D, T - t_d^+)$ is the option price derived by the Black-Scholes equation with asset price $S_d^- - D$ and time to maturity $T - t_d^-$. When D is large enough, there will be a S_f satisfying:

$$C_{BS}(S_f - D, T - t_d) = S_f - E \tag{3.12}$$

S_f in (3.12) is a critical point that, if $S_d^- < S_f$ it is never optimal to exercise the American call prematurely. Actually, the holder of an American call option with discrete dividend would only exercise the option immediately before the ex-dividend date provided both $D > E(1 - e^{-r(T-t_d)})$ and $S_d^- \geq S_f$.

Early exercise policies for American put options
The holder of American put which pays single or multiple dividends tend to hold the option until the ex-dividend date in order to benefit from the decline of the asset price. From the date of the last ex-dividend, the American put will behave like a European put.

For the one-dividend American put option, let the ex-dividend date be t_d, the expiry date T and the dividend amount D. Before the ex-dividend date, to exercise the option before t_d or not depends on the strike price E and D. The interest rate income from t to t_d is $E(e^{r(t_d-t)} - 1)$, where r is the risk-free interest rate. If $E(e^{r(t_d-t)} - 1) < D$, then it is not optimal to exercise the option before t_d. In this case, the holder will gain more benefit from the decline of the asset price than the interest income.

Observing that the interest rate income $E(e^{r(t_d-t)} - 1)$ depends on $t_d - t$. So there is a critical value t_s so that

$$E(e^{r(t_d-t_s)} - 1) = D \tag{3.13}$$

Solve the equation,

$$t_s = t_d - \frac{1}{r} ln(1 + \frac{D}{E}) \tag{3.14}$$

When $t < t_s$, we have $E(e^{r(t_d-t)} - 1) > D$, and under such condition, early exercise may become optimal if the asset price is below a critical value.

Let $\tau = T - t$, the optimal exercise boundary $S_f(\tau)$ of an American put option with single dividend follows the criteria:

1. $S_f(\tau)$ is a continuous decreasing function of τ with $S_f(0) = E$ when $\tau \in [0, T - t_d)$
2. at $T - t_d$, $S_f(\tau)$ jumps to zero and remains to be zero till $T - t_s$
3. after $T - t_s$, $S_f(\tau)$ increases smoothly from zero to a peak value then decreases smoothly to a certain value at a higher value of τ

Chapter 4

Finite-difference methods for pricing options

In this chapter we report the basics of the finite difference methods in space and time. Finite-difference methods are elementary approaches to approximate partial differential equations and linear complementarity problems. To simplify the approach to price the options, diffusion equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

would be solved firstly, then the solution transformed back to the Black-Scholes variables

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

with $\tau = \frac{1}{2}\sigma^2(T-t)$, $k = r/\frac{1}{2}\sigma^2$, similar to (2.20), the option price is

$$V = E^{\frac{1}{2}(1+k)} S^{\frac{1}{2}(1-k)} e^{\frac{1}{4}(1+k)^2\tau} u(log(S/E), \tau)$$

4.1 Difference Approximations

The basic idea of finite-difference methods is based on Taylor series expansions, such as

$$\frac{\partial u}{\partial \tau}(x,\tau) = \frac{u(x,\tau+\Delta\tau) - u(x,\tau)}{\Delta\tau} + O(\Delta\tau) \tag{4.1}$$

which is a finite-difference approximation of $\partial u/\partial \tau$. (2.1) is called *forward difference*, because differencing is in the forward τ direction.

Similar to the forward difference, the *backward difference* is

$$\frac{\partial u}{\partial \tau}(x,\tau) = \frac{u(x,\tau) - u(x,\tau-\Delta\tau)}{\Delta\tau} + O(\Delta\tau) \tag{4.2}$$

The second partial derivative $\partial^2 u/\partial x^2$ can be approximated by *symmetric central-difference* approximation.

$$\frac{\partial^2 u}{\partial x^2}(x,\tau) = \frac{u(x+\Delta x,\tau) - 2u(x,\tau) + u(x-\Delta x,\tau)}{(\Delta x)^2} + O((\Delta x)^2) \quad (4.3)$$

4.2 Explicit Method

Apply forward difference (4.1) for $\partial u/\partial\tau$ and symmetric central differences (4.3) for $\partial^2 u/\partial x^2$, discarding the error term leads to

$$\frac{u_n^{m+1} - u_n^m}{\Delta\tau} = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\Delta x)^2} \quad (4.4)$$

Rearrange (4.4), we obtain

$$u_n^{m+1} = \lambda u_{n+1}^m + (1-2\lambda)u_n^m + \lambda u_{n-1}^m \quad (4.5)$$

where

$$\lambda = \frac{\Delta\tau}{(\Delta x)^2} \quad (4.6)$$

At timestep m, if u_n^m for all n are known, then u_n^{m+1} can be calculated explicitly.

For $u^{(m)} = (u_{N^-+1}^m, \ldots u_{N^+-1}^m)^{tr}$, where $-N^-$ and N^+ are large positive numbers, it is possible to implement the method. Define A as

$$A := \begin{pmatrix} 1-2\lambda & \lambda & 0 & \ldots & 0 \\ \lambda & 1-2\lambda & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda \\ 0 & \ldots & 0 & \lambda & \ddots \end{pmatrix} \quad (4.7)$$

In order to calculate the option price, the non-dimensional time is divided into M equal time steps

$$\Delta\tau = \frac{1}{2}\sigma^2 T/M$$

So that we solve the partial differential equation for $N^- < n < N^+$ and $0 < m \le M$, and the boundary conditions are

$$u_{N^-}^m \approx u_{-\infty}(N^-\Delta x, m\Delta\tau), \quad 0 < m \le M \quad (4.8)$$
$$u_{N^+}^m \approx u_{\infty}(N^+\Delta x, m\Delta\tau), \quad 0 < m \le M \quad (4.9)$$

And the initial condition is

$$u_n^0 = u_0(n\Delta x), \quad N^- \le n \le N^+ \quad (4.10)$$

Now the explicit method can be given in the matrix notation:

$$u^{(m+1)} = Au^{(m)} \tag{4.11}$$

Investigating the truncated error of the numerical solution and the theoretical solution of (4.11)

$$e^{(m)} = \bar{u}^{(m)} - u^{(m)}$$

where the $\bar{u}$ is the theoretical solution. Obviously, there is a relation such as

$$Ae^{(m)} = A\bar{u}^{(m)} - Au^{(m)} = \bar{u}^{(m+1)} - u^{(m+1)} = e^{(m+1)}$$

then apply it repeatedly, we get

$$e^{(m)} = A^m e^{(0)}$$

For the method to be stable, previous errors must be damped, which leads to $A^m e^{(0)} \to 0$ for $m \to \infty$. It can be easily proved that the stability criterion is

$$0 < \frac{\Delta\tau}{(\Delta x)^2} \leq 0$$

4.3 Implicit Method

The fully-implicit finite-difference method uses the backward-difference approximation (4.2) for $\partial u/\partial\tau$ and symmetric central difference (4.3) for $\partial^2 u/\partial x^2$, which leads to

$$\frac{u_n^m - u_n^{m-1}}{\Delta\tau} = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\Delta x)^2} \tag{4.12}$$

which yields the alternative to (4.5)

$$-\lambda u_{n-1}^m + (1+2\lambda)u_n^m - \lambda u_{n+1}^m = u_n^{m-1} \tag{4.13}$$

In (4.13), only the right-hand side value u_n^{m-1} is known, whereas the three unknown values of u on the left-hand side wait to be calculated. Similar to the explicit counterpart, define A as

$$A := \begin{pmatrix} 2\lambda+1 & -\lambda & 0 & \dots & 0 \\ -\lambda & 2\lambda+1 & -\lambda & \ddots & 0 \\ 0 & -\lambda & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & -\lambda \\ 0 & 0 & & -\lambda & 1+2\lambda \end{pmatrix} \tag{4.14}$$

To be more specific, write (4.13) as

$$\begin{pmatrix} 2\lambda+1 & -\lambda & 0 & \dots & 0 \\ -\lambda & 2\lambda+1 & -\lambda & \ddots & 0 \\ 0 & -\lambda & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & -\lambda \\ 0 & 0 & & -\lambda & 1+2\lambda \end{pmatrix} \begin{pmatrix} u^m_{N^-+1} \\ \vdots \\ u^m_0 \\ \vdots \\ u^m_{N^+-1} \end{pmatrix}$$

$$= \begin{pmatrix} u^{m-1}_{N^-+1} \\ \vdots \\ u^{m-1}_0 \\ \vdots \\ u^{m-1}_{N^+-1} \end{pmatrix} + \lambda \begin{pmatrix} u^m_{N^-} \\ 0 \\ \vdots \\ 0 \\ u^m_{N^+} \end{pmatrix} = \begin{pmatrix} b^m_{N^-+1} \\ \vdots \\ b^m_0 \\ \vdots \\ b^m_{N^+-1} \end{pmatrix}$$

Then the compact form of the matrix notation is

$$Au^{(m)} = b^{(m)} \tag{4.15}$$

Here $b^{(m)} = u^{(m-1)} + \lambda(u^m_{N^-}, 0, ...0, u^m_{N^+})$. So in principle, the $u^{(m)}$ can be solved by the inverse of A and $b^{(m)}$, such as

$$u^{(m)} = A^{-1}b^{(m)} \tag{4.16}$$

For A is symmetric and tridiagonal, both LU and SOR algorithms are usually applied to solve (4.15).

The LU Method The LU decomposition is trying to find a lower triangular matrix L and a upper triangular matrix U to make $A = LU$.

$$\begin{pmatrix} 2\lambda+1 & -\lambda & 0 & \dots & 0 \\ -\lambda & 2\lambda+1 & -\lambda & \ddots & 0 \\ 0 & -\lambda & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & -\lambda \\ 0 & 0 & & -\lambda & 1+2\lambda \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ l_{N^-+1} & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & l_{N^+-2} & 1 \end{pmatrix} \begin{pmatrix} y_{N^-+1} & z_{N^-+1} & 0 & \dots & 0 \\ 0 & y_{N^-+2} & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & z_{N^+-2} \\ 0 & \dots & 0 & 0 & y_{N^+-1} \end{pmatrix}$$

After simple multiplication, the l_n, y_n and z_n can be given as follows:

$$y_{N^-+1} = 1 + 2\lambda$$

$$y_n = (1+2\lambda) - \lambda^2/y_{n-1}, \quad n = N^- + 2, ..., N^+ - 1$$

$$z_n = -\lambda, l_n = -\lambda/y_n, \qquad n = N^- + 1, ..., N^+ - 2$$

So we only need to calculate y_n, $n = N^- + 1, ..., N^+ - 1$. Given $A = LU$, rewrite $Au^{(m)} = b^{(m)}$ as

$$Lq^{(m)} = b^{(m)}, \quad Uu^{(m)} = q^{(m)}$$

$q^{(m)}$ is an intermediate vector. Using y_n to eliminate l_n and z_n, we get

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -\frac{\lambda}{y_{N^-+1}} & 1 & 0 & & \vdots \\ 0 & -\frac{\lambda}{y_{N^-+2}} & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -\frac{\lambda}{y_{N^+-2}} & 1 \end{pmatrix} \begin{pmatrix} q^m_{N^-+1} \\ q^m_{N^-+2} \\ \vdots \\ q^m_{N^+-2} \\ q^m_{N^+-1} \end{pmatrix} = \begin{pmatrix} b^m_{N^-+1} \\ b^m_{N^-+2} \\ \vdots \\ b^m_{N^+-2} \\ b^m_{N^+-1} \end{pmatrix}$$

and

$$\begin{pmatrix} y_{N^-+1} & -\lambda & 0 & \dots & 0 \\ 0 & y_{N^-+2} & -\lambda & & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & & \ddots & y_{N^+-2} & 0 \\ 0 & \dots & 0 & 0 & y_{N^+-1} \end{pmatrix} \begin{pmatrix} u^m_{N^-+1} \\ u^m_{N^-+2} \\ \vdots \\ u^m_{N^+-2} \\ u^m_{N^+-1} \end{pmatrix} = \begin{pmatrix} q^m_{N^-+1} \\ q^m_{N^-+2} \\ \vdots \\ q^m_{N^+-2} \\ q^m_{N^+-1} \end{pmatrix}$$

Then the intermediate q^m_n can be solved as

$$q^m_{N^-+1} = b^m_{N^-+1}, \quad q^m_n = b^m_n + \frac{\lambda q^m_{n-1}}{y_{n-1}}, \quad n = N^- + 2, ..., N^+ - 1$$

Similar for all the u^m_n,

$$u^m_{N^+-1} = \frac{q^m_{N^+-1}}{y^m_{N^+-1}}, \quad u^m_n = \frac{q^m_n + \lambda u^m_{n+1}}{y_n}, \quad n = N^+ - 2, ..., N^- + 1$$

The LU method is a direct method which aims to find the unknowns without iteration. This kind of methods is fast. And one of the drawback is that they cannot include easily the transaction costs in the calculation.

The SOR Method SOR is short for Successive Over Relaxation, which is an algorithm of iterative method. The basic idea of SOR is to search the approximated solution near the exact solution by iteration. Rearrange (4.13)

$$u^m_n = \frac{1}{1+2\lambda}(b^m_n + \lambda(u^m_{n-1} + u^m_{n-1})) \tag{4.17}$$

The SOR method is a refinement of the Gauss-Seidel method and, the Gauss-Seidel method is a development of the Jacobi method. The basic idea of the Jacobi method is to derive some a guess u_n^m by some initial guess of u which is substituted into the right-hand side of (4.17). Generally, the value of previous step is a sound one, such as

$$u_n^{m,k+1} = \frac{1}{1+2\lambda}(b_n^m + \lambda(u_{n-1}^{m,k} + u_{n-1}^{m,k})) \tag{4.18}$$

Where $u^{m,k+1}$ is the $k+1$-th iteration of u_n^m. One expects that $u_n^{m,k} \to u_n^m$ as $k \to \infty$. Set a small number ϵ, as long as the measure

$$\sum_n (u_n^{m,k+1} - u_n^{m,k})^2 < \epsilon$$

then stop the iteration and let $u_n^m = u_n^{m,k+1}$.

