Sensitivity Analysis of Model Output

Performance of various local and global sensitivity measures on reliability problems

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9th October 2000

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Abstract

Mathematical models of real life processes, such as physical, social and economic processes, mostly consist of large sets of coupled, non-linear equations. In general, these equations consist of a number of uncertain variables, and an important issue is to know the sensitivities of their solutions to variations of the input variables.

This report describes the results of a study to sensitivity measures, which are tools to determine the influence of the input parameters. The purpose is to use the sensitivity measures on reliability problems. To the present day, first order reliability methods (FORM) and Monte Carlo simulation have been used. Since both methods have some disadvantages and some problems could not be solved with these methods, other techniques have been investigated.

During the project the following goals have been handled:

1. Comparison of the sensitivities resulting from Monte Carlo and FORM methods.

2. Investigation of existing local and global sensitivity measures. Development of new sensitivity measures.

The emphasis of the project has been put on the development of a new sensitivity measure and a broad investigation of the existing sensitivity measures. The sensitivity measure should meet the following criteria: computationally efficient, robust, accurate and model independent.

During a literature survey a number of sensitivity measures have been found. These methods have been tested on various test cases, including some cases, where both FORM and Monte Carlo give no good answers. Two methods, namely the correlation ratio and the FAST-method performed best and have also been tested on some more complex problems. One of these problems considered the dike-ring reliability of the Hoeksche Waard dike-ring. To compare the results and to judge the correctness of the results a referee has been developed.

Both the FAST method and the correlation ratio gave good results on the test cases and can be a valuable supplement to the existing techniques.

Keywords: Sensitivity Analysis, Sensitivity Measures, Correlation Ratio, FAST, FORM.

Preface

Delft, August 2000

This report describes the results of a graduation project, carried out at TNO Building and Constructions Research Rijswijk in the period February 2000 till October 2000, as part of the study applied mathematics at Delft University of Technology. The motive for this research project was a reliability study of the Hoeksche Waard dike-ring carried out with The Dutch Ministry of Transport, Public Works and Water Management and Delft University of Technology.

Since this report describes my graduation work, it fully describes both the mathematical foundation of the applied methods and the results. People more interested in the practical application of the theory may skip the Sections 2.2 till 2.3 and 3.3 till 3.6.

I would like to thank the following people for their help and support during my internship at TNO Bouw: Roger Cooke, Ton Vrouwenvelder, Paul Waarts, Sten de Wit, Dorota Kurowicka and Robert-Jan van Egmond.

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1 Introduction

1.1 Why sensitivity analysis

Computer-based models can approximate real life processes, such as economical, social or physical processes. These models mostly consist of equations with a number of uncertain variables. An important issue is to determine the influence of variation of the model-input variables on the variation of the output quantity. In other words, to determine the sensitivity of the model output to variations of the input variables. Suppose for instance that we are considering the failure probability of a dike with respect to overtopping. The failure probability of the dike can be modelled as a function of input factors, such as the wind-velocity, the discharge of the river and the height of the dike. The output is, for instance, a distribution of the failure frequency of the dike. Suppose now, that someone offers 1000 guilders to reduce the uncertainty of one variable. This can be achieved by studying or measuring the variable. Then we are interested in which variable will give us the greatest uncertainty reduction in the distribution of the failure frequency of the dike. In other words, we are interested in which variable we should study or measure.

During a recent research project for The Dutch Ministry of Transport, Public Works and Water Management, considering the failure probability of the Hoeksche Waard dike ring, several problems considering the sensitivities of the input factors came around. Two methods, namely Monte Carlo Simulation [32] and First Order Reliability Methods (FORM) [46] were used to calculate the sensitivities. The sensitivities were sometimes very different, although the failure probabilities agreed very well. It would be useful to have an accurate method to calculate the sensitivities, suitable for all kind of models.

1.2 The common methods

To the present day the following methods have been used on reliability problems to calculate failure probabilities and the corresponding sensitivity indices: First Order Sensitivity Methods (FORM) [47], Second Order Sensitivity Measures (SORM) [47], Monte Carlo Simulation (MC) [32] and Local Probabilistic Sensitivity Measure (LPSM) [5]. These methods compute the probability of failure and the sensitivities given an output function and stochastic parameters. These methods have some limitations, which will be described hereafter.

The standard method to calculate the failure probability and sensitivities is the FORM-method. The sensitivities given by this method are both 'local' and 'global'. Three different problem settings can be distinguished, namely parameter screening, local sensitivity analysis and global sensitivity analysis. Parameter screening identifies the important factors, but does not quantify how much of the variation of the output is caused by variation of the input variables. Since we are not only interested in the ranking of the variables, but also in the amount of their influence, we have not paid attention to screening methods. In local sensitivity analysis (SA) one is interested in the sensitivities at a certain value of the output function whereas in global SA one is interested in the sensitivities over the entire range of output values. There are four main problems involved with the use of this method (FORM).

1. The result is not accurate when the output function is very non-linear.

2. The sensitivities are calculated in the so-called design-point. The optimisation procedure used to find the design-point finds a local minimum, which can be another minimum than the global minimum we are looking for.

3. If the design-point is not unique, the results of FORM can be completely wrong.

4. The FORM method can not handle series systems. The various limit-state functions are therefore combined with the method of Hohenbichler [47]. This is an approximation, which also introduces errors.

The SORM method consists of a FORM-calculation followed by a second order correction. The mentioned difficulties still exist, but can be decreased in some cases.

With crude Monte Carlo simulation very accurate results can be achieved with respect to the probability of failure, but at a cost of a lot of computational effort. Crude Monte Carlo simulation only results in a failure probability and gives no sensitivities, apart from exceptions, cf. [29]. That is why the local probabilistic sensitivity measure (LPSM) has been developed. This measure has some specific calculation problems.

Table 1 summarises the advantages and disadvantages of the standard methods. We can conclude that the accurate Monte Carlo method takes too much computational effort, whereas the faster FORM method gives less accurate results.

Measure	**Sensitivities**	**Advantage**	**Disadvantage**
FORM	Influence factors	Sensitivities can be calculated quickly	• Linearisation • Determination design-point • Hohenbichler-error • Results can be wrong if design-point not unique
SORM	Influence factors	Sensitivities can be calculated quickly	Output function has to satisfy certain conditions
Crude Monte Carlo	No sensitivities	Accurate	Large amount of model evaluations
LPSM	Local sensitivities	Can be used on non-linear output functions	Difficult to calculate, due to derivatives

Table 1: Advantages and disadvantages of the standard methods

1.3 A 'new' sensitivity measure

Since there are a number of situations in which both FORM/SORM and Monte Carlo with LPSM do not give good results, it is sensible to investigate whether there are other sensitivity measures which can solve these cases satisfactory, or to develop a new sensitivity measure to solve these cases.

There are some properties we would like the sensitivity measure to have, namely:

1. Computationally efficient

2. Robust

3. Reproducible

4. Accurate

5. Model independent

Furthermore, we wish the measure to have a sound mathematical foundation and we want the sensitivity measure to correspond to the influence factors given by a FORM-analysis in the linear case. The FORM method linearizes the output function in the design-point and then calculates the sensitivities. If the output function is indeed linear then the influence factors correspond to the Pearson product-moment correlation (cf. [30]), which gives the percentage of the variance of Z that is explained by the corresponding input factor.