For the Gauss-Seidel method, one replaces $u_{n-1}^{m,k}$ by $u_{n-1}^{m,k+1}$ in (4.18). So that

$$u_n^{m,k+1} = \frac{1}{1+2\lambda}(b_n^m + \lambda(u_{n-1}^{m,k+1} + u_{n-1}^{m,k})) \tag{4.19}$$

The Gauss-Seidel method is more efficient than the Jacobi method, because the Gauss-Seidel method uses the most recent guess of u_n^m.

The SOR method is based on the Gauss-Seidel method, while the SOR method is a little bit complex. Write $u_n^{m,k+1}$ as following:

$$u_n^{m,k+1} = u_n^{m,k} + (u_n^{m,k+1} - u_n^{m,k})$$

where $(u_n^{m,k+1} - u_n^{m,k})$ can be thought as an error term or a correction term. Put

$$\begin{aligned} y_n^{m,k+1} &= \frac{1}{1+2\lambda}(b_n^m + \lambda(u_{n-1}^{m,k+1} + u_{n+1}^{m,k})) \\ u_n^{m,k+1} &= u_n^{m,k} + \omega(y_n^{m,k+1} - u_n^{m,k}) \end{aligned} \tag{4.20}$$

When $\omega > 1$, it is called the *over relaxation* parameter. Here $y_n^{m,k+1}$ is derived by the Gauss-Seidel method, and $u_n^{m,k+1}$ can be obtained by the correction of $(y_n^{m,k+1} - u_n^{m,k})$. The SOR method can be proved to converge when $0 < \omega < 2$, while $0 < \omega < 1$ is called *under relaxation.* It is also can be shown that there is one optimal value of ω when $1 < \omega < 2$, which makes the convergence most rapidly. Actually, in the numerical implementation, one can change ω each iteration to accelerate the SOR approach.

The implicit method is stable for all λ, so one can apply larger time step to compensate the heavier task in each iteration.

4.4 The Crank-Nicolson Method

Both the explict method and the implicit method produce the order $\Delta\tau$ accurate approximation to $\frac{\partial u}{\partial \tau}$. One expects a higher order $\Delta\tau^2$ accuracy of the time discretization of $\frac{\partial u}{\partial \tau}$ with the unconditional stability. Crank and Nicolson adviced the average of the forward difference and the backward difference. Then firstly apply the forward difference to derive the explicit scheme

$$\frac{u_n^{m+1} - u_n^m}{\Delta\tau} = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\Delta x)^2}$$

then apply the backward difference to derive the implicit scheme

$$\frac{u_n^m - u_n^{m-1}}{\Delta\tau} = \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\Delta x)^2}$$

Take the average of the two equations, gives

$$\frac{u_n^m - u_n^{m-1}}{\Delta\tau} = \frac{1}{2}\left(\frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\Delta x)^2} + \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\Delta x)^2}\right) \tag{4.21}$$

Rearrange (4.21), the Crank-Nicolson method can be written as:

$$u_n^{m+1} - \frac{1}{2}\lambda(u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}) = u_n^m + \frac{1}{2}\lambda(u_{n+1}^m - 2u_n^m + u_{n-1}^m) \tag{4.22}$$

Where u_{n-1}^m, u_n^m and u_{n+1}^m are known, which are used to calculate u_{n-1}^{m+1}, u_n^{m+1} and u_{n+1}^{m+1}. λ here is still $\Delta\tau/(\Delta x)^2$. Because the right-hand side of (4.22) can be solved explicitly, we perform the Crank-Nicolson method in two steps, firstly calculate

$$Z_n^m = (1-\lambda)u_n^m + \frac{1}{2}\lambda(u_{n-1}^m + u_{n+1}^m) \tag{4.23}$$

and then calculate

$$(1+\lambda)u_n^{m+1} - \frac{1}{2}\lambda(u_{n-1}^{m+1} + u_{n+1}^{m+1}) = Z_n^m \tag{4.24}$$

Write (4.23) and (4.24) in matrix notation,

$$Au^{(m)} = Bu^{(m+1)} \tag{4.25}$$

where

$$A = \begin{pmatrix} 1-\lambda & \frac{\lambda}{2} & & & 0 \\ \frac{\lambda}{2} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & \ddots \end{pmatrix}, \quad B = \begin{pmatrix} 1+\lambda & -\frac{\lambda}{2} & & & 0 \\ -\frac{\lambda}{2} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & \ddots \end{pmatrix}$$

Similar to the implicit method, define $b^{(m)}$

$$b^{(m)} = \begin{pmatrix} Z^m_{N^-+1} \\ \vdots \\ Z^m_0 \\ \vdots \\ Z^m_{N^+-1} \end{pmatrix} + \frac{1}{2}\lambda \begin{pmatrix} u^{m+1}_{N^-} \\ 0 \\ \vdots \\ 0 \\ u^{m+1}_{N^+} \end{pmatrix} \tag{4.26}$$

Then we get

$$Bu^{(m+1)} = b^{(m)} \tag{4.27}$$

It is obvious that the only difference between the Crank-Nicolson method and the fully implicit method is that λ in the fully implicit method is the substitution of $\frac{1}{2}\lambda$ in the Crank-Nicolson method.

So the left-hand side of (4.25) can be solved explicitly similar to the forward difference scheme, while the right-hand side of (4.25) can be derived similar to the backward difference scheme using either LU method or SOR method given the boundary conditions and the initial condition described in (4.8), (4.9) and (4.10).

4.5 Discretization of the general form of the PDE

In our code, a general form of a parabolic partial differential equation with non-constant coefficients, Dirichlet boundary conditions and an initial condition, reads

$$\frac{\partial u}{\partial t} = \alpha(x)\frac{\partial^2 u}{\partial x^2} + \beta(x)\frac{\partial u}{\partial x} + \gamma(x)u(x,t) + f(x,t) \tag{4.28}$$

$$u(a,t) = L(t) \tag{4.29}$$

$$u(b,t) = R(t) \tag{4.30}$$

$$u(x,0) = \phi(x) \tag{4.31}$$

These equations are solved on a grid with N points on interval $[a, b]$. Let $x_i = a + ih$, where $h = (b - a)/N$. Rewrite (4.28) in matrix form:

$$\frac{du}{dt} = Au + b(t) + f(t) \tag{4.32}$$

4.5.1 Fourth order accuracy

Apply Taylor's expansion with the neighbor points, we can obtain the fourth order accurate discretization of (4.28). Write $\Delta x = h$ for ease, in the points $x_{i\pm1}$,

$$
\begin{aligned}
u(x_{i\pm1}) = u(x_i) \pm h\frac{\partial u}{\partial x}|_{x_i} + \frac{1}{2}h^2\frac{\partial^2 u}{\partial x^2}|_{x_i} \pm \frac{1}{6}h^3\frac{\partial^3 u}{\partial x^3}|_{x_i} \\
+\frac{1}{24}h^4\frac{\partial^4 u}{\partial x^4}|_{x_i} \pm \frac{1}{120}h^5\frac{\partial^5 u}{\partial x^5}|_{x_i} + O(h^6)
\end{aligned}
\tag{4.33}
$$

And in the points $x_{i\pm2}$,

$$
\begin{aligned}
u(x_{i\pm2}) = u(x_i) \pm 2h\frac{\partial u}{\partial x}|_{x_i} + 2h^2\frac{\partial^2 u}{\partial x^2}|_{x_i} \pm \frac{4}{3}h^3\frac{\partial^3 u}{\partial x^3}|_{x_i} \\
+\frac{2}{3}h^4\frac{\partial^4 u}{\partial x^4}|_{x_i} \pm \frac{4}{15}h^5\frac{\partial^5 u}{\partial x^5}|_{x_i} + O(h^6)
\end{aligned}
\tag{4.34}
$$

With (4.33) and (4.34) and assuming that all derivatives exist, the fourth order approximations of the derivatives are

$$
\frac{\partial^2 u}{\partial x^2}|_{x_i} + O(h^4) = \frac{1}{12h^2}(-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}) \tag{4.35}
$$

$$
\frac{\partial u}{\partial x}|_{x_i} + O(h^4) = \frac{1}{12h}(-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}) \tag{4.36}
$$

Combine (4.35) and (4.36), we obtain

$$
\begin{aligned}
\frac{\partial u_i}{\partial t} = \frac{1}{12h^2}(-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}) \\
+\frac{1}{12h}(-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}) + \gamma_i u_i + f_i(t)
\end{aligned}
\tag{4.37}
$$

In the fourth order scheme, points x_1 and x_2 need special treatment on the left boundary and x_{N-1} and x_{N-2} need special treatment on the right boundary. The equation of the system at point x_2 is:

$$
\begin{aligned}
\frac{\partial u_2}{\partial t} = \frac{1}{12h^2}\alpha_2(-u_4 + 16u_3 - 30u_2 + 16u_1 - u_0) \\
+\frac{1}{12h}\beta_2(-u_4 + 8u_3 - 8u_1 + u_0) + \gamma_2 u_2 + f_2(t)
\end{aligned}
\tag{4.38}
$$

and at point x_1:

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} = \frac{1}{12h^2}\alpha_2(-u_3 + 16u_2 - 30u_1 + 16u_0 - u_{-1}) \\
+\frac{1}{12h}\beta_2(-u_3 + 8u_2 - 8u_0 + u_{-1}) + \gamma_1 u_1 + f_1(t)
\end{aligned}
\tag{4.39}
$$

The fourth order ghost value u_{-1} can be derived by extrapolation:

$$
u_{-1} = 4u_0 - 6u_1 + 4u_2 - u_3 + O(h^4) \tag{4.40}
$$

The situation for the points x_{N-2} and x_{N-1} on the right boundary is similar as the points on the left boundary.

The elements in matrix A of (4.32) read:

$$\begin{aligned}
a_{ii} &= -\frac{15}{4h^2}\alpha_i + \gamma_i \\
a_{ii+1} &= \frac{4}{3h^2}\alpha_i + \frac{4}{h}\beta_i \\
a_{ii-1} &= \frac{4}{3h^2}\alpha_i - \frac{4}{h}\beta_i \\
a_{ii+2} &= -\frac{1}{12h^2}\alpha_i - \frac{1}{12h}\beta_i \\
a_{ii-2} &= -\frac{1}{12h^2}\alpha_i + \frac{1}{12h}\beta_i
\end{aligned} \tag{4.41}$$

Correct the first and last row from (4.41)

$$\begin{aligned}
a_{11} &= -\frac{2}{h^2}\alpha(a+h) - \frac{1}{2h}\beta(a+h) + \gamma(a+h) \\
a_{12} &= \frac{1}{h^2}\alpha(a+h) + \frac{1}{h}\beta(a+h) \\
a_{13} &= -\frac{1}{6h}\beta(a+h) \\
a_{N-1,N-1} &= -\frac{2}{h^2}\alpha(b-h) + \frac{1}{2h}\beta(b-h) + \gamma(b-h) \\
a_{N-1,N-2} &= \frac{1}{h^2}\alpha(b-h) - \frac{1}{h}\beta(b-h) \\
a_{N-1,N-3} &= \frac{1}{6h}\beta(b-h)
\end{aligned} \tag{4.42}$$

Apply (4.40), we obtain the vector b in (4.32):

$$b_i = \begin{cases}
\left(\frac{\alpha(a+h)}{h^2} - \frac{\beta(a+h)}{3h}\right)L(t) & i = 1 \\
\left(-\frac{\alpha(a+2h)}{12h^2} + \frac{\beta(a+2h)}{12h}\right)L(t) & i = 2 \\
0 & 3 \le i \le N-3 \\
\left(-\frac{\alpha(b-2h)}{12h^2} - \frac{\beta(b-2h)}{12h}\right)R(t) & i = N-2 \\
\left(\frac{\alpha(b-h)}{h^2} + \frac{\beta(b-h)}{3h}\right)R(t) & i = N-1
\end{cases} \tag{4.43}$$

4.5.2 Coordinate transformation with stretching

The finite difference methods are based on the derivatives provided by the Taylor expansion, however, in option pricing the final conditions are typically not differentiable. A local grid refinement around the non-differentiable payoff condition can improve the accuracy. The basic idea of the refinement is using more points near the non-differentiable condition. A *coordinate transformation with stretching* will be applied. After an analytic coordinate transformation, the original differential equation changes. An equidistant grid is then used in the discretization in (4.28), while this discretization can also be used after the transformation, as only the coefficients of the derivatives change.

Make a coordinate transformation $y = \psi(x)$, with the inverse $x = \varphi(y) = \psi^{-1}(y)$, by the chain rule we obtain the first order derivative:

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{1}{\varphi'(y)}\frac{du}{dy} \tag{4.44}$$

and the second order derivative reads:

$$\frac{d^2u}{dx^2} = \frac{1}{(\varphi'(y))^2}\frac{d^2u}{dy^2} - \frac{\varphi''(y)}{(\varphi'(y))^3}\frac{du}{dy} \tag{4.45}$$

Put (4.44) and (4.45) into (4.28), the transformation of the coefficients α, β and γ will be:

$$\begin{aligned} \widehat{\alpha}(y) &= \frac{\alpha\varphi(y)}{(\varphi'(y))^2} \\ \widehat{\beta}(y) &= \frac{\beta\varphi(y)}{\varphi'(y)} - \alpha(\varphi(y))\frac{\varphi''(y)}{(\varphi'(y))^3} \\ \widehat{\gamma}(y) &= \gamma(\varphi(y)) \end{aligned} \tag{4.46}$$

The left and right boundaries are transformed to $\psi(a)$ and $\psi(b)$. Then the new step of the transformed equation is $\hat{h} = ((\psi(b) - \psi(a)))/N$. In option pricing, the transformation

$$y = \psi(S) = sinh^{-1}(S - E) + sinh^{-1}E \tag{4.47}$$

is used here, where ψ is a monotonically increasing function. In this case, a specific point can be focused on by the grid refinement, for instance at the strike price E. The transformation applied in this thesis is

$$y = \psi(S) = sinh^{-1}(\mu(S - E)) + sinh^{-1}(\mu E) \tag{4.48}$$

where μ is the rate of stretching.

$\varphi(y)$ and its derivatives can be derived as following:

$$\varphi(y) = \frac{1}{\mu}sinh(y - sinh^{-1}(\mu S_0)) + S_0 \tag{4.49}$$

$$\varphi'(y) = \frac{1}{\mu}cosh(y - sinh^{-1}(\mu S_0)) \tag{4.50}$$

$$\varphi''(y) = \frac{1}{\mu}sinh(y - sinh^{-1}(\mu S_0)) \tag{4.51}$$

The time grid can also be generated by the fourth order discretization method. Apply a transformation of the equation which is backward in time into an equation forward in time, we get

$$(\frac{25}{12}I - kL)u^{m+1} = 4u^m - 3u^{m-1} + \frac{4}{3}u^{m-2} + \frac{1}{4}u^{m-3} \tag{4.52}$$

with the time step k, the identity matrix I. And L here represents the discrete version of

$$L = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} - r$$

With the fourth order approximation, the result can be obtained accurately. The detail of the fourth order Crank-Nicolson method is omitted for its similarity of the second order scheme.

4.6 Methods for American Options

The situation of calculating American options is not as straightforward as their European counterparts. Since there can be early exercised, which give rise to a free boundary problem. To simplify the free boundary problem, we try to transform the original problem to a fixed boundary problem, and deal with the free boundary afterwards. Recall Chapter 3, the American option pricing problem can be given as the linear complementary from:

$$\begin{gathered}(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2})(u(x,\tau) - g(x,\tau)) = 0 \\ (\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}) \geq 0, \quad (u(x,\tau) - g(x,\tau)) \geq 0\end{gathered} \tag{4.53}$$

where the transformed payoff function $g(x,\tau)$ is given by

$$g(x,\tau) = e^{\frac{1}{2}(k+1)^2\tau} max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0) \tag{4.54}$$

for the American put option, and the payoff function for the American call option is similar,

$$g(x,\tau) = e^{\frac{1}{2}(k+1)^2\tau} max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0) \tag{4.55}$$

The initial and fixed boundary conditions are

$$\begin{gathered}u(x,0) = g(x,0), \\ u(x,\tau) \text{ is continuous}, \\ \frac{\partial u}{\partial x}(x,\tau) \text{ is as continuous as } g(x,\tau), \\ \lim_{x\to\pm\infty} u(x,\tau) = \lim_{x\to\pm\infty} g(x,\tau)\end{gathered} \tag{4.56}$$

The linear complementary formulation (4.53) does not treat the free boundary explicitly. If one solves (4.53), then one can find the free boundary $x = x_f(\tau)$ by the following conditions:

$$u(x_f(\tau), \tau) = g(x_f(\tau), \tau), \quad \text{but for } x > x_f(\tau), u(x, \tau) > g(x, \tau)$$

for the American put option, and

$$u(x_f(\tau), \tau) = g(x_f(\tau), \tau), \quad \text{but for } x > x_f(\tau), u(x, \tau) < g(x, \tau)$$

for the American call option.

The procedure to solve the linear complementary problem is as follows.

Divide the (x, τ)-plane into a regular finite grid as usual. Then apply the Crank-Nicolson method to the inequality of

$$(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}) \geq 0$$

then

$$\frac{u_n^{m+1} - u_n^m}{\Delta \tau} \geq \frac{1}{2}(\frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\Delta x)^2} + \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\Delta x)^2}) \tag{4.57}$$

rearrange (4.57),

$$u_n^{m+1} - \frac{1}{2}\lambda(u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}) \geq u_n^m + \frac{1}{2}\lambda(u_{n+1}^m - 2u_n^m + u_{n-1}^m) \tag{4.58}$$

where $\lambda = \Delta\tau/(\Delta)^2$. Then define

$$g_n^m = g(n\Delta x, m\Delta\tau) \tag{4.59}$$

Discretise the condition $u(x, \tau) \geq g(x, \tau)$ as

$$u_n^m \geq g_n^m \quad \text{for } m \geq 1 \tag{4.60}$$

And the initial and boundary conditions are

$$u_{N^-}^m = g_{N^-}^m, \quad u_{N^+}^m = g_{N^+}^m, \quad u_n^0 = g_n^0 \tag{4.61}$$

where $-N^-$ and N^+ are both large numbers satisfiying

$$N^-\Delta x \leq x = n\Delta x \leq N^+\Delta x$$

Define

$$Z_n^m = (1-\lambda)u_n^m + \frac{1}{2}\lambda(u_{n-1}^m + u_{n+1}^m) \tag{4.62}$$

Then (4.58) becomes

$$(1+\lambda)u_n^{m+1} - \frac{1}{2}\lambda(u_{n-1}^{m+1} + u_{n+1}^{m+1}) \geq Z_n^m \tag{4.63}$$

At time-step $(m+1)\Delta\tau$, one can find Z_n^m explicitly, for one has already knows u_n^m. Equation

$$(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2})(u(x,\tau) - g(x,\tau)) = 0 \tag{4.64}$$

can be approximated by

$$((1+\lambda)u_n^{m+1} - \frac{1}{2}\lambda(u_{n-1}^{m+1} + u_{n+1}^{m+1}) - Z_n^m)(u_n^{m+1} - g_n^{m+1}) = 0 \tag{4.65}$$

Now we give the matrix notation of the

$$u^m = \begin{pmatrix} u_{N^-+1}^m \\ \vdots \\ u_{N^+-1}^m \end{pmatrix}, \quad g^m = \begin{pmatrix} g_{N^-+1}^m \\ \vdots \\ g_{N^+-1}^m \end{pmatrix} \tag{4.66}$$

Discarding the boundary values $u_{N^-}^m$ and $u_{N^+}^m$ as they can be determined by the boundary conditions. Let b^m defined by

$$b^m = \begin{pmatrix} b_{N^-+1}^m \\ \vdots \\ b_0^m \\ \vdots \\ b_{N^+-1}^m \end{pmatrix} = \begin{pmatrix} Z_{N^-+1}^m \\ \vdots \\ Z_0^m \\ \vdots \\ Z_{N^+-1}^m \end{pmatrix} + \frac{1}{2}\lambda \begin{pmatrix} g_{N^-}^{m+1} \\ 0 \\ \vdots \\ 0 \\ b_{N^+}^{m+1} \end{pmatrix} \tag{4.67}$$

If one introduces the $(N^+ - N^- - 1)$, with the symmetric tridiagonal matrix

$$C = \begin{pmatrix} 1+\lambda & -\frac{\lambda}{2} & & & 0 \\ -\frac{\lambda}{2} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & \ddots \end{pmatrix} \tag{4.68}$$

then the linear complementary problem (4.53) can be given by the matrix notation:

$$\begin{gathered} (u^{(m+1)} - g^{(m+1)})(Cu^{(m+1)} - b^{(m)}) = 0 \\ Cu^{(m+1)} \geq b^{(m)}, \quad u^{(m+1)} \geq g^{(m+1)} \end{gathered} \tag{4.69}$$

The time-stepping is inherent in the approach. Similar to the SOR method, one can use the *projected SOR* method which is a modification of SOR to solve (4.69).

Apply in the Crank-Nicolson method the SOR algorithm we obtain

$$\begin{aligned} y_n^{m+1,k+1} &= \frac{1}{1+\lambda}(b_n^m + \frac{1}{2}\lambda(u_{n-1}^{m+1,k+1} + u_{n+1}^{m+1,k})) \\ u_n^{m+1,k+1} &= u_n^{m+1,k} + \omega(y_n^{m+1,k+1} - u_n^{m+1,k}) \end{aligned} \tag{4.70}$$

To satisfy the constraint $u^{(m+1)} \geq g^{(m+1)}$, rewrite the second equation of (4.70),

$$\begin{aligned} y_n^{m+1,k+1} &= \frac{1}{1+\lambda}(b_n^m + \frac{1}{2}\lambda(u_{n-1}^{m+1,k+1} + u_{n+1}^{m+1,k})) \\ u_n^{m+1,k+1} &= max(u_n^{m+1,k} + \omega(y_n^{m+1,k+1} - u_n^{m+1,k}), g_n^{m+1}) \end{aligned} \tag{4.71}$$

Where $u_n^{m+1,k+1}$ is used immediately to calculate $u_{n+1}^{m+1,k+1}$. And put $u^{m+1} = u^{m+1,k+1}$ when $\parallel u^{m+1,k+1} - u^{m+1,k} \parallel < \epsilon$, where ϵ is a small number.

Chapter 5

Parameter Study based on Black-Scholes Equation

There are several factors influencing the option price, such as volatility, dividend, time to maturity, strike price and risk-free interest rate. In this chapter, only dividend amount and interest rate will be discussed for the American options.

5.1 Effect of single dividend and interest rate

With one discrete dividend D, the solution of the European call can be written as

$$C_E(S,t) = (S - De^{-r(t_d - t)})N(d_1) - Ee^{-r(T-t)}N(d_2) \tag{5.1}$$

where t_d is the ex-dividend date. And the solution to the European put is

$$P_E(S,t) = Ee^{-r(T-t)}N(-d_2) - (S - De^{-r(t_d - t)})N(-d_1) \tag{5.2}$$

In order to examine the behavior of the option price provided the change of dividend amount D and interest rate r, put the parameters needed by the Black-Scholes equation as strike price $E = 22$, time to maturity $T = 1$, ex-dividend date $t_d = 0.5$ and implied volatility $\sigma = 0.25$.

For the American put, when the dividend amount D is fixed, the option prices decrease as interest rate r increases, see Table (5.1). Since the discounted strike price $Ee^{-r(T-t)}$ and the discounted dividend amount $De^{-r(t_d-t)}$ becomes smaller when r goes up, the American put price given by (5.2) goes down.

Secondly, we fix the interest rate r. As the dividend amount D goes up, the American put price P also goes up, see Table (5.2). That is because when

Grid	20 by 20	40 by 40	80 by 80
r=0.015	P=2.5513	P=2.5563	P=2.5568
r=0.020	P=2.5009	P=2.5046	P=2.5052
r=0.025	P=2.4513	P=2.4544	P=2.4522
r=0.015	C=1.9621	C=1.9621	C=1.9626
r=0.020	C=1.9983	C=1.9990	C=1.9991
r=0.025	C=2.0347	C=2.0362	C=2.0360

Table 5.1: Test for American option for fixed D=1, changing r, where C for call option price, P for put option price

Grid	20 by 20	40 by 40	80 by 80
D=0.9	P=2.4463	P=2.4498	P=2.4504
D=1.0	P=2.5009	P=2.5046	P=2.5052
D=1.1	P=2.5564	P=2.5601	P=2.5607
D=0.9	C=2.0229	C=2.0251	C=2.0253
D=1.0	C=1.9983	C=1.9990	C=1.9991
D=1.1	C=1.9743	C=1.9744	C=1.9748

Table 5.2: Test for American option for fixed r=0.02, changing D, where C for call option price, P for put option price

D increases, the put price given by (5.2)increases as the the discounted dividend amount $De^{-r(t_d-t)}$ increases and other parts remain the same.

Contrary to the American put, the American call prices C increase as the interest rate increases, see Table (5.1). Applying (5.1), as r goes up, the discounted strike price $Ee^{-r(T-t)}$ and the discounted dividend amount $De^{-r(t_d-t)}$ go down, so that the call option prices go up.

When D increases, the American call prices decrease since S becomes smaller, for the discounted dividend amount $De^{-r(t_d-t)}$ also gets larger with other parts unchanged, see Table (5.2).

The call option price rises when the interest rate rises. That is because options are priced on a risk-neutral basis, i.e. on a delta-neutral basis. So a long call would be paired with a short-stock, and a short-stock position generates interest revenue. That makes the call option worth more. If interest rates go up, the interest revenue from the short stock position increases, which makes the call worth still more. Note that for put options it works the opposite way. Dividends also work in the opposite direction.

grid	r	$D_{1,2} = 0.5, 0.5$	$D_{1,2} = 0.45, 0.55$	$D_{1,2} = 0.55, 0.45$
	r=0.015	2.5436	2.5414	2.5458
20 by 20	r=0.020	2.4869	2.4847	2.4890
	r=0.025	2.4309	2.4288	2.4330
	r=0.015	2.5457	2.5434	2.5480
40 by 40	r=0.020	2.4890	2.4868	2.4912
	r=0.025	2.4336	2.4314	2.4357
	r=0.015	2.5462	2.5439	2.5485
80 by 80	r=0.020	2.4896	2.4874	2.4918
	r=0.025	2.4342	2.4320	2.4363

Table 5.3: Test for American put price P

So, in an optimization strategy where r and D are parameters to be optimized multiple optimal solutions (r up, D down, or vice versa) are expected.

Table (5.1) and (5.2) show that we obtain stable, converged put and call prices with 20-80 grid points in space and time. We will use 40 points in the calibration to follow.