Robustness means here the ability of the method to be used on all kind of problems, complex or containing many variables. Reproducibility means the ability to replicate its results with different input samples taken from the same population. Accuracy means the extent in which the sensitivity estimates are close to the true values at a given sample size N. The accuracy gives an indication about the error between the estimated sensitivities and the true values. Usually there is a trade-off between accuracy and computational cost.

1.4 Discussion

In general, sensitivity analysis of model output aims to quantify how a model depends on its input parameters. Local sensitivity analysis is concerned with the influence of a given parameter on the output in a certain point, or a number of points, of the multidimensional input space, whereas global sensitivity analysis is concerned with apportioning the output uncertainty to the uncertainty of the different input variables. Figure 1 shows the general scheme of global sensitivity analysis, where the total variance of the output is apportioned to the various input factors, as shown by the pie diagram (σ_Z denotes the standard deviation of the output Z). There are no explicit directions when local SA is more suited to the problem and when it is better to perform global SA. If we are interested in the effect of fluctuations in the variance of the various input factors on the variance of the whole model, we should use global SA techniques. On the other hand, if we are interested in the sensitivities at a certain value or certain values of the output we should look at local SA techniques.

In our problem setting we are especially interested in the sensitivities at the failure region, actually the failure contour, which points to local SA. However, local SA mostly considers one specific point of the output, whereas we are interested in the whole failure contour. Furthermore, it is also interesting to know how uncertainties in the input variables affect our model and how we can get the greatest profit in terms of uncertainty reduction by measuring one of the input variables.

In this study attention is paid to both methods. By the interpretation of the results it should be kept in mind which method is used and if that method really provides us the sensitivities we are looking for. It is useful to compare the results to some kind of referee. The referee developed in this study is based on the idea that the probability of failure changes if the input factors are changed. The factor that causes the greatest change in the probability of failure should be identified as being the most important.

1.5 Goals and approach

During the graduation project the following goals will be handled:

1. Comparison of the sensitivities resulting from Monte Carlo and FORM methods.

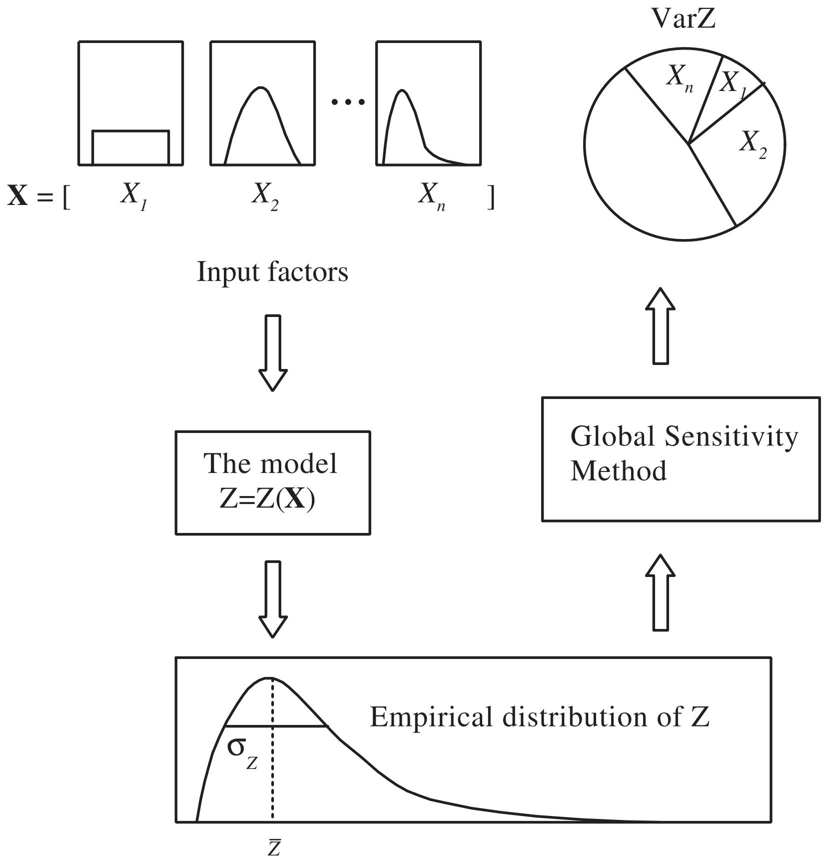

Figure 1: General Scheme of a Quantitative Sensitivity Analysis

2. Investigation of existing local and global sensitivity measures; Development of new sensitivity measures.

The first goal also includes the comparison of sensitivities of series systems obtained with the FORM method together with the method of Hohenbichler and other methods. The emphasis of the project will be on the development of a new sensitivity measure and a broad investigation of the existing sensitivity measures. The sensitivity measures should be tested on cases, where FORM and Monte Carlo do not give good answers. It is also desirable to obtain a certain reference point, which tells us if the results are correct.

During the project some assumptions have been made, namely:

- The quality of the random generator does not influence the results.
- Errors introduced by finite machine precision are considered negligible.

We want the sensitivity measure to be suitable for all kind of output functions, so we do not make restrictions to the output function, which implies that the output function may be non-linear, non-continuous or deterministic. Since local sensitivity measures require the output function to be continuously differentiable, this research focuses on global sensitivity measures.

1.6 Outline of the report

First, a literature survey has been carried out. Chapter 2, together with Chapter 3 and 4, gives a brief overview of the importance measures found in the literature. First, it describes local SA measures, which are in fact derivatives. Second, it describes the FORM sensitivity measure, which is both a local and a global measure. Then, it describes the regression-based methods, which are global SA methods. Special attention is paid to one class of global sensitivity measures, namely the variance-based methods. These are described more extensively in the chapters 3 and 4. Chapter 3 describes the correlation ratio and how this measure can be computed. This chapter also describes the other variance-based methods and their relation to the correlation ratio. Chapter 4 describes the FAST-method, which is a sensitivity measure based on Fourier Analysis. Chapter 5 starts with an overview of the described methods, where attention is paid to what extent the methods satisfy the properties mentioned earlier. A number of the global sensitivity measures have been tested on several cases. First, a referee is derived, to be able to judge the results of the test cases and then the results of the test cases are given. The measures that performed best have been tested on more complex problems. Chapter 6 gives the results of these problems. Finally, conclusions and recommendations are given in Chapter 7.

2 FORM, Differential Analysis and Regression-based methods

In this project we are interested in the sensitivities of the model output of a reliability problem. A reliability problem is defined by an output function $Z(\mathbf{X})$ (cf. [48]) and a set of k random variables $\mathbf{X}$. In civil engineering the output function of a reliability problem is called the limit state function (LSF). In general, the limit state function describes the strength of a construction minus the stress. Failure is defined by the event $Z(\mathbf{X}) \leq 0$. The equality $Z(\mathbf{X}) = 0$ is called the limit state equation.

The failure probability can be expressed by:

$$P_f = P(Z(\mathbf{X}) \leq 0) = \int_{Z(\mathbf{X})\leq 0} f_{\mathbf{X}}(\xi)\mathrm{d}\xi \tag{1}$$

where $f_{\mathbf{X}}(\xi)$ the joint probability density of $\mathbf{X}$.