5.2 Effect of two dividends and interest rate

The solution to the two dividends Black-Scholes model is similar to that of the single dividend model, such as

$$C_E(S,t) = (S - D_1 e^{-r(t_{d1}-t)} - D_2 e^{-r(t_{d2}-t)})N(d_1) - Ee^{-r(T-t)}N(d_2) \quad (5.3)$$

where t_{d1} and t_{d2} are the ex-dividend dates and D_1, D_2 are the dividend amounts. The solution to the European put is

$$P_E(S,t) = Ee^{-r(T-t)}N(-d_2) - (S - D_1 e^{-r(t_{d1}-t)} - D_2 e^{-r(t_{d2}-t)})N(-d_1) \quad (5.4)$$

To illustrate how two dividends D_1, D_2 influence the option price with risk-free interest rate, we put $E = 22$, $t_{d1} = 0.25$, $t_{d2} = 0.75$, $\sigma = 0.25$ and $T = 1$, and change D_1, D_2 simultaneously with different r. Let $D = D_1 + D_2 = 1$. Table (5.3) illustrates that the American put, as r increases, the option prices decrease no matter how D_1, D_2 change. As $D = D_1 + D_2$ holds constant, D_1 and D_2 behave reversely. When t_{d1} and t_{d2} are fixed, the option price given by $D_1 > D_2$ is higher than that given by $D_1 < D_2$. When applying (5.4), put our attention to $D_1 e^{-r(t_{d1}-t)} + D_2 e^{-r(t_{d2}-t)}$, we define a function of D_1 as

$$f(D_1) = D_1 e^{-r(t_{d1}-t)} + (1 - D_1)e^{-r(t_{d2}-t)} \quad (5.5)$$

grid	r	$D_{1,2} = 0.5, 0.5$	$D_{1,2} = 0.45, 0.55$	$D_{1,2} = 0.55, 0.45$
	r=0.015	1.9351	1.9212	1.9493
20 by 20	r=0.020	1.9742	1.9601	1.9885
	r=0.025	2.0136	1.9993	2.0282
	r=0.015	1.9366	1.9225	1.9511
40 by 40	r=0.020	1.9751	1.9605	1.9904
	r=0.025	2.0146	2.0000	2.0301
	r=0.015	1.9366	1.9226	1.9515
80 by 80	r=0.020	1.9753	1.9610	1.9905
	r=0.025	2.0148	2.0003	2.0302

Table 5.4: Test for American call price C

r	$D_1 = 0.4$	$D_1 = 0.5$	$D_1 = 0.6$
r=0.015	C=1.9632	C=1.9454	C=1.9304
r=0.020	C=2.0024	C=1.9845	C=1.9687
r=0.025	C=2.0428	C=2.0241	C=2.0074

Table 5.5: Test for American call price for fixed r and D_2, changing D_1

where $e^{-r(t_{d1}-t)}$ and $e^{-r(t_{d2}-t)}$ are fixed. As $t < t_{d1} < t_{d2}$, $e^{-r(t_{d1}-t)} > e^{-r(t_{d2}-t)}$. Rewrite (5.5)

$$f(D_1) = (e^{-r(t_{d1}-t)} - e^{-r(t_{d2}-t)})D_1 + e^{-r(t_{d2}-t)} \tag{5.6}$$

So $f(D_1)$ is an monotonically increasing function, then the put option prices go up when D_1 goes up.

Table (5.4) illustrates that the American call, as r increases, the option prices increase no matter how D_1, D_2 change. For the American put, if there are two dividend payments with a constant sum during the option's lifetime, the larger the first amount of the dividend D_1, the bigger the option price is. However, this result is confusing because as D_1 goes up, the call option prices should go down. To illustrate this, we fix $D_2 = 0.5$, In this case, the call option prices go down when D_1 goes up, See Table (5.5).

In the case of American call calculated by the 40 by 40 grid with the parameter $r = 0.02, D1 = 0.49, D2 = 0.51$, we find the option price $C = 1.9717$. It is quite close to the case of $r = 0.02, D1 = D2 = 0.5$, which is $C = 1.9751$.

In the view of accuracy, the solution given by the fine 80 by 80 grid can be regarded as the standard result. And the coarse 20 by 20 grid does not perform as good as the 40 by 40 grid. To save CPU time, the calibration

Grid	20 by 20	40 by 40	80 by 80
r=0.015;	P=2.5006	P=2.5044	P=2.5051
r=0.020;	P=2.4490	P=2.4523	P=2.4531
r=0.025;	P=2.3992	P=2.4020	P=2.4026
r=0.015;	C=1.9035	C=1.9621	C=1.9057
r=0.020;	C=1.9411	C=1.9425	C=1.9427
r=0.025;	C=1.9789	C=1.9057	C=1.9802

Table 5.6: Test for American option for fixed D=1, changing r with volatility correction (2.33), where C for call option price, P for put option price

methods introduced in chapter 6 will apply the 40 by 40 grid to balance the performance and the accuracy.

5.3 Effect of the parameters using volatility adjustment

5.3.1 Effect of single dividend and interest rate using volatility adjustment

During the life time of the option we have a single dividend payment D at t_d. Due to the absence of arbitrage:

$$S(t_d^+) = S(t_d^-) - D \tag{5.7}$$

where t_d^+ and t_d^- are the instants immediately before and after the ex-dividend date. The value V of the option must be smooth as a function of time over the time of payment

$$V(S(t_d^+), t_d^+) = V(S(t_d^-), t_d^-) \tag{5.8}$$

We distinguish the following approaches to include a discrete dividend.

The volatility correction (2.33) is volatility correction after t_d and (2.34) is volatility correction before t_d which are both applied in the following results. The result given by (2.33) see Tables (5.6), (5.7). The result given by (2.34) see Tables (5.8), (5.9).

Both the volatility correction models behave similarly to the classical Wilmott's model. The volatility correction method before t_d provides the largest option values while the volatility correction method after t_d provides the smallest values, and Wilmott's model gives the results which are almost the average of the former two.

Grid	20 by 20	40 by 40	80 by 80
D=0.9	P=2.3995	P=2.4027	P=2.4036
D=1.0	P=2.4490	P=2.4523	P=2.4531
D=1.1	P=2.4993	P=2.5026	P=2.5034
D=0.9	C=1.9740	C=1.9747	C=1.9753
D=1.0	C=1.9411	C=1.9425	C=1.9427
D=1.1	C=1.9087	C=1.9116	C=1.9120

Table 5.7: Test for American option for fixed r=0.02, changing D with volatility correction (2.33), where C for call option price, P for put option price

Grid	20 by 20	40 by 40	80 by 80
r=0.015;	P=2.5977	P=2.6015	P=2.6021
r=0.020;	P=2.5450	P=2.5490	P=2.5497
r=0.025;	P=2.4948	P=2.4980	P=2.4989
r=0.015;	C=1.9995	C=2.0002	C=2.0007
r=0.020;	C=2.0364	C=2.0367	C=2.0375
r=0.025;	C=2.0737	C=2.0747	C=2.0751

Table 5.8: Test for American option for fixed D=1, changing r with volatility correction (2.34), where C for call option price, P for put option price

Grid	20 by 20	40 by 40	80 by 80
D=0.9	P=2.4861	P=2.4899	P=2.4907
D=1.0	P=2.5450	P=2.5490	P=2.5497
D=1.1	P=2.6046	P=2.6087	P=2.6094
D=0.9	C=2.0593	C=2.0611	C=2.0611
D=1.0	C=2.0364	C=2.0367	C=2.0375
D=1.1	C=2.0141	C=2.0154	C=2.0156

Table 5.9: Test for American option for fixed r=0.02, changing D with volatility correction (2.34), where C for call option price, P for put option price

grid	r	$D_{1,2} = 0.5, 0.5$	$D_{1,2} = 0.45, 0.55$	$D_{1,2} = 0.55, 0.45$
20 by 20	r=0.015	P=2.5048	P=2.5038	P=2.5058
	r=0.020	P=2.4478	P=2.4469	P=2.4488
	r=0.025	P=2.3921	P=2.3912	P=2.3930
40 by 40	r=0.015	P=2.5078	P=2.5067	P=2.5088
	r=0.020	P=2.4512	P=2.4501	P=2.4522
	r=0.025	P=2.3957	P=2.3947	P=2.3966
80 by 80	r=0.015	P=2.5086	P=2.5075	P=2.5096
	r=0.020	P=2.4518	P=2.4508	P=2.4529
	r=0.025	P=2.3963	P=2.3954	P=2.3973

Table 5.10: Test for American put with volatility correction (2.33)

grid	r	$D_{1,2} = 0.5, 0.5$	$D_{1,2} = 0.45, 0.55$	$D_{1,2} = 0.55, 0.45$
20 by 20	r=0.015	C=1.8938	C=1.8815	C=1.9087
	r=0.020	C=1.9326	C=1.9201	C=1.9477
	r=0.025	C=1.9729	C=1.9604	C=1.9871
40 by 40	r=0.015	C=1.8970	C=1.8846	C=1.9110
	r=0.020	C=1.9358	C=1.9231	C=1.9499
	r=0.025	C=1.9755	C=1.9627	C=1.9896
80 by 80	r=0.015	C=1.8976	C=1.8851	C=1.9112
	r=0.020	C=1.9363	C=1.9235	C=1.9503
	r=0.025	C=1.9761	C=1.9629	C=1.9902

Table 5.11: Test for American call with volatility correction (2.33)

5.3.2 Effect of two dividends and interest rate using volatility correction

The situation of two discrete dividends model using volatility correction is similar to the non-volatility correction model. So we just put the results here. The result given by (2.33) see Table (5.10) and (5.11). The result given by (2.34) see Table (5.12) and (5.13).

Although the interest rate and the dividend are not the primary factors affecting an option's price, the option trader should still be aware of their effects. In fact, the primary drawback we have seen in many of the option analysis tools available is that they use a simple Black Scholes model which can only give the analytical solution to the European style options. While in the real market, the options traded are usually American style options. It is better to derive much accurate tools to model the American options by checking the possibility of early exercise.

grid	r	$D_{1,2} = 0.5, 0.5$	$D_{1,2} = 0.45, 0.55$	$D_{1,2} = 0.55, 0.45$
20 by 20	r=0.015	P=2.5538	P=2.5504	P=2.5571
	r=0.020	P=2.4965	P=2.4933	P=2.4998
	r=0.025	P=2.4409	P=2.4377	P=2.4441
40 by 40	r=0.015	P=2.5565	P=2.5531	P=2.5599
	r=0.020	P=2.4998	P=2.4965	P=2.5031
	r=0.025	P=2.4443	P=2.4411	P=2.4475
80 by 80	r=0.015	P=2.5573	P=2.5539	P=2.5607
	r=0.020	P=2.5006	P=2.4973	P=2.5039
	r=0.025	P=2.4449	P=2.4417	P=2.4482

Table 5.12: Test for American put with volatility correction (2.34)

grid	r	$D_{1,2} = 0.5, 0.5$	$D_{1,2} = 0.45, 0.55$	$D_{1,2} = 0.55, 0.45$
20 by 20	r=0.015	C=1.9420	C=1.9277	C=1.9591
	r=0.020	C=1.9816	C=1.9670	C=1.9979
	r=0.025	C=2.0219	C=2.0071	C=2.0370
40 by 40	r=0.015	C=1.9454	C=1.9308	C=1.9614
	r=0.020	C=1.9845	C=1.9696	C=2.0001
	r=0.025	C=2.0241	C=2.0089	C=2.0400
80 by 80	r=0.015	C=1.9460	C=1.9313	C=1.9617
	r=0.020	C=1.9849	C=1.9698	C=2.0008
	r=0.025	C=2.0245	C=2.0092	C=2.0406

Table 5.13: Test for American call with volatility correction (2.34)

Chapter 6

Calibration of the Implied Variables

We use the data of the ING Groep NV to calibrate the implied interest rate, implied dividend and implied volatility. And the data from Fortis Bank will be used to test the calibration approach. The data is in given the form of the following format for one time point in a specific day:

22.140				
Strike price	CallBid	CallAsk	PutBid	PutAsk
15.000	7.150	7.250	0.000	0.050
16.000	6.150	6.300	0.000	0.100
17.000	5.200	5.300	0.050	0.100
18.000	4.200	4.300	0.100	0.150
19.000	3.250	3.350	0.150	0.250
20.000	2.400	2.450	0.300	0.400
21.000	1.550	1.650	0.550	0.650
22.000	0.900	0.950	0.950	1.050
23.000	0.450	0.500	1.550	1.650
24.000	0.150	0.250	2.350	2.450
25.000	0.050	0.150	3.250	3.350

Table 6.1: Data of ING on 03-Feb-2005

In Table (6.1), the value 22.140 is the asset closing price at 03-Feb-2005. The values of the first column from 15.000 to 25.000 are the strike prices. The second column is the call option price for bid, the third column is the call option price for ask, the fourth column is the put option price for bid and the last column is the put option price for ask.

To do the calibration, we use two approaches supplied by Matlab. One is fminsearch, the other one is fmincon. Both of these methods will be discussed in the following sections.

6.1 Implied volatility and implied dividend

Implied volatility is a theoretical value designed to represent the volatility of the security underlying an option as determined by the price of the option. In general, implied volatility decreases when the market is *bullish* (means people have an optimistic outlook for the market) and increases when the market is *bearish* (means people have a pessimistic outlook for the market). This is due to the common belief that bearish markets are more risky than bullish markets. Implied volatilities are often referred to as a "market consensus", which is an indication of risk that combines the insights of many market participants. However, implied volatilities are essentially parameters. They can be biased for some instances. The factors that affect implied volatility are the exercise price, the risk-free interest rate, the expiry date, the asset price and the price of the option.

Paying dividend is the one of most important ways of the business to fulfill the mission of creating profit for the owners. When a company earns a profit, some of this money is typically reinvested in the business and called retained earnings, and some of it can be paid to its shareholders as a dividend. The amount of the dividend is determined every year at the company's annual general meeting, and declared as either a cash amount or a percentage of the company's profit. When a share is sold shortly before the dividend is to be paid, the seller rather than the buyer is entitled to the dividend. At the point at which the buyer is not entitled to the dividend if the share is sold, the share is said to go ex-dividend. This is usually two business days before the dividend is to be paid, depending on the rules of the stock exchange. When a share goes ex-dividend, its price will generally fall by the amount of the dividend. **Implied dividend** in this paper is the market expectation of the dividend, which we want to test if it is close to the dividend having been announced by the company.