As described in the introduction we are looking for a sensitivity measure with the following properties: computationally efficient, robust, reproducible, accurate and model independent. The measure should have a sound mathematical foundation and it is desirable that it corresponds to the FORM influence factors in the linear case.

There are several ways to indicate the importance for input factors, such as partial derivatives, standardised regression coefficients, correlation coefficients, partial correlation coefficients, variance-based methods and regression-based methods. First, a division in local and global measures can be made. In a local sensitivity analysis (SA) one is interested in the behaviour of a function in a specific region, whereas a global sensitivity analysis focuses on the output uncertainty over the entire range of values of the input parameters. At this point, we are interested in both local and global measures. A special case of SA is given by screening methods, which only identify influential input variables.

Standardised regression coefficients, correlation coefficients (Pearson) and partial correlation coefficients rely on the assumption that the output and the input factors are near-linearly related. To satisfy the goals described in the previous chapter the method should also be able to handle non-linear Z-functions. Therefore these methods have not been investigated.

The rank equivalent methods such as the standardised rank regression coefficients, Spearman correlation, partial rank correlation coefficients rely on the assumption that output and input are near-monotonically related and have therefore also not been investigated.

This overview is not a complete resume of all existing sensitivity measures, but a summary of the most important techniques.

2.1 First Order Reliability Methods (FORM)

Suppose the limit state function $Z(X_1, ..., X_k)$, which expresses the reliability of a dike section, is a 'deterministic' function of the random vector $\mathbf{X} = (X_1, ...X_k)$ [46]. Assuming that Z is analytic, we can linearize in the neighbourhood of some point $x^* = (x_1^*, ..., x_k^*)$,

$$Z(\mathbf{X}) = Z(\mathbf{x}^*) + \sum_{i=1}^{k} \frac{\partial Z}{\partial X_i}(X_i - x_i^*) + \text{HOT (higher order terms)}. \tag{2}$$

The point $\mathbf{x}^*$ is chosen as the 'design point', which is defined as the point with greatest probability density, satisfying $Z(\mathbf{x}^*) = 0$. This point is found by an optimisation procedure.

Let μ_i and σ_i denote the mean and standard deviation of X_i, respectively. Neglecting the higher order terms and assuming the X_i's are all independent, we have

$$
\begin{aligned}
Z(\mathbf{X}) &\simeq Z(\mathbf{x}^*) + \sum_{i=1}^{k}(X_i - x_i^*)\frac{\partial Z(\mathbf{x}^*)}{\partial X_i}, & (3)\\
\mu_Z &\simeq Z(\mathbf{x}^*) + \sum_{i=1}^{k}(\mu_i - x_i^*)\frac{\partial Z(\mathbf{x}^*)}{\partial X_i}, & (4)\\
\sigma_Z &\simeq \left\{\sum_{i=1}^{k}\sigma_i^2(\frac{\partial Z}{\partial X_i})^2\right\}^{\frac{1}{2}}. & (5)
\end{aligned}
$$

We can express σ_Z as a linear function of the σ_i, $\sigma_Z = \sum_{i=1}^{k}\alpha_i\frac{\partial Z}{\partial X_i}\sigma_i$, where

$$
\alpha_i = \frac{\frac{\partial Z}{\partial X_i}\sigma_i}{\left\{\sum_{i=1}^{k}\sigma_i^2(\frac{\partial Z}{\partial X_i})^2\right\}^{\frac{1}{2}}} = \frac{\frac{\partial Z}{\partial X_i}\sigma_i}{\sigma_Z}. \tag{6}
$$

The factor α_i gives the influence of the standard deviation of variable X_i on the standard deviation of Z. It follows that: $\sum_{i=1}^{k}\alpha_i^2 = 1$.

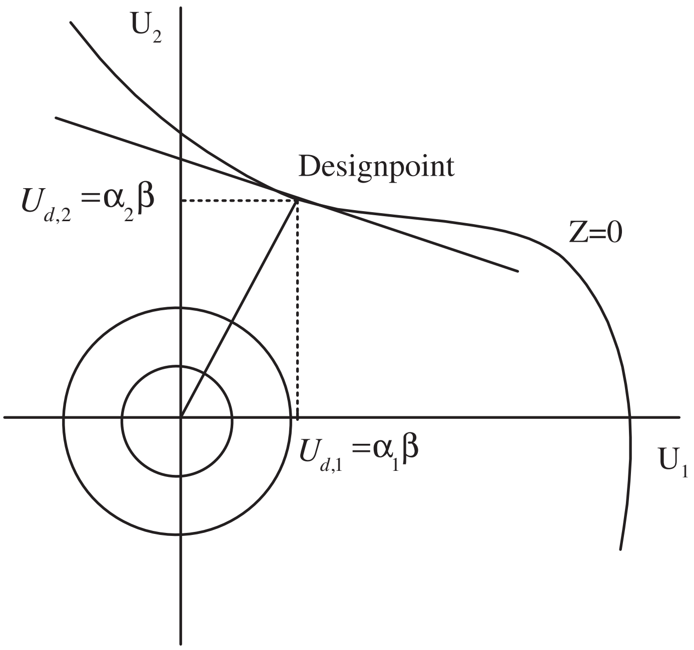

Figure 2: Definition of the design point in the U-space

So, the sensitivities in a FORM-analysis are given by the α_i's (cf. [47]). With these α_i's, the limit-state function can be written as a function of standard normally distributed variables U_i, in a standardised form.

$$
Z_L = \beta + \alpha_1 U_1 + \alpha_2 U_2 + ... \tag{7}
$$

The co-ordinates of the design point are given by $U_{d,i} = -\alpha_i \beta$.

The FORM-method has the following disadvantages. The failure probability is not accurate if the limit-state function is strongly non-linear and the values of the α_i's depend on the point of linearisation x^*. Furthermore, if the design point is not unique the sensitivities can be completely wrong.

Cawlfield and Wu [3] suggest to use a measure in the space of the original uncertain parameters. They use a γ-measure (Der Kiureghian and Liu (1985)) defined by:

$$\gamma = \frac{\Gamma_0^T \alpha}{|\Gamma_0^T \alpha|} \tag{8}$$

where Γ_0 transforms the original variables to standard normal uncorrelated variables. γ is a vector containing the different sensitivities. As in the previous problem, $Z = Z(\mathbf{X})$, with X_i the uncertain variables of the problem. The uncertain variables X_i are transformed into the space of uncorrelated standard normal variables ($\mathbf{U}$) by:

$$\mathbf{U} = \Gamma_0 \mathbf{Z} \tag{9}$$

where $\mathbf{Z}$ is a vector of correlated standard normal variates obtained by $Z_i = \Phi^{-1}[F_{X_i}(X_i)]$. Γ_0 is a lower-triangular matrix resulting from the decomposition of the correlation matrix of $\mathbf{Z}$.

Cawlfield and Wu discuss examples where the γ-measure provides better sensitivity information than the α's, but this method obviously has the same limitations as a standard FORM-analysis.