In the option pricing formulas, such as the Black Scholes formula, the only unknown parameter is the implied volatility of the underlying stock. Our purpose in option pricing is to find the implied volatility, given the observed price quoted in the market. For example, given C_0, a price of a call option, the following equation should be solved for the value of σ:

$$C_0 = C_{BS}(S, E, r, \sigma, T) \tag{6.1}$$

Actually, it is an inverse problem to solve σ in (6.1), and this equation has no closed form solution, which means the equation need to be numerically solved.

If there is a dividend payment during the option's lifetime, we can use an adjusted form of (6.1):

$$C_0 = C_{BS}(S - D, E, r, \sigma \frac{S}{S - D}, T) \tag{6.2}$$

$$C_0 = C_{BS}(S - e^{-rt_d}D, E, r, \sigma \frac{S}{S - e^{-rt_d}D}, T) \tag{6.3}$$

Adjustments (6.2) and (6.3) are similar, and the only difference is that (6.3) uses a discounted dividend D instead of D itself.

In this case, the unknown variables are both D and σ, and not only σ mentioned in (6.1), because we assume the dividend payment is also implied in the real market. So an optimization method has been derived to solve the two unknown variables problem:

$$\min_{(\sigma,D)} \sum_{i=1}^{N} |C_{market} - C(S, E_i, T, t_d, (\sigma_{imp}, D_{imp}))|^2 \tag{6.4}$$

Formula (6.4) is minimized to find the most suitable (σ, D) in our analytical model to fit the market option price.

6.2 Calibration Methods

I. fminsearch

fminsearch finds the minimum of a scalar function of several variables, starting at an initial estimate. This is generally referred to as unconstrained nonlinear optimization. To do this, using $x = fminsearch(fun, x0)$, which starts at the initial point $x0$ and finds a local minimum x of the function described in fun.

fminsearch uses the simplex search method named the Nelder-Mead algorithm [2]. Four scalar parameters must be specified to define a complete Nelder-Mead method: coefficients of reflection ρ, expansion χ, contraction γ, and shrinkage $\tilde{\sigma}$. According to the original Nelder-Mead paper, these parameters should satisfy

$$\rho > 0, \quad \chi > 1, \quad \chi > \rho, \quad 0 < \gamma < 1 \quad \text{and} \quad 0 < \tilde{\sigma} < 1$$

At the beginning of the kth iteration, $k \geq 0$, a nondegenerate simplex (a geometrical figure consisting of $N+1$ vertices in N dimensions, whereas the $N+1$ vertices span a N-dimensional vector space) Δk is given, along with its $n+1$ vertices, each of which is a point in R^n. It is always assumed that iteration k begins by ordering and labelling these vertices as $x_1^{(k)}, ..., x_{n+1}^{(k)}$, such that

$$f_1^{(k)} \leq f_2^{(k)} \leq ... \leq f_{n+1}^{(k)}$$

Where $f_i^{(k)}$ denotes $f(x_i^{(k)})$. The kth iteration generates a set of $n+1$ vertices that define a different simplex for the next iteration, so that $\Delta_{k+1} \neq \Delta_k$. Because we seek to minimize f, we refer to $x_1^{(k)}$ as the best point, to $x_{n+1}^{(k)}$ as the worst point. Similarly, we refer to $f_{n+1}^{(k)}$ as the worst function value, and so on.

One iteration of the Nelder-Mead algorithm:

1. Order. Order the $n+1$ vertices to satisfy $f(x_1) \leq f(x_2) \leq ... \leq f(x_{n+1})$, using the tie-breaking rules given below.

2. Reflection. Compute the reflection point x_r from

$$x_r = \bar{x} + \rho(\bar{x} - x_n + 1) = (1+\rho)\bar{x} - \rho x_{n+1}$$

where $\bar{x} = (1/n)\sum_{i=1}^{n} x_i$ is the centroid of the n best points (all points except for x_{n+1}).
Evaluate $f_r = f(x_r)$, if $f_1 \leq f_r < f_n$, accept the reflected point x_r and terminate the iteration.

3. Expand. If $f_r < f_1$, calculate the expansion point x_e,

$$x_e = \bar{x} + \chi(x_r - \bar{x}) = \bar{x} + \rho\chi(\bar{x} - x_{n+1}) = (1+\rho\chi)\bar{x} - \rho\chi x_{n+1}$$

and evaluate $f_e = f(x_e)$. If $f_e < f_r$, accept x_e and terminate the iteration; if $f_e \geq f_r$, accept x_r and terminate the iteration.

4.Contract. If $f_r \geq f_n$, perform a contraction between $\bar{x}$ and the better of x_{n+1} and x_r.

a.Outside. If $f_n \leq f_r < f_n + 1$, perform an outside contraction: calculate

$$x_c = \bar{x} + \gamma(x_r - \bar{x}) = \bar{x} + \gamma\rho(\bar{x} - x_{n+1}) = (1+\rho\gamma)\bar{x} - \rho\gamma x_{n+1}$$

Evaluate $f_c = f(x_c)$. If $f_c \leq f_r$, accept x_c and terminate the iteration; otherwise, go to step 5.
b.Inside. If $f_r \geq f_{n+1}$, perform an inside contraction:

$$x_{cc} = \bar{x} - \gamma(\bar{x} - x_{n+1}) = (1-\gamma)\bar{x} + \gamma x_{n+1}$$

Evaluate $f_{cc} = f(x_{cc})$. If $f_{cc} < f_{n+1}$, accept x_{cc} and terminate the iteration; otherwise go to step 5.

5. Perform a shrink step. Evaluate f at the n points $v_i = x_1 + \sigma(x_i - x_1)$, $i = 2, 3, ..., n+1$. The (unordered) vertices of the simplex at the next iteration consist of $x_1, v_2, ..., v_n + 1$.

The limitation is that fminsearch can only give local solutions, and fminsearch is a quite slow algorithm.

II. fimincon

fmincon attempts to find a constrained minimum of a scalar function of several variables starting at an initial estimate. This is generally referred to as constrained nonlinear optimization or nonlinear programming. In $x = fmincon(fun, x0, A, b, Aeq, beq, lb, ub)$, fun is the objective function, $x0$ is the starting point. A, b present the constrained linear inequalities $Ax \leq b$. Aeq, beq present the constrained linear equalities $Aeq \cdot x = beq$. And lb, ub give the lower and upper bounds of x, respectively.

In our situation, the medium-scale algorithm of fmincon is used, which uses the sequential quadratic programming (SQP) method with BFGS method [3].

The SQP implementation consists of three main stage:

1. Updating the Hessian Matrix

At each major iteration a positive definite quasi-Newton approximation of the Hessian of the Lagrangian function, H, is calculated using the BFGS method, where $\lambda_i (i = 1, ..., m)$ is an estimate of the Lagrange multipliers.

$$H_{k+1} = H_k + \frac{q_k q_k^T}{q_k^T s_k} - \frac{H_k^T H_k}{s_k^T H_k s_k} \quad \text{where}$$

$$s_k = x_{k+1} - x_k$$

$$q_k = \nabla f(x_{k+1}) + \sum_{i=1}^{n} \lambda_i \nabla g_i(x_{k+1}) - (\nabla f(x_k) + \sum_{i=1}^{n} \lambda_i \nabla g_i(x_k))$$

2.Quadratic Programming Solution

At each major iteration of the SQP method, a QP problem of the following form is solved

$$\begin{aligned} \text{minimize} \qquad & q(d) = \frac{1}{2} d^T H d + c^T d \\ & A_i d = b_i \quad i = 1, ..., m_e \\ & A_i d \leq b_i \quad i = m_e + 1, ..., m \end{aligned}$$

3.Line Search and Merit Function
The solution to the QP subproblem produces a vector d_k, which is used to form a new iterate

$$x_{k+1} = x_k + \alpha_k d_k$$

The step length parameter α_k is determined in order to produce a sufficient decrease in a merit function, given by the following implementation.

$$\Psi(x) = f(x) + \sum_{i=1}^{m_e} r_i g_i(x) + \sum_{i=m_e+1}^{m} r_i max(0, g_i(x))$$

In our model, only the upper and lower bounds of the variables are defined, such that implied interest rate $0.01 \leq r \leq 1$, implied dividend $0.01 \leq D \leq S0$ and implied volatility $0.01 \leq \sigma \leq 0.5$.

6.3 Comparison of fmincon and fminsearch

Compare the results derived by fminsearch and fmincon by taking the example of Table (6.1):

fminsearch							
r	D						
0.01886	0.89670						
sigma							
0.25688	0.21438	0.18943	0.18019	0.17082	0.16451	0.15506	0.15633
error							
0.00016							

Table 6.2: Test for fminsearch

fmincon							
r	D						
0.01883	0.89621						
sigma							
0.25650	0.21434	0.18944	0.18018	0.17079	0.16451	0.15507	0.15631
error							
0.00016							

Table 6.3: Test for fmincon

The CPU time for the fminsearch is $1442.648s$ and for fmincon is $232.664s$, and the calibrated results of these two methods are nearly the same. Given the similar results, fmincon performs much faster than fminsearch. So we choose fmincon as our main optimization approach.

6.4 Algorithm of the whole approach

The algorithm of the whole approach can be described by the picture below. Firstly, the program reads the all of the data from the directory; then we set the initial guess of the parameters which need to be calibrated; after that we use Black-Scholes equation to calculate the option price given the initial parameters; finally, fmincon is used to find the optimal solution of the parameters in such a way that if the value of the objective function (in this case it is the difference between the calibrated option prices and the market prices) is larger than a certain constant (in our case it is 10^{-6}), fmincon will update the initial parameters. The updated parameters will be set as the initial parameters and put into the algorithm to produce another value of the objective function and so on.

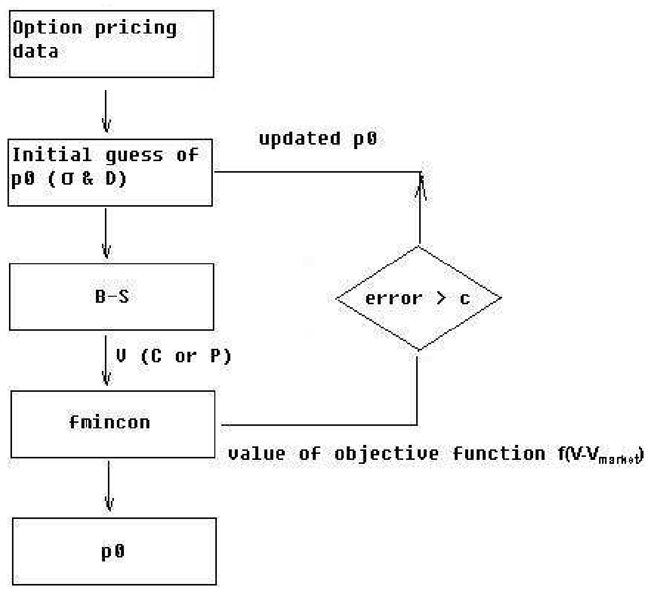

Figure 6.1: Algorithm of the whole approach

6.5 Objective Functions

The basic objective function we use is the Sum of Relative Error (SRE):

$$MSE = \sum_{i=1}^{n} \frac{[C^i_{calib} - C^i_{market}]^2}{C^i_{market}} + \sum_{i=1}^{n} \frac{[P^i_{calib} - P^i_{market}]^2}{P^i_{market}} \tag{6.5}$$

Where $C_{market} = (CallAsk+CallBid)/2$ and $P_{market} = (PutAsk+PutBid)/2$. Take the third row in Table (6.1) for example, where the strike price $E = 20$, and $CallBid = 2.600$, $CallAsk = 2.700$, $PutBid = 0.850$ and $PutAsk = 0.950$.Traders who make markets routinely quote two prices, one to buy (bid) and one to sell (ask), where his selling price is always higher than his buying price. This difference is known as the Bid-Ask spread. Essentially, the dealer is offering a put to sell to him at his bid price and a call to buy from him at his ask price. The price he charges for these two options is the Bid-Ask spread.

According to the bid-ask spread, we add some extra features to the objective function. In more detail, if the calibrated option price falls into the interval of $[Bid, Ask]$, we put a weight $\mu \in (0,1)$ in front of $\frac{[C^i_{calib} - C^i_{market}]^2}{C^i_{market}}$ and $\frac{[P^i_{calib} - P^i_{market}]^2}{P^i_{market}}$.

This objective function (6.5) does not perform very well when the calibrated option price may lay outside the interval of $[Bid, Ask]$. Because it is only using the average of Ask and Bid, while in the real market, the option price is sometimes close to Ask and sometimes close to Bid but outside the interval. An example will be illustrated later. So we derive another objective function to deal with the situation that the calibrated option prices lay outside the interval of $[Bid, Ask]$:

$$MSE = \sum_{i=1}^{n} C^i_{MSE} + \sum_{i=1}^{n} P^i_{MSE} \tag{6.6}$$

where

$$C^i_{MSE} = \begin{cases} \frac{[C^i_{calib} - CallBid^i]^2}{C^i_{market}} & C^i_{calib} < CallBid \\ \frac{[C^i_{calib} - CallAsk^i]^2}{C^i_{market}} & C^i_{calib} > CallAsk \\ \mu\frac{[C^i_{calib} - C^i_{market}]^2}{C^i_{market}} & C^i_{calib} \in [CallBid, CallAsk] \end{cases} \tag{6.7}$$

$$P^i_{MSE} = \begin{cases} \dfrac{[P^i_{calib} - PutBid^i]^2}{P^i_{market}} & P^i_{calib} < PutBid \\ \dfrac{[P^i_{calib} - PutAsk^i]^2}{P^i_{market}} & P^i_{calib} < PutAsk \\ \mu\dfrac{[P^i_{calib} - P^i_{market}]^2}{P^i_{market}} & P^i_{calib} \in [PutBid, PutAsk] \end{cases} \quad (6.8)$$

6.6 Calibration Results

6.6.1 Single dividend from ING

We first examine the single dividend case, and aim to deduce from the data when the dividend is announced and how the dividend amount of the ING behaves in 2005. The option data has been collected day by day, starting from 22-Jan-2005, and in this test the options expire on 17-Jun-2005. The date string 22-Jan-2005 can be transformed to the number 38373 by Matlab.