2.2 Differential Analysis

In local sensitivity analysis we are interested in the rate of change of the output at a certain value of the output-function. In fact, we are interested in the derivative (or Jacobian) of the model output with respect to the model input, possibly normalised by the means or standard deviations of the input/output variables themselves.

The determination of partial derivatives is usually not easy. To facilitate the calculation of these derivatives a number of specialised techniques have been developed, such as the adjoint method and Green's function techniques. A description of these techniques follows in the next sections.

2.2.1 Local Probabilistic Sensitivity Measure

Cooke and Van Noortwijk [5] propose a local sensitivity measure that measures the rate of change with respect to Z of (some function of) $\mathbf{X}|Z$ at $Z = 0$. The Z-function is the limit-state function. They suggest a local probabilistic sensitivity measure proportional to

$$\frac{\partial E(X_i|Z = z)}{\partial z}|_{z=0}. \tag{10}$$

When the linear model holds, the local probabilistic sensitivity measure (LPSM) should equal $\rho(X_i, Z)$, in agreement with the FORM measure. Therefore, LPSM_i is defined by:

$$\frac{\sigma_Z \partial E(X_i|Z = z)}{\sigma_i \partial z}. \tag{11}$$

See also [5].

2.2.2 Green's function techniques

As mentioned above, specific methods can be used to calculate the partial derivatives. In fact, in FORM and SORM-analysis Taylor series are used. It is also possible to use Green's function techniques or adjoint techniques. This section describes the Green's function techniques (cf. [49]).

Suppose we consider a system of the following form:

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{a}, t) \quad \text{and} \quad \mathbf{y}(t_0) = \mathbf{y}_0 \tag{12}$$

where $\mathbf{y}$ is a vector of n variables and $\mathbf{y} = \mathbf{y}(\mathbf{a}, t)$, $\mathbf{a}$ is a vector containing m parameters, and t is time. So, $\mathbf{f}$ is a function which maps $\mathbb{R}^{n+m+1}$ into $\mathbb{R}^n$.

Then

$$\frac{\partial \dot{\mathbf{y}}}{\partial a_j} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\right) \cdot \frac{\partial \mathbf{y}}{\partial a_j} + \frac{\partial \mathbf{f}}{\partial a_j},$$

and the corresponding sensitivity equation is

$$\dot{\mathbf{y}}_{a_j}(t) = \mathbf{J}(t)\mathbf{y}_{a_j}(t) + \mathbf{f}_{a_j}(t) \quad \text{and} \quad \mathbf{y}_{a_j}(t_0) = \delta_{a_j y_{0i}} \tag{13}$$

where $\mathbf{y}_{a_j} = \partial \mathbf{y}/\partial a_j$, $\mathbf{J}(t)$ (an $n \times n$ matrix) the Jacobian, $J_{ij} = \partial f_i/\partial y_j$, $\mathbf{f}_{a_j} = \partial \mathbf{f}/\partial a_j$, and $\delta_{a_j y_{0i}}$ is an n vector with all components being 0, except the ith component being 1 if $a_j = y_{0i}$. The solution of such a problem is given by

$$\mathbf{y}_{a_j}(t) = \mathbf{K}(t, t_0)\mathbf{y}_{a_j}(t_0) + \int_{t_0}^{t} \mathbf{K}(t, \tau)\mathbf{f}_{a_j}(\tau)\mathrm{d}\tau \tag{14}$$

$\mathbf{K}(t, \tau)$ is an $n \times n$ matrix called the kernel or Green's function of the problem. In this case sensitivities to the parameters a_j are calculated.

A Green's functions approach uses the above formulation and a specific approach to solve for $\mathbf{K}(t, \tau)$ to solve the sensitivity equation, however in most cases a numerical integration is required. Then having solved the system, the sensitivities are given by the derivatives of $\mathbf{y}$ with respect to the a_j's.

2.2.3 Adjoint method

In the adjoint sensitivity method, the original system, called the primary problem, is transformed into an adjoint system. With the adjoint method, a single solution of the primary problem for a given system is required for the calculation of local sensitivity derivatives of the performance measure to all the model parameters (cf. [42]).

The following conditions are sufficient and jointly necessary for the application of the adjoint sensitivity formulation:

1. The state variable must be continuous throughout the spatial domain.

2. The state variable must satisfy all spatial boundary conditions.

3. The model parameters a_k must all be defined.

Remark that the form of the problem does not matter.

Continuous form
Consider, for instance, a physical problem, that can be represented by the following system of K coupled non-linear equations, written in operator form as (cf. [1], [2]) :

$$\mathbf{N}[u(x), a(x)] = \mathbf{Q}[a(x), x], , \tag{15}$$

where

1. $x = (x_1, ..., x_J)$ the phase-space position vector, $x \in \Omega \subset \mathrm{IR}^J$,
2. $u(x) = [u_1(x), ..., u_K(x)]$ the state vector; $u(x) \in E_u$, where E_u is a normed linear space,
3. $a(x) = [a_1(x), ..., a_L(x)]$ the vector of system parameters; $a \in E_a$ also a normed linear space,
4. $\mathbf{Q}[a(x), x] = [Q_1(a, x), ..., Q_K(a, x)]^T$ a vector containing inhomogeneous source terms; $\mathbf{Q} \in E_Q$, where E_Q again a normed linear space. The components of $\mathbf{Q}$ are operators acting on $a(x)$ and x.
5. $\mathbf{N} = [N_1(u, a), ..., N_K(u, a)]^T$, a vector containing non-linear operators acting on the state vector $u(x)$, and in general also on the vector of system parameters $a(x)$.

If differential operators appear in equation (15), then it is essential to define boundary conditions. These can be represented by

$$\{\mathbf{B}(e) - \mathbf{A}(a)\}_{\partial\Omega} = \mathbf{O} \tag{16}$$

where $\mathbf{A}$ and $\mathbf{B}$ operators, $e = (u, a)$ and $\partial\Omega$ the boundary of Ω. The operator $\mathbf{A}(a)$ represents all inhomogeneous boundary terms.

The performance measure (ν) is defined as some function of the solution and the system parameters a_k (so, sensitivities to the system parameters will be calculated). The performance measure must be associated with the problem modelled by equations (15) and (16). In general, the performance measure is a non-linear functional of e, represented by the mapping

$$\nu(e) = D \subset E \to \Lambda,$$

Where D a finitely open subset of E and Λ the underlying scalar field of real numbers. For example,

$$\nu(\mathbf{u_i}, a_k) = \int\int g(\mathbf{u_i}, a_k) d\Omega, \tag{17}$$

where g is some function of the solution, for instance the velocity at some point.

The sensitivity of the performance measure to the parameters a_k is defined by the derivative $\partial\nu/\partial a_k$. Differentiating (15)-(17) and setting $\partial u_i/\partial a_k = \psi_i$ gives a set of equations in ψ_i given a solution of u_i. The adjoint equations are developed by multiplying the equations by an arbitrary function ψ_i^* and integrating over the area of the domain. Actually, the adjoint problem is a variation formulation of the original problem.

The calculation of the sensitivities requires the following steps [42]:

1. Solve (15) with (16) for u_i.

2. Choose a performance measure ν.

3. Formulate and solve the equations for ψ_i^*

4. Calculate the derivatives with respect to the parameter of interest in the adjoint problem.

5. Use the adjoint problem to calculate the sensitivities. Sensitivities with respect to additional parameters can be obtained by recalculating the derivatives in the adjoint problem.

The adjoint method can also be applied to a finite element or matrix form. Applying the adjoint method to the discrete form has several advantages, such as straightforward computer implementation. When the primary and adjoint problem are in discretised form, the solutions are values at a set of points throughout the domain. The sensitivities are given by the sum of the local contributions from each of these points. The spatial distribution of the local contributions shows how the sensitivity to a parameter varies over the domain.

On the other hand, the boundary and initial conditions can be more readily developed from the continuous form. In both forms, a non-linear primary problem, results in a linear adjoint problem. This linear adjoint system can be easily and cheaply solved, computationally more efficient than sampling and perturbation methods. This methodology can also be used to examine the physical processes, which control the system behaviour.

The limit-state function for which we finally want to determine the sensitivities, is not a smooth continuous function. The adjoint sensitivity method is not suitable for non-continuous functions, and therefore we won't further investigate this method.

2.3 Regression-based methods

Regression-based methods are based on the assumption of a linear model. These methods form an important class of statistical methods. The regression-based methods are often applied to rank-transformed data. They have desirable properties, but departure from linearity can cause serious difficulties and inaccuracies. The validity of the assumptions of a linear model is of importance in evaluating the results.

2.3.1 Linear regression

A reasonable measure of the importance R of the different input variables is given by [27]:

$$R_i^2 = b_i^2 V[X_i]/V[Z]. \tag{18}$$

Where a linear approximation for Z is assumed, namely

$$Z = b_0 + \sum_{i=1}^{k} b_i X_i + \epsilon \tag{19}$$

and where V denotes variance, b_0 is a constant, ϵ denotes the error and the X_i are assumed to be independent. This term measures the contribution of input X_i to the variance of the output Z under the linear analysis model assumptions.

For a subset X^s of s input variables, the measure of importance from the linear regression is a (multiple) correlation coefficient R^2 of the form

$$R^2 = \sum_{k \in I_s} (\hat{b}_k^s)^2 V[X_k] / \hat{V}[Z], \tag{20}$$

where $I_s = \{i_1, i_2, ..., i_s\}$ the set of subscripts of the elements of the set and $\hat{b}_k^s$ the estimator of the vector $(b_{i_1}, b_{i_2}, ..., b_{i_s})$. Equation (20) gives the fraction of the variance of Z, 'explained' by the regression. The variance of X^s comes from its probability distribution rather than being estimated from a sample of values.

This method is suitable if indeed the linear model is appropriate, so if Z indeed can be written as a linear combination of the X_i. This measure can be used to determine the linearity of the model under consideration.

2.3.2 Non-linear regression

Non-linear regression methods are in fact variance-based methods, which are described in the next sections. Three variance-based methods: Correlation Ratio, Sobol' Indices and Fourier Amplitude Sensitivity Test (FAST) are discussed in [35]. These methods and modified versions of these methods are discussed here. The first two can be calculated with Monte Carlo simulation. In [34] an Iterated Fractional Factorial Design is suggested instead of random sampling. Chapter 3 describes the Correlation Ratio and some other variance based methods. In Chapter 4 pays special attention to one variance-based method, namely the FAST method.

3 Correlation Ratio and other variance-based methods

The previous chapter stated that non-linear regression based methods are variance-based methods. Since we are looking for a importance measure that does not assume the model to be linear these methods are interesting. This chapter describes one of the variance-based sensitivity measures, namely the correlation ratio. It describes the mathematical foundation and various calculation methods. Further, it describes other variance-based methods, that is the Hora and Iman Importance measure, the Sobol' indices, the Total Effect indices of Homma and Saltelli and Jansen's measure. These measures appear to be equal to the correlation ratio or an extension of it. One variance-based method, the FAST-method is described more extensively in the next chapter.

3.1 Definition of the Correlation Ratio

Let the output variable Z be a function of k variables (cf. [33], [35]):

$$Z = Z(X_1, X_2, ..., X_k). \tag{21}$$

Assuming that the input is composed of independent random variables the joint probability density function of the input is

$$f(X_1, X_2, ..., X_k) = \prod_{j=1}^{k} f_j(X_j). \tag{22}$$

The unconditional and conditional variance of Z are given by

$$V(Z) = \int_{\Omega^k} Z(\mathbf{X})^2 \prod_{j=1}^{k} f_j(X_j)\mathrm{d}X_j - [E(Z)]^2 \tag{23}$$

and

$$\begin{aligned} V(Z|X_i = x_i) &= \int ... \int (z(x_i) - E(Z|X_i = x_i))^2 \prod_{j \neq i} f_j(X_j)\mathrm{d}X_j \\ &= \int ... \int z(x_i)^2 \prod_{j \neq i} f(X_j)\mathrm{d}X_j - [E(Z|X_i = x_i)]^2, \end{aligned} \tag{24}$$

where

$$E(Z|X_i = x_i) = \int ... \int z(x_i) \prod_{j \neq i} f_j(X_j)\mathrm{d}X_j \tag{25}$$

and

$$z(x_i) = Z(X_1, ..., X_{i-1}, x_i, X_{i+1}, ..., X_k). \tag{26}$$

The dependence of the variance $V(Z|X_i)$ upon the specific value x_i can be eliminated by averaging $V(Z|X_i)$ according to $f(X_i)$, to yield

$$E[V(Z|X_i)] = \int V(Z|X_i) f_i(X_i)\mathrm{d}X_i. \tag{27}$$

Substituting (24) in (27) gives

$$E[V(Z|X_i)] = \int Z(\mathbf{X})^2 \prod_{j=1}^{k} f_j(X_j)\mathrm{d}X_j - \int [E(Z|X_i)]^2 f_i(X_i)\mathrm{d}X_i. \tag{28}$$

Comparing (23) and (28) leads to the following relation

$$\begin{aligned} V(Z) - E[V(Z|X_i)] &= \int [E(Z|X_i)]^2 f_i(X_i)\mathrm{d}X_i - [E(Z)]^2 &(29)\\ V(Z) &= \int (E(Z|X_i) - E(Z))^2 f_i(X_i)\mathrm{d}X_i + E[V(Z|X_i)] \\ &= V[E(Z|X_i)] + E[V(Z|X_i)]. &(30) \end{aligned}$$

The first term of the variance decomposition (30) is called the Variance Conditional Expectation (VCE) and the second the residual part. The VCE is a suitable measure of the importance of X_i since the importance of X_i relates to how well X_i drives or controls z, that is, how well $E(Z|X_i = x)$ mimics z. So, if $V(E(Z|X_i = x))$ resembles $V(Z)$, X_i is very important.

The correlation ratio ([30], [25]) is defined by:

$$\eta_i^2 = \frac{V(E(Z|X_i = x))}{V(Z)}. \tag{31}$$

This seems to be a useful measure and it can be proved that in the linear case it corresponds to α_i^2, with α_i as defined in equation (6). The value of η_i lies between 0 and 1; $\eta_i = 1$ implies that X_i is very important, whereas $\eta_i = 0$ implies that X_i is not important.