As we have already know that the ING Groep pays the dividend twice a year, in 2005 on 28-Apr-2005 and 12-Aug-2005. The dividend are €0.58 and €0.49, respectly. The dividend announcement date is 17-Feb-2005.

There are some aims we want to achieve:

- The first implied dividend amount after 17-Feb-2005 should be around 0.58
- The dividend announcement date should be made visible, if the market had a different guess of the size of the dividend
- The errors in the optimization should be reasonably small
- The calibrated results should be stable over several days

In the first test, all of the parameters are calibrated simultaneously. The results are in Table (6.4), where $Sigma_{15}$ is according to the implied volatility for the strike price 15. The calibrated risk-free interest rate is around 0.02, while the interest rate given by the European Central Bank is also around 0.02 in the first six months in 2005. To simplify the problem and to avoid multiple optimal solutions with unstable parameters among r and D, the risk-free interest rate r will be fixed at 0.02 in later calculations.

Using the objective function with (6.5), we obtain the results in Table (6.5). The implied dividend amount, one of the implied volatilities and the errors are given with the initial date, the expiry date and the fixed risk-free interest rate. It is obvious that there is a jump in the dividend amount at

Initial date	Expiry date	Interest rate	Dividend	$Sigma_{15}$	Error
38373	38527	0.021939	0.515946	0.296387	0.000155
38376	38527	0.020103	0.490714	0.307422	0.000364
38377	38527	0.020962	0.493104	0.279730	0.000085
38378	38527	0.021357	0.505352	0.276019	0.000341
38379	38527	0.021553	0.512003	0.305813	0.000197
38380	38527	0.018829	0.439513	0.276437	0.000063
38383	38527	0.025857	0.515767	0.281370	0.000081
38384	38527	0.021164	0.507776	0.318473	0.000496
38385	38527	0.018968	0.499693	0.286197	0.000115
38386	38527	0.023126	0.497379	0.287140	0.000278
38387	38527	0.022551	0.511048	0.292522	0.000377
38390	38527	0.020036	0.511128	0.300797	0.000318
38392	38527	0.021459	0.533613	0.298096	0.000275
38393	38527	0.022667	0.508263	0.298962	0.000456
38394	38527	0.020933	0.487833	0.312863	0.000156
38397	38527	0.022944	0.506561	0.318011	0.000143
38398	38527	0.019561	0.487557	0.322057	0.000221
38399	38527	0.018607	0.467708	0.318818	0.000125
38400	38527	0.020575	0.558542	0.324225	0.000273
38401	38527	0.021078	0.583934	0.322581	0.000381
38404	38527	0.022588	0.574570	0.324551	0.000784
38405	38527	0.021921	0.584837	0.320452	0.000396
38406	38527	0.022080	0.575445	0.320655	0.000303

Table 6.4: ING option with all parameters calibrated

38399 of the initial date, which is 17-Feb-2005. Before 17-Feb-2005, the dividend amount is around 0.50 which is the dividend amount paid by ING in 2004, while after 17-Feb-2005, the dividend amount moves around 0.58. The errors are always less than 0.01, so the calibrated results can be accepted. And the results can be thought stable in some sense.

Table 6.5: ING option with fixed interest rate

Initial date	Expiry date	Interest rate	Dividend	$Sigma_{15}$	Error
38373	38527	0.02	0.508066	0.294281	0.000132
38376	38527	0.02	0.499899	0.305448	0.000382
38377	38527	0.02	0.493971	0.277986	0.000095
38378	38527	0.02	0.502031	0.274090	0.000332
38379	38527	0.02	0.508028	0.303789	0.000177
38380	38527	0.02	0.509099	0.273610	0.000598

38383	38527	0.02	0.470633	0.279811	0.000162
38384	38527	0.02	0.507680	0.316680	0.000528
38385	38527	0.02	0.521320	0.284080	0.000209
38386	38527	0.02	0.478650	0.285420	0.000273
38387	38527	0.02	0.499800	0.290870	0.000387
38390	38527	0.02	0.478660	0.299560	0.000310
38392	38527	0.02	0.489680	0.296750	0.000347
38393	38527	0.02	0.471200	0.297490	0.000490
38394	38527	0.02	0.492670	0.310990	0.000156
38397	38527	0.02	0.493430	0.316120	0.000146
38398	38527	0.02	0.504317	0.319456	0.000274
38399	38527	0.02	0.508441	0.316489	0.000440
38400	38527	0.02	0.571386	0.321308	0.000293
38401	38527	0.02	0.551329	0.320453	0.000448
38404	38527	0.02	0.572366	0.322161	0.000994
38405	38527	0.02	0.574956	0.318423	0.000284
38406	38527	0.02	0.579970	0.318630	0.000364
38407	38527	0.02	0.608180	0.318177	0.000734
38408	38527	0.02	0.567505	0.332179	0.000213
38411	38527	0.02	0.602965	0.344433	0.001275
38412	38527	0.02	0.593130	0.350710	0.000454
38413	38527	0.02	0.565210	0.356530	0.000248
38414	38527	0.02	0.573970	0.362660	0.000109
38415	38527	0.02	0.583940	0.370070	0.000672
38418	38527	0.02	0.562920	0.378490	0.001567
38419	38527	0.02	0.562630	0.373690	0.000517
38420	38527	0.02	0.588990	0.375460	0.000518
38421	38527	0.02	0.576980	0.367810	0.000637
38422	38527	0.02	0.543080	0.370860	0.000289
38427	38527	0.02	0.570239	0.371011	0.000258
38428	38527	0.02	0.576388	0.369521	0.000440
38429	38527	0.02	0.586651	0.376288	0.000760
38432	38527	0.02	0.568003	0.383667	0.000534
38433	38527	0.02	0.575822	0.378273	0.000588
38434	38527	0.02	0.578276	0.380609	0.000482
38435	38527	0.02	0.577024	0.393358	0.000748
38436	38527	0.02	0.572623	0.396551	0.000649
38439	38527	0.02	0.564013	0.403160	0.000497
38440	38527	0.02	0.597608	0.404688	0.001470
38441	38527	0.02	0.587600	0.402521	0.000685
38442	38527	0.02	0.607204	0.404909	0.001813
38443	38527	0.02	0.576860	0.405670	0.000442
38446	38527	0.02	0.559460	0.403110	0.000357
38447	38527	0.02	0.573040	0.402880	0.000479

38448	38527	0.02	0.567140	0.411290	0.000792
38449	38527	0.02	0.584720	0.424580	0.000417
38450	38527	0.02	0.569100	0.423700	0.000373
38454	38527	0.02	0.593440	0.425780	0.001237
38455	38527	0.02	0.596220	0.437540	0.000568
38457	38527	0.02	0.582936	0.432096	0.000464
38460	38527	0.02	0.595361	0.406600	0.001145
38461	38527	0.02	0.606763	0.393427	0.002463
38462	38527	0.02	0.570960	0.395335	0.000653
38463	38527	0.02	0.612593	0.401268	0.000469
38464	38527	0.02	0.588554	0.415965	0.000397
38467	38527	0.02	0.560588	0.436865	0.000728
38469	38527	0.02	0.573752	0.410784	0.001139

However, using the objective function with (6.5), the implied volatility becomes larger and larger when the option is close to its maturity, which can be shown in the Table (6.6).

As we all know that the implied volatility cannot go beyond 1 in the

Initial date	Expiry date	Interest rate	$Sigma_{15}$	$Sigma_{25}$	Error
38492	38527	0.02	0.624756	0.239844	0.001750
38495	38527	0.02	0.629482	0.263826	0.001326
38496	38527	0.02	0.623689	0.269366	0.001013
38497	38527	0.02	0.652232	0.262197	0.001044
38498	38527	0.02	0.678379	0.256337	0.001086
38499	38527	0.02	0.695198	0.261571	0.000711
38509	38527	0.02	0.844717	0.327253	0.007731
38510	38527	0.02	0.864703	0.296577	0.008725
38511	38527	0.02	0.893884	0.313207	0.001650
38512	38527	0.02	0.936581	0.316499	0.001880
38513	38527	0.02	0.979555	0.328685	0.010421
38516	38527	0.02	1.080919	0.366025	0.009178
38517	38527	0.02	1.075210	0.374596	0.022102
38518	38527	0.02	1.145459	0.389684	0.028781
38519	38527	0.02	1.240903	0.396630	0.050459
38520	38527	0.02	1.355987	0.414168	0.110484

Table 6.6: ING option close to maturity with the objective function (6.5)

real market, so the calibrated implied volatility is clearly too high. The calibrated implied volatility which is according to the strike price 15 is increasing to 1.355987. Take the data of the day 38520 for example, see Table (6.7):

23.240				
Strike price	CallBid	CallAsk	PutBid	PutAsk
15.000	8.200	8.300	0.000	0.050
16.000	7.200	7.300	0.000	0.050
17.000	6.200	6.300	0.000	0.050
18.000	5.200	5.250	0.000	0.050
19.000	4.200	4.300	0.000	0.050
20.000	3.200	3.250	0.000	0.050
21.000	2.200	2.250	0.000	0.050
22.000	1.200	1.250	0.000	0.050
23.000	0.200	0.250	0.050	0.050
24.000	0.000	0.050	0.700	0.800
25.000	0.000	0.050	1.700	1.800

Table 6.7: Data of ING on 17-Jun-2005

We can see that when the option is close to its maturity, only as the strike prices are around the asset price, the option price for bid is not zero. The reason is simple, the market assumes that the asset price cannot move too high or too low in a few days, so the option cannot be exercised when the strike prices are far away from the asset price. When the strike price is 15, the option put price is close to 0, while the strike price is 22, the option put price will be close to 0.05. So applying the objective function with (6.5), we cannot calibrate the option price very well, since it is only calculate the average of Bid and Ask. To improve the results, the objective function (6.6) will be used, and the results are presented in Table (6.8):

Using the objective function (6.6), the calibrated implied volatility according to the strike price 15 becomes quite stable around 0.4. However, the calibrated implied volatility according to the strike price 25 is not yet stable, for sometimes it reaches the lower bound 0.05 of the fmincon optimization method, and at this time, the some errors are larger than 0.01.

To avoid the situation that calibrated parameters reach their lower or upper bound, we set the calibrated parameters $p0(t)$ of data(t) as the initial guess of date(t+1). We set $p0(t) - bI$ as the lower bound for date(t+1) and $p0(t)+bI$ as the upper bound for date (t), where b is a small value depending on how the parameters behave and I is the vector with all elements equal

Initial date	Expiry date	Interest rate	$Sigma_{15}$	$Sigma_{25}$	Error
38492	38527	0.02	0.400001	0.160288	0.000042
38495	38527	0.02	0.399997	0.050000	0.010014
38496	38527	0.02	0.400001	0.050000	0.010010
38497	38527	0.02	0.400000	0.050000	0.013651
38498	38527	0.02	0.400004	0.171814	0.000011
38499	38527	0.02	0.399994	0.195138	0.000009
38509	38527	0.02	0.400000	0.050000	0.004256
38510	38527	0.02	0.399999	0.050000	0.010783
38511	38527	0.02	0.400000	0.256493	0.000111
38512	38527	0.02	0.400000	0.050003	0.010064
38513	38527	0.02	0.399999	0.050029	0.010067
38516	38527	0.02	0.399999	0.276694	0.000115
38517	38527	0.02	0.400000	0.282023	0.000120
38518	38527	0.02	0.399999	0.296542	0.000061
38519	38527	0.02	0.400000	0.310489	0.000211
38520	38527	0.02	0.400005	0.330762	0.047473

Table 6.8: ING option close to maturity with the objective function (6.6)

to 1. So the calibrated parameters $p0(t+1)$ is bounded by a small range derived by $p0(t)$. With the updated $p0$, the results become much better, which can be seen in Table (6.6.1):

Using the objective function (6.6) with the updated $p0$, the calibrated implied volatility according to the strike price 15 becomes quite stable around 0.41944. The calibrated implied volatility according to the strike price 22 does not reach its lower bound any longer. Also the error remaining after the optimization with objective function (6.6) and updated $p0$ performs much better than the results given by objective function (6.5) as well as objective function (6.6) without updated $p0$. The only drawback is that the error of the last day is larger than 0.01.

With the updated $p0$, the CPU time can be saved because the difference between two arbitrary adjacent $p0$ is small, so fmincon can easily find $p0(t+1)$ given $p0(t)$. How much CPU time can saved with updated $p0$ depends on the initial guess of $p0$ in a time step. In my case, I put the initial guess as $p0 = [0.4, 0.38, 0.36, ..., 0.2]$ for each step. While using the updated $p0$, the gain in CPU time is 20%-25%. If the initial guess is far away from the calibrated result, updated $p0$ can save even more CPU time.