It can be proved that the maximum of the correlation between the function $Z(\mathbf{X})$ and some other function of the variables is taken by $E(Z|\mathbf{X})$. The proof of this lemma (111) is given in Appendix A.1. From this lemma it follows that the correlation ratio is a good indicator that measures the correlation between $Z(\mathbf{X})$ and its variables X_i.

We can estimate η_i in various ways, but we cannot compute it in a straightforward way by MC methods. A variance components based method (due to McKay, [27]), which uses a single r-LHS (Latin Hypercubic Sample) of size m with r replicates (see [35]), can be used to estimate η^2. This method is computationally expensive, as it requires $rm \times (k+1)$ model evaluations, where k is the number of inputs. However, Ishigami and Homma ([21], [22]) proposed a more efficient way (based on the Hora and Iman Importance Measure) to evaluate $E(Z|X_i = x_i)$, reducing the number of model evaluations to $N \times (k+1)$, with N the sample size.

It can be shown that the measure of importance of single input factors based on correlation ratios is equivalent to the Sobol' and FAST first order sensitivity indices, which is discussed in the next sections.

In case of a linear model, it can be shown that the correlation ratio corresponds to the linear regression correlation coefficient. In the linear regression model the sensitivity measure is given by (18), where a linear approximation for Z is assumed, namely (19). It follows that:

$$\begin{aligned} E(Z|X_i = x_i) &= b_0 + E(\sum_{j\neq i}^{k} b_j X_j) + E(\epsilon) + b_i x_i \\ V(E(Z|X_i)) &= b_i^2 V(X_i) \\ \Rightarrow R_i^2 &= \frac{V(E(Z|X_i))}{V(Z)} = \eta_i^2. \end{aligned}$$

So, R_i^2 corresponds to the correlation ratio if the model is indeed linear. We have $R_i^2 = \rho^2(Z, X_i)$ if the variables are independent and the linear model holds. $R^2 = \sum_{i=1}^{k} \rho^2(Z, X_i)$ is the percentage of the variance of Z that is explained by the linear model with independent variables.

For a subset X^s we can calculate a multiple regression coefficient, as defined by (20). This measure is in fact an extension of the measure the linear regression correlation coefficient.

In this project three methods have been used to compute the correlation ratio. These are described in the next subsections. In most cases we cannot calculate the ratio (31) exactly and we have to estimate it then. An estimator of η_i^2 is given by a ratio of two estimators, namely

$$\hat{\eta_i}^2 = \frac{\widehat{VCE}(X_i)}{\hat{V}(Z)}, \tag{32}$$

where $VCE(X_i)$ is the Variance Conditional Expectation, defined by

$$VCE(X_i) = V(E(Z|X_i)). \tag{33}$$

The following sub-sections describe three ways to estimate the correlation ratio. The first is an estimator, defined by McKay, based on a Latin Hypercubic Sampling scheme. The second is a sampling scheme defined by Ishigami and Homma based on the importance measure of Hora and Iman and the third is a method called Averaged Sampling.

3.2 Calculation of the Correlation Ratio

3.2.1 Variance components based method/Latin Hypercubic sampling

An efficient estimation of the variance components for all the individual inputs can be achieved by using a Latin Hypercubic Sampling plan (LHS). A LHS of size m for k inputs is denoted by the matrix

$$D_0 = [\mathbf{X_1}, \mathbf{X_2}, ..., \mathbf{X_k}], \tag{34}$$

where each column vector $\mathbf{X}_i$ contains m values x_{ji} (for $j = 1, ..., m$), being samples of the following procedure: The range of each variable is divided into m disjunct intervals of equal probability and these intervals are randomised as to the position in the vector. For each interval a realisation of the variable is drawn. So, for each variable we get a set of realisations in the different intervals. Then the m values obtained for X_1 are combined in a random manner with the m values of X_2, etc. to form m k-tuples.

For instance, consider a variable which is uniformly distributed on the interval [0,1] and suppose m is 10. We have to draw a value in the 10 intervals $(\frac{i-1}{10}, \frac{i}{10})$, for $i = 1, ..., 10$ and then randomise at the position at the vector, which can be achieved with a random permutation. Then we get, for instance, the following vector (0.0910, 0.4014, 0.9663, 0.7782, 0.1623, 0.3978, 0.8126, 0.5416, 0.6488, 0.2003). For each variable, the vectors are calculated and together they form the columns of the D_0-matrix. A design matrix in an LHS with r replicates and m intervals ($r - LHS_m$) is obtained by r independent permutations of the columns, according to some specific conditions.

Due to the random pairing of intervals it is possible that undesired pairwise correlation's among some of the variables could be introduced. This is more likely if m is small. On the other hand, it is possible to obtain a specific rank correlation structure by selecting the proper combination of intervals as described in [19]. However, this method is only suitable if rank correlation's are used and if the complete rank correlation matrix is specified.

Suppose we want to sample two variables in three intervals. There are six interval combinations possible, which are given by:

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 1 \end{pmatrix},$$

where the first column represents the first variable and 1 indicates the first interval, and so on. The first interval pairing has rank correlation 1 and the second -1. These interval combinations can be interchanged. So, $\begin{pmatrix} 2 & 2 \\ 1 & 1 \\ 3 & 3 \end{pmatrix}$ is the same pairing as $\begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{pmatrix}$.

It is possible to draw values in equiprobable intervals for all kind of probability distributions. This can be achieved by first drawing values of the uniform distribution by

$$u_i' = \frac{i-1}{m} + \frac{1}{m} U_k, \tag{35}$$

where U_k is drawn from the uniform distribution on (0,1) and m the number of intervals and $i = 1, ..., m$. Then, we can get the desired values by transforming these numbers into the desired distribution by calculating $F^{-1}(u')$, where F is the cumulative probability function of the specific variable.

After pairing the intervals r times, we get a design matrix D given by:

$$\begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_r \end{pmatrix}.$$

The model outputs are denoted by z_{jl}. So, each D_l yields a vector of m output values z_j. The estimator of the output variance is given by

$$V(\hat{Z}) = \frac{1}{mr} \sum_{j=1}^{m} \sum_{l=1}^{r} (z_{jl} - \bar{z})^2, \tag{36}$$

where $\bar{z} = \frac{1}{mr} \sum_{j=1}^{m} \sum_{l=1}^{r} z_{jl}$ is the grand mean. This estimator is called the Total Mean Square.

McKay (1995) [27] shows that an biased estimate for η^2 is given by

$$\eta(\hat{X}_i) = \frac{r\sum_{j=1}^{m}(\bar{z}_{j.} - \bar{z})^2}{\sum_{j=1}^{m}\sum_{l=1}^{r}(z_{jl} - \bar{z})^2}, \tag{37}$$

where $z_{j.} = \frac{1}{r}\sum_{l=1}^{r} z_{jl}$. McKay shows that this estimator converges slowly to the analytical values by increasing the number of replicates.

3.2.2 Resampling based method

To explain the sampling scheme of Ishigami and Homma we will first derive the Homma and Iman Importance Measure I_i, on which the sampling scheme is based.