The implied volatility can be shown by the pictures as Figure (6.2)-(6.5):

6.6.2 Two dividends from ING

For the ING Groep pays dividend twice in 2005, we calibrate next the second dividend amount of the ING stock. All options we used here will expire on 16-Dec-2005 (38701). Firstly, both of the first dividend and the second dividend will be calibrated simultaneously, and the results are put in Table (6.10). In this table, the dividend announcement date cannot be found in the column of Dividend 1. And after the dividend announcement date (38399), the first dividend does not move around 0.58, meanwhile the second dividend does not move around 0.49, although the error performs well. This is due to the existence of multiple optimal solutions if both D_1 and D_2 are chosen as parameters to be optimized.

Therefore we fix the first dividend amount given by Table (6.5), and we only calibrate the second dividend amount and the implied volatilities. This time the amount of the second dividend goes round 0.49 with the error also performing well. The results are presented in Table (6.11).

Initial date	Expiry date	Interest rate	$Sigma_{15}$	$Sigma_{25}$	Error
38492	38527	0.02	0.41944	0.24428	1.67E-05
38495	38527	0.02	0.41944	0.26386	6.98E-06
38496	38527	0.02	0.41944	0.26936	0.000113
38497	38527	0.02	0.41944	0.26215	4.64E-06
38498	38527	0.02	0.41945	0.25653	1.26E-05
38499	38527	0.02	0.41945	0.26158	9.80E-06
38509	38527	0.02	0.41945	0.32742	0.000717
38510	38527	0.02	0.41945	0.29669	0.000693
38511	38527	0.02	0.41945	0.31324	7.31E-06
38512	38527	0.02	0.41945	0.31652	1.23E-05
38513	38527	0.02	0.41945	0.32905	9.85E-05
38516	38527	0.02	0.41945	0.35405	9.70E-05
38517	38527	0.02	0.41945	0.35869	9.25E-05
38518	38527	0.02	0.41945	0.37623	4.38E-05
38519	38527	0.02	0.41945	0.39153	0.000126
38520	38527	0.02	0.41945	0.41505	0.046697

Table 6.9: ING option close to maturity with the objective function (6.6) and updated $p0$

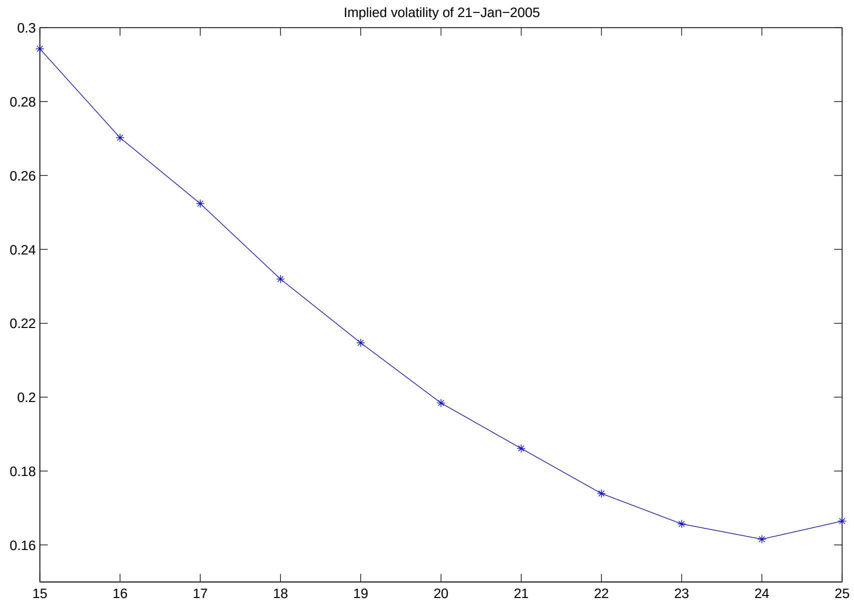

Figure 6.2: Implied volatility of ING on 21-Jan-2005

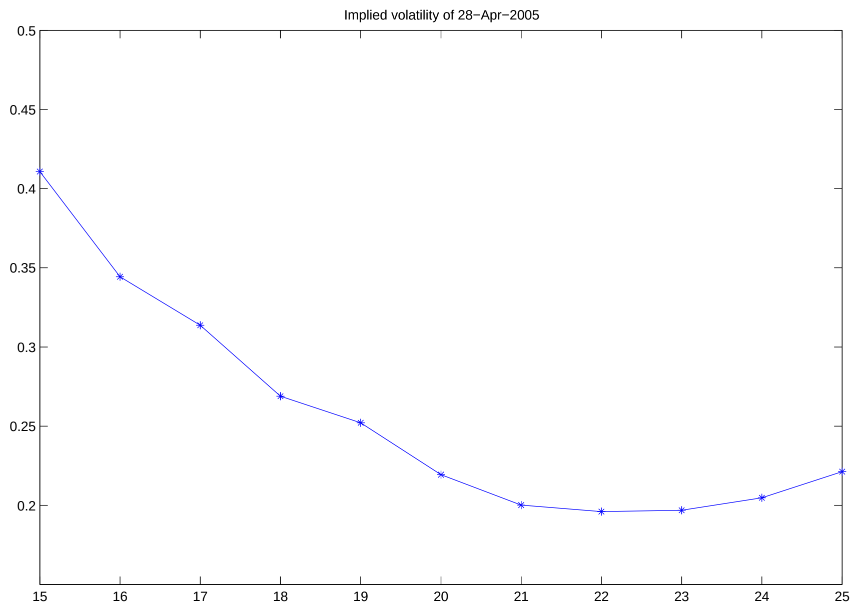

Figure 6.3: Implied volatility of ING on 28-Apr-2005

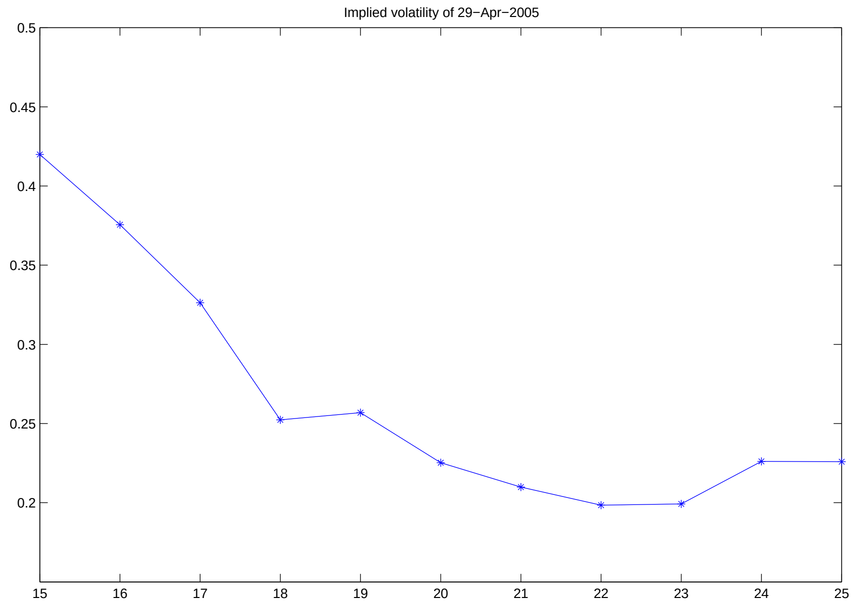

Figure 6.4: Implied volatility of ING on 29-Apr-2005

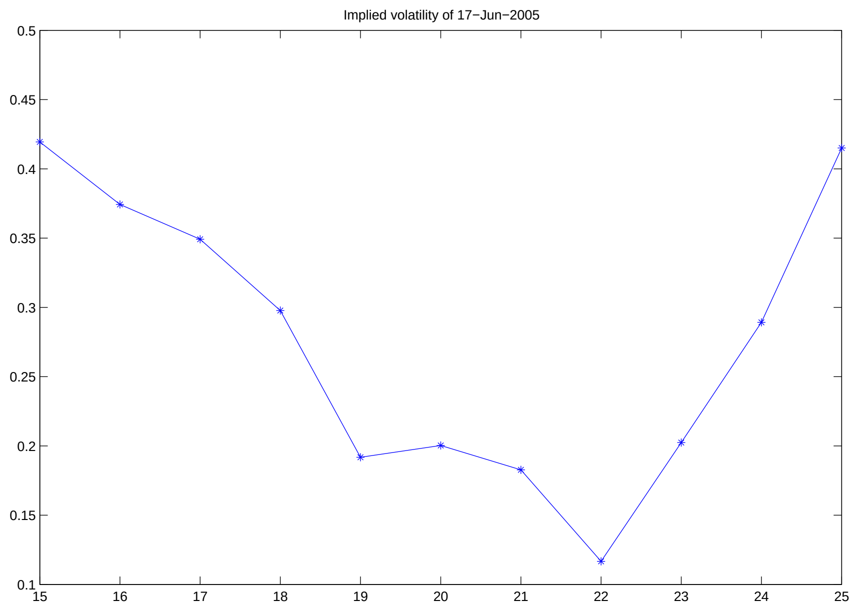

Figure 6.5: Implied volatility of ING on 17-Jun-2005

Initial date	Expiry date	Dividend 1	Dividend 2	Error
38373	38701	0.482873	0.480260	0.000017
38376	38701	0.474995	0.484836	0.000161
38377	38701	0.475825	0.516397	0.000136
38378	38701	0.468500	0.488297	0.000505
38379	38701	0.476493	0.515834	0.000431
38380	38701	0.507224	0.497765	0.000441
38383	38701	0.472092	0.476802	0.000236
38384	38701	0.432050	0.534076	0.000059
38385	38701	0.499337	0.500311	0.000168
38386	38701	0.431131	0.522670	0.000264
38387	38701	0.446490	0.508677	0.000227
38390	38701	0.477264	0.481752	0.000277
38392	38701	0.463684	0.496962	0.000281
38393	38701	0.459182	0.484533	0.000442
38394	38701	0.441031	0.541017	0.000059
38397	38701	0.443271	0.527188	0.000318
38398	38701	0.441893	0.546166	0.000263
38399	38701	0.461823	0.525720	0.000221
38400	38701	0.449991	0.610474	0.000182
38401	38701	0.444835	0.598729	0.000220
38404	38701	0.514305	0.564121	0.000214
38405	38701	0.499460	0.571135	0.000270
38406	38701	0.503694	0.573947	0.000252

Table 6.10: ING option with two dividends calibrated simultaneously

Initial date	Expiry date	Dividend 1	Dividend 2	Error
38373	38701	0.508066	0.459462	0.000048
38376	38701	0.499899	0.464669	0.000133
38377	38701	0.493971	0.506920	0.000158
38378	38701	0.502031	0.457849	0.000384
38379	38701	0.508028	0.490483	0.000368
38380	38701	0.509099	0.498679	0.000316
38383	38701	0.470633	0.484704	0.000221
38384	38701	0.507680	0.468061	0.000051
38385	38701	0.521320	0.479604	0.000108
38386	38701	0.478650	0.484413	0.000226
38387	38701	0.499800	0.459513	0.000121
38390	38701	0.478660	0.484386	0.000168
38392	38701	0.489680	0.476040	0.000245
38393	38701	0.471200	0.479320	0.000365
38394	38701	0.492670	0.503361	0.000062
38397	38701	0.493430	0.486962	0.000323
38398	38701	0.504317	0.499030	0.000277
38399	38701	0.508441	0.488948	0.000239
38400	38701	0.571386	0.508508	0.000217
38401	38701	0.551329	0.511860	0.000276
38404	38701	0.572366	0.484170	0.000198
38405	38701	0.574956	0.504191	0.000077
38406	38701	0.579970	0.510321	0.000267

Table 6.11: ING option the 2nd dividend is calibrated with the 1st dividend fixed

The calibration results given in this section are calculated by the objective function (6.5) without the updated $p0$. We find that when the option is far from maturity, the objective function (6.5) performs well. Without the updated $p0$, we can investigate how the implied dividend behaves in a much wider range.

6.6.3 Calibrated parameters for Fortis option

Fortis Bank pays one dividend on 27-May-2005, and the dividend amount is €1.17 with the dividend announcement date 10-Mar-2005 (38420). The objective function (6.6) with the updated $p0$ is applied, the results are shown in Table (6.13). The risk-free interest rate is fixed to 0.02, which is not shown in Table (6.13). And the data from Fortis are presented in Table (6.12).