From (28) it follows that

$$V(Z) - E[V(Z|X_i)] = \int [E(Z|X_i)]^2 f_i(X_i)\mathrm{d}X_i - [E(Z)]^2 = U_i - [E(Z)]^2, \tag{38}$$

where

$$U_i = \int [E(Z|X_i)]^2 f_i(X_i)\mathrm{d}X_i. \tag{39}$$

Hora and Iman [33] define a measure of importance as

$$I_i = \sqrt{V(Z) - E[V(Z|X_i)]} = \sqrt{U_i - [E(Z)]^2}. \tag{40}$$

Since the quantity $E(Z)$ is a constant, the factor ranking is based on the values of U_i, which implies that a factor k is more important than factor l if $U_k > U_l$. The correlation ratio can be written as $\frac{I_i^2}{V(Z)}$.

In order to find an effective computation scheme we can rewrite U_i as

$$\begin{aligned} U_i &= \int \left\{ \int ... \int z(x_i) \prod_{j \neq i} f_j(X_j)\mathrm{d}X_i \right\}^2 f_i(x_i)\mathrm{d}x_i \\ &= \int\int ... \int Z(X_1, X_2, ..., x_i, ..., X_k) \times Z(X_1', X_2', ..., x_i, ..., X_k') \\ &\quad \times \left(\prod_{j \neq i} f_j(X_j)\mathrm{d}X_j \right) \left(\prod_{j \neq i} f_j(X_j')\mathrm{d}X_j' \right) f_i(x_i)\mathrm{d}x_i \\ &= \int\int ... \int Z(X_1, X_2, ..., X_k) \times Z(X_1', X_2', ..., X_i, ..., X_k') \\ &\quad \times \left(\prod_{j=1}^{k} f_j(X_j) \right) \left(\prod_{j \neq i} f_j(X_j')\mathrm{d}X_j' \right), \end{aligned}$$

where $z(x_i) = Z(X_1, ..., X_{i-1}, x_i, X_{i+1}, ..., X_k)$, where X_i' is a resampling of X_i and $\times$ indicates multiplication. From the above equations it follows that U_i is the expectation value of the function

$$\begin{aligned} &H(X_1, X_2, ..., X_k, X_1', X_2', ..., X_{i-1}', X_{i+1}', ..., X_k') = Z \times Z^i \\ &= Z(X_1, X_2, ..., X_k) \times Z(X_1', X_2', ..., X_{i-1}', X_i, X_{i+1}', ..., X_k'), \end{aligned} \tag{41}$$

of a set of $(2k-1)$ independent variables.

If N is the sample size used to estimate the expectation value of the output, the number of model evaluations needed to compute the expected value of this function is just $2N$, since the function H is expressed as a twofold product of the original model function Z. H can be evaluated by multiplying Z, computed from a sampled vector of the k input variables, with Z^i, which is computed by resampling all input variables, except the ith one. The total number of model evaluations needed to rank all the variables is $N \times (k+1)$ (cf. [33]).

So, an estimator for U_i is given by

$$\hat{U}_i = \frac{1}{N}\sum_{j=1}^{N} Z_j Z_j^i \tag{42}$$

Then, the correlation ratio can be estimated by

$$\frac{\hat{U}_i - \hat{E(Z)}^2}{\hat{V(Z)}} \tag{43}$$

where $E(Z)$ and $V(Z)$ can be estimated with, for instance, crude Monte Carlo simulation or from the first sample of size N (without resamples).

3.2.3 Averaged Sampling

This section describes another way to compute the correlation ratio. Consider $100N$, with N an integer, samples of X_1 till X_k. Calculate the corresponding output values Z. Consider the first variable X_1. Order (X_1, Z) according to increasing values of X_1. Construct 100 batches of size N. Each batch corresponds to a percentile of X_1. For the kth batch calculate the average of the Z values. This average equals $\frac{\sum Z|\overline{X_{1(k\%)}}}{N} \sim E(Z|\overline{X_{1(k\%)}})$. Calculate these averages for all $k \in (1, ..., 100)$. So, the averages $E(Z|\overline{X_{1(1\%)}})$ till $E(Z|\overline{X_{1(100\%)}})$ are calculated. The variance of these 100 values converges to the Variance Conditional Expectation $V(E(Z|X_i))$ for N big enough. The same procedure can be repeated for all variables. This method takes $100N$ model evaluations.

3.3 Hora and Iman Importance measure

Since the correlation ratio appears to lack robustness and can be highly influenced by outliers associated with long tailed input distributions, Hora and Iman (cf. [36]) propose a new statistic, which is based on estimating the quantity:

$$V[E(\log Z|X_j)]/V[\log Z] \tag{44}$$

where V stands for variance over all the possible values of X_j and $E[\log Z|X_j]$ is estimated using linear regression. This statistic is indeed robust and also computationally efficient, but conclusions drawn on log$Z$ are not easily converted back to Z.

As shown in section 3.2.2 Hora and Iman (cf. [33]) also define a measure of importance as

$$HI_i = \sqrt{V(Z) - E[V(Z|X_i)]} = \sqrt{U_i - [E(Z)]^2} = \sqrt{V(E(Z|X_i))}, \tag{45}$$

where

$$U_i = \int [E(Z|X_i)]^2 f_i(X_i) \mathrm{d}X_i, \tag{46}$$

and we have the following relationship $\eta_i^2 = \frac{HI_i^2}{V(Z)}$.

The HIIM (Hora and Iman Importance Measure) is however also not a robust measure. Hora and Iman (cf. [18]) suggest to replace the raw values by their ranks to improve robustness. However, a disadvantage of the rank version is that it does not weight the real fraction of the total variance accounted for by the single input variables.

3.4 Method of Sobol

Let $Z = g(\mathbf{X})$ be the function of a set of input factors, $\mathbf{X}$. The main idea of Sobol' approach is to decompose the $g(\mathbf{X})$ into summands of increasing dimensionality. The g-function is assumed to be defined on the k-dimensional unit cube:

$$K^k = \left\{ \mathbf{X} | 0 \leq X_i \leq 1; \quad i = 1, ..., k \right\} \tag{47}$$

Under assumptions described in [41] every g-function can be decomposed in summands of increasing dimensionality:

$$g(X_1, ..X_k) = g_0 + \sum_{i=1}^{k} g_i(X_i) + \sum_{1 \leq i < j \leq k} g_{ij}(X_i, X_j) + ... + g_{1,2,..,k}(X_1, ..., X_k). \tag{48}$$

For (48) to hold g_0 must be a constant, and the integrals of every summand over any of its own variables must be zero, i.e.

$$\int_0^1 g_{i_1,...,i_s}(X_{i_1}, ..., X_{i_s}) dX_{i_l} = 0, \quad \text{if} \quad 1 \leq l \leq s. \tag{49}$$

The decomposition has some additional properties. All the summands in (48) are orthogonal, i.e. if $(i_1, ..., i_s) \neq (j_1, ..., j_s)$, then

$$\int_{\Omega^k} g_{i_1,...,i_s} g_{j_1,...,j_s} d\mathbf{X} = 0. \tag{50}$$

Since at least one of the indices $(i_1, ..., i_s)$, $(j_1, ..., j_s)$ is not repeated twice the integral vanishes due to definition (49). Another property is that

$$g_0 = \int_{\Omega^k} g(\mathbf{X}) d\mathbf{X}. \tag{51}$$

Sobol' showed that the decomposition (48) is unique and that all the terms in (48) can be evaluated via multidimensional integrals, namely

$$g_i(X_i) \quad = \quad -g_0 + \int_0^1 ... \int_0^1 g(\mathbf{X}) d\mathbf{X}_{\setminus i}, \tag{52}$$

$$g_{ij}(X_i, X_j) \quad = \quad -g_0 - g_i(X_i) - g_j(X_j) + \int_0^1 ... \int_0^1 g(\mathbf{X}) d\mathbf{X}_{\setminus (ij)}, \tag{53}$$

where $d\mathbf{X}_{\setminus i}$, $d\mathbf{X}_{\setminus(ij)}$ denote integration over all variables except X_i, and X_i and X_j, respectively.

The sensitivity estimates $S_{i_1,...,i_s}$ are:

$$S_{i_1,...,i_s} = \frac{D_{i_1,...,i_s}}{D} \tag{54}$$

where

$$D = \int_{\Omega^k} g^2(\mathbf{X})d\mathbf{X} - g_0^2 \tag{55}$$

and

$$D_{i_1,...,i_s} = \int_0^1 ... \int_o^1 g^2_{i_1,...,i_s}(X_1, ..., X_s)dX_{i_1}, ..., dX_{i_s} \tag{56}$$

where $1 \leq i_1 < ... < i_s \leq k$ and $s = 1, ..., k$. S_i is called the first order sensitivity index for factor X_i, which measures the main effect of X_i on the output (the contribution of X_i to the variance of $f(\mathbf{X})$). S_{ij} for $i \neq j$, is called the second order sensitivity index which measures the interaction effect, and so on.

By squaring and integrating (48) over K^k and using the orthogonality property it can be proved that

$$D = \sum_{i=1}^{k} D_i + \sum_{1\leq i<j\leq k} D_{ij} + ... + D_{1,2,...,k}. \tag{57}$$

From equations (54) - (57) it follows that:

$$\sum_{i=1}^{k} S_i + \sum_{1\leq i<j\leq k} S_{ij} + ... + S_{1,2,...,k} = 1. \tag{58}$$

This decomposition method can give valuable insight into the problem, but it becomes very impractical for systems with many factors. We are actually interested in a measure that is also suitable for large, complex systems, so this is a real disadvantage. The number of terms can be as high as $2^k - 1$. Then $N \times (2^k)$ model evaluations are needed, where N is the sample size. If only the first indices are considered, then $N \times (k + 1)$ model evaluations are needed. The main drawback in the use of $S_{i_1,...,i_s}$, is the large sample size needed for their evaluation (cf. [17]). However, the Sobol' indices are model independent and accurate.

It can be shown that the first order indices are equal to the correlation ratio. We have:

$$Z(X_1, ..X_k) = g_0 + \sum_{i=1}^{k} g_i(X_i) + \sum_{1\leq i<j\leq k} g_{ij}(X_i, X_j) + ... + g_{1,2,..,k}(X_1, ..., X_k)$$

It follows that

$$g_i(X_i) = E(Z|X_i) - g_0 \tag{59}$$

where $g_0 = E(Z(\mathbf{X})) = E(Z)$. In the same way

$$\begin{aligned} g_{ij}(X_i, X_j) &= E(Z|X_i, X_j) - g_i(X_i) - g_j(X_j) - g_0 \\ &= E(Z|X_i, X_j) - E(Z|X_i) - E(Z|X_j) + E(Z) \end{aligned}$$

and

$$\begin{aligned} g_{ijk}(X_i, X_j, X_k) &= E(g|X_i, X_j, X_k) - g_{ij}(X_i, X_j) - g_{ik}(X_i, X_k) \\ &\quad -g_{jk}(X_j, X_k) - g_i(X_i) - g_j(X_j) - g_k(X_k) - g_0 \\ &= E(Z|X_i, X_j, X_k) - E(Z|X_i, X_j) - E(Z|X_i, X_k) - E(Z|X_j, X_k) \\ &\quad +E(Z|X_i) + E(Z|X_j) + E(Z|X_k) - E(Z). \end{aligned}$$

The partial variances are defined by:

$$D_{i_1,...,i_s} = \int_0^1 ... \int_o^1 g^2_{i_1,...,i_s}(X_1, ..., X_s)dX_{i_1}, ..., dX_{i_s}, \tag{60}$$

where $1 \leq i_1 < ... < i_s \leq k$ and $s = 1, ..., k$. and the total variance is given by

$$D = \int_{\Omega^k} g^2(\mathbf{X})d\mathbf{X} - g_0^2. \tag{61}$$

For the first order indices we get:

$$\begin{aligned} D_i &= \int g_i^2(X_i)dX_i = \int \left[\int ... \int g(\mathbf{X})\{d\mathbf{X}/dX_i\} - g_0 \right]^2 dX_i \\ &= g_0^2 - 2g_0 \int g(\mathbf{X})d\mathbf{X} + \int \left[\int ... \int g(\mathbf{X})\{d\mathbf{X}/dX_i\} \right. \\ &\quad \left. \times \int ... \int g(\mathbf{X})d\mathbf{X}/dX_i\} \right] dX_i \\ &= -g_0^2 + \int ... \int g(\mathbf{X}_{\setminus i}, X_i)g(\mathbf{X'}_{\setminus i}, X_i)dX_i d\mathbf{X}_{\setminus i}d\mathbf{X'}_{\setminus i} \\ &= V(E(Z|X_i)). \end{aligned} \tag{62}$$

So, the first order indices are given by:

$$S_i = \frac{D_i}{D} = \frac{V(E(Z|X_i))}{V(Z)} = \eta_i^2. \tag{63}$$

The first order indices correspond to the correlation ratio. The higher order sensitivity measures are given by (54). For instance, the second order indices measure the interaction effect due to X_i and X_j, which cannot be explained by the sum of the individual effects of X_i and X_j.

3.5 Homma and Saltelli Importance Measure/Total Effect Indices

In [17] Homma and Saltelli propose a measure for global sensitivity analysis of non-linear models based on the Sobol' sensitivity indices. They partition the variables into two subsets, one containing only the given variable X_i, and the complementary set $\mathbf{X}_{\setminus i}$ containing all the X_j with $j \neq i$. The decomposition of $g(X)$ turns into:

$$g(X) = g_0 + g_i(X_i) + g_{\setminus i}(\mathbf{X}_{\setminus i}) + g_{i,\setminus i}(X_i, \mathbf{X}_{\setminus i}), \tag{64}$$

where $g_i(X_i)$ denotes terms only involving X_i, $g_{\backslash i}(\mathbf{X}_{\backslash i})$ denotes the terms considering all variables except X_i and $g_{i,\backslash i}(X_i, \mathbf{X}_{\backslash i})$ denotes the terms representing the interaction between X_i and $\mathbf{X}_{\backslash i}$. For instance, suppose that

$$g(X_1, X_2, X_3) = 2X_1 + X_2 + 3X_3 + X_1^2 + X_1X_2 + X_1X_3 + X_2X_3 + X_1X_2X_3.$$

Then

$$\begin{aligned} g