21.320				
Strike price	CallBid	CallAsk	PutBid	PutAsk
10.000	11.35	11.45	0.000	0.050
15.000	6.400	6.500	0.050	0.100
16.000	5.400	5.500	0.050	0.100
17.000	4.450	4.550	0.100	0.150
18.000	3.500	3.550	0.200	0.250
19.000	2.550	2.650	0.350	0.450
20.000	1.750	1.800	0.650	0.700
21.000	1.000	1.100	1.100	1.150
22.000	0.500	0.600	1.750	1.850
23.000	0.250	0.300	2.550	2.650
25.000	0.000	0.100	4.400	4.500

Table 6.12: Data of Fortis on 21-Jan-2005

Table 6.13: Fortis option with fixed interest rate

Initial date	Expiry date	Dividend	$Sigma_{10}$	$Sigma_{25}$	Error
38373	38527	0.934826	0.479580	0.155082	0.000003
38376	38527	0.933933	0.479580	0.158532	0.000009
38377	38527	0.906764	0.479583	0.147468	0.000005
38378	38527	0.907638	0.479584	0.154727	0.000008
38379	38527	0.922589	0.479584	0.157866	0.000004
38380	38527	0.942229	0.479585	0.157668	0.000005
38383	38527	0.934152	0.479585	0.163426	0.000010
38384	38527	0.934003	0.479586	0.170400	0.000008
38385	38527	0.933845	0.479586	0.164047	0.000011
38386	38527	0.932407	0.479587	0.160272	0.000010
38387	38527	0.922614	0.479587	0.158921	0.000003
38390	38527	0.925925	0.479587	0.160187	0.000004
38392	38527	0.925607	0.479587	0.152542	0.000003
38393	38527	0.912323	0.479588	0.151124	0.000013
38394	38527	0.952026	0.479588	0.142895	0.000005
38397	38527	0.949771	0.479589	0.149649	0.000023
38398	38527	0.958215	0.479596	0.139104	0.000007
38399	38527	0.957089	0.479597	0.141944	0.000013
38400	38527	0.960785	0.479597	0.144045	0.000006
38401	38527	0.927932	0.479599	0.142476	0.000006
38404	38527	0.939053	0.479599	0.149856	0.000001
38405	38527	0.926781	0.479599	0.156725	0.000104
38406	38527	0.927193	0.479599	0.165605	0.000002

38407	38527	0.926990	0.479599	0.162830	0.000003
38408	38527	0.927654	0.479599	0.162117	0.000009
38411	38527	0.929799	0.479599	0.158942	0.000013
38412	38527	0.930128	0.479599	0.166958	0.000024
38413	38527	0.929641	0.479599	0.167150	0.000002
38414	38527	0.931700	0.479599	0.165220	0.000009
38415	38527	0.964984	0.479599	0.160406	0.000049
38418	38527	0.976377	0.479599	0.159571	0.000004
38419	38527	1.000629	0.479594	0.154133	0.000003
38420	38527	1.011569	0.479594	0.147061	0.000004
38421	38527	1.010606	0.479594	0.152109	0.000005
38422	38527	1.010516	0.479594	0.154522	0.000002
38427	38527	1.009660	0.479594	0.164579	0.000004
38428	38527	1.009484	0.479594	0.164527	0.000010
38429	38527	1.019464	0.479595	0.149867	0.000001
38432	38527	1.033535	0.479594	0.145934	0.000003
38433	38527	1.032625	0.479594	0.146997	0.000005
38434	38527	1.032301	0.479594	0.158290	0.000006
38435	38527	1.030352	0.479593	0.150290	0.000018
38436	38527	1.030852	0.479593	0.151442	0.000004
38439	38527	1.031113	0.479593	0.154709	0.000004
38440	38527	1.081113	0.479591	0.151861	0.000006
38441	38527	1.031113	0.479588	0.165305	0.000050
38442	38527	1.030962	0.479588	0.161890	0.000003
38443	38527	1.030629	0.479588	0.164521	0.000003
38446	38527	1.030070	0.479588	0.177992	0.000002
38447	38527	1.027633	0.479587	0.171208	0.000006
38448	38527	1.021150	0.479586	0.172900	0.000002
38449	38527	1.035994	0.479582	0.168697	0.000002
38450	38527	1.008158	0.479582	0.163340	0.000005
38454	38527	1.008731	0.479581	0.174451	0.000008
38455	38527	1.004585	0.479581	0.170320	0.000003
38456	38527	1.004665	0.479581	0.177057	0.000002
38457	38527	1.005419	0.479581	0.184053	0.000008
38460	38527	1.039402	0.479581	0.213607	0.000011
38461	38527	1.039944	0.479581	0.214393	0.000007
38462	38527	1.039629	0.479581	0.221684	0.000002
38463	38527	1.032678	0.479580	0.225774	0.000002
38464	38527	1.045315	0.479580	0.224126	0.000004
38467	38527	1.062951	0.479580	0.225187	0.000010
38469	38527	1.045876	0.479580	0.241575	0.000005
38470	38527	1.057039	0.479575	0.250408	0.000100
38471	38527	1.057493	0.479575	0.253198	0.000020
38474	38527	1.056956	0.479575	0.247895	0.000012

38475	38527	1.056514	0.479575	0.248678	0.000005
38476	38527	1.042808	0.479575	0.240271	0.000007
38477	38527	1.042729	0.479575	0.233832	0.000011
38478	38527	1.029811	0.479575	0.234915	0.000004
38481	38527	1.028795	0.479574	0.240241	0.000005
38482	38527	1.023615	0.479574	0.245488	0.000013
38483	38527	1.021258	0.479574	0.255283	0.000007
38484	38527	1.031844	0.479574	0.252362	0.000010
38485	38527	1.029856	0.479574	0.260979	0.000004
38488	38527	1.042315	0.479574	0.268128	0.000020
38489	38527	1.042897	0.479574	0.265446	0.000004
38490	38527	1.040857	0.479574	0.256947	0.000005
38491	38527	1.043940	0.479574	0.243960	0.000010
38492	38527	1.068721	0.479574	0.245937	0.000022
38495	38527	1.062535	0.479574	0.247910	0.000011
38496	38527	1.072398	0.479574	0.250730	0.000031
38497	38527	1.052399	0.479574	0.253154	0.000034
38498	38527	1.050820	0.479574	0.254159	0.000053
38499	38527	0.000000	0.479574	0.246641	0.000010
38509	38527	0.000000	0.479574	0.285461	0.000002
38510	38527	0.000000	0.479548	0.270867	0.000057
38511	38527	0.000000	0.479548	0.280724	0.000033
38512	38527	0.000000	0.479548	0.275813	0.000723
38513	38527	0.000000	0.479548	0.284427	0.000071
38516	38527	0.000000	0.479548	0.307466	0.000005
38517	38527	0.000000	0.479548	0.298895	0.000136
38518	38527	0.000000	0.479548	0.314530	0.000006
38519	38527	0.000000	0.479548	0.320255	0.000109
38520	38527	0.000000	0.479548	0.315301	0.010113

In Table (6.13), the implied dividend moves up to 1 on the date (38419) which is one day before the dividend announcement. This situation happening may be resulted from the market having expected that the dividend of Fortis bank will be announced. And the calibrated implied dividend is around 1.05 which is a little lower than the announcement. The implied volatility according to both strike price 15 and 25 both perform quite well. And most of the errors are smaller than 0.01 except the last one.

The implied volatility of Fortis can be shown by the pictures as Figure (6.6)-(6.9):

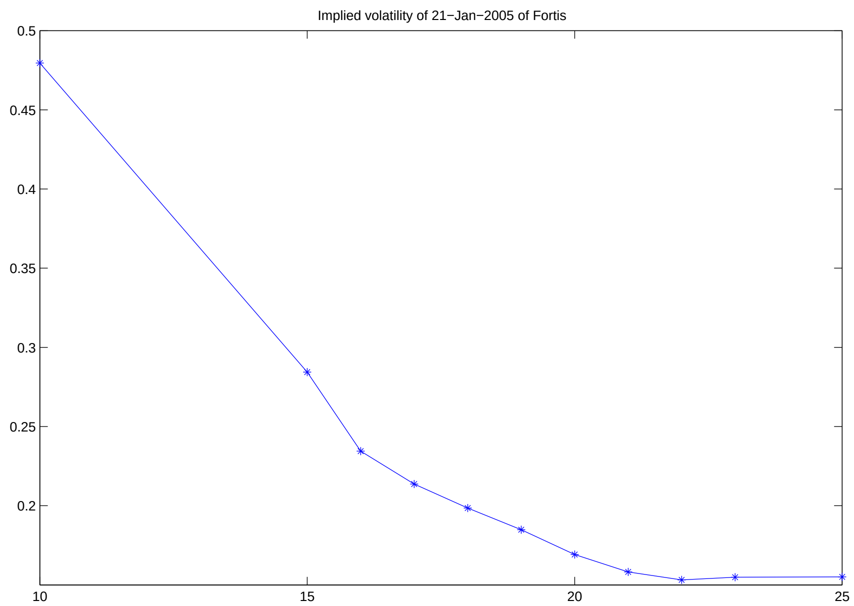

Figure 6.6: Implied volatility of Fortis on 21-Jan-2005

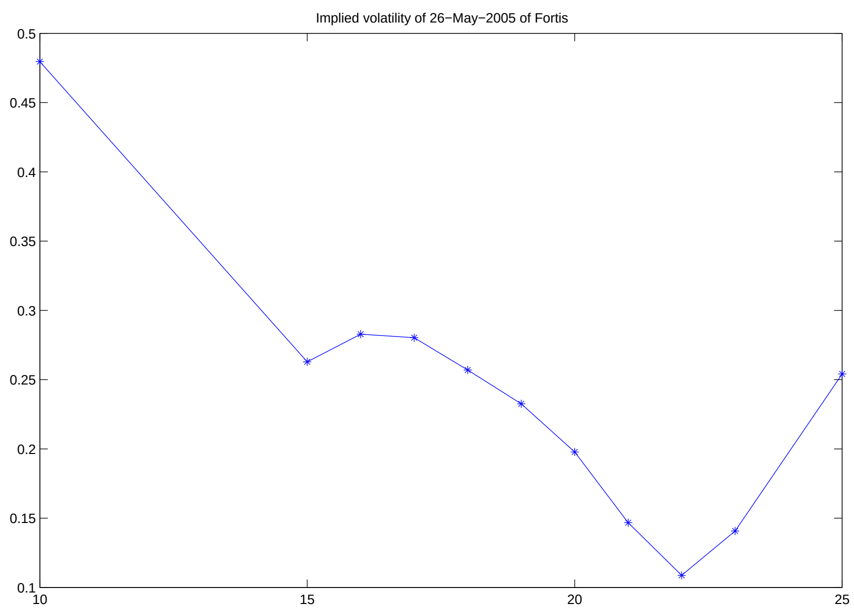

Figure 6.7: Implied volatility of Fortis on 26-May-2005

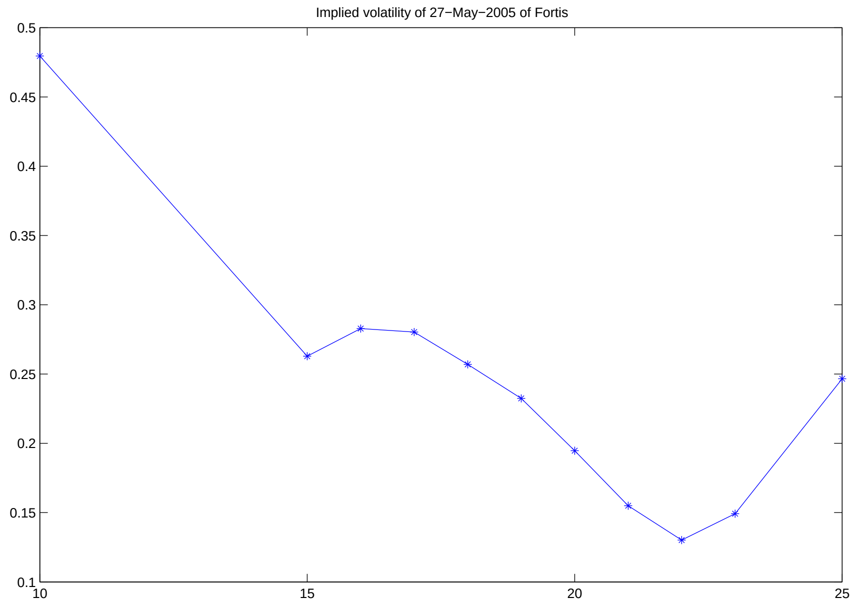

Figure 6.8: Implied volatility of Fortis on 27-May-2005

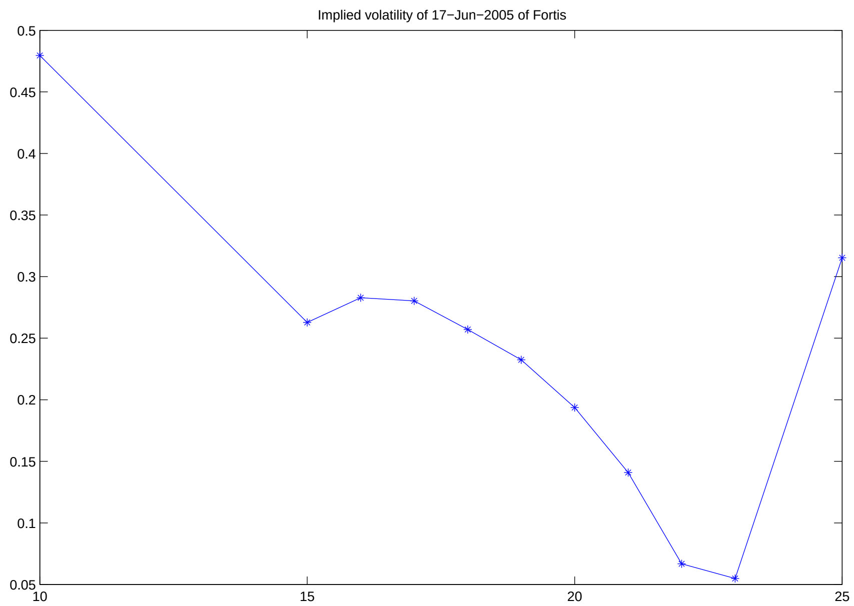

Figure 6.9: Implied volatility of Fortis on 17-Jun-2005

Chapter 7

Conclusion

The Black-Scholes equation is a useful and fundamental tool to calculate the option prices. Since the stock options in the real market are all of American type, there is no closed form to solve the Black-Scholes equation. In our code, the fourth order finite difference method has been applied to discretize the partial differential equation with stretching. In the Black-Scholes equation, there are several parameters. The implied volatility and the implied dividend are unknown, while the risk free-interest rate, the stock price, the strike price and the lifetime of the option are all known.

Given the market prices of the ING and Fortis options, the two unknown parameters implied volatility and the implied dividend can be calibrated by the other known parameters using the Matlab's optimization functions. It has been shown that there is no unique solution of the calibration approach, but we can obtain some useful results and insights:

- The implied dividend amount can be calibrated to a value around the amount given by the company
- The dividend announcement date can be visualized for ING and for Fortis
- The errors are sufficiently small except for the data of the last option day
- The calibrated results are stable

There are some aspects need to be improved in the approach:

- The risk-free interest rate changes a little weekly. To obtain more accurate results, it is better use the interest rate provided by LIBOR
- Close to the option's maturity, the implied volatility becomes unstable, the volatility correction model could be used to handle this problem

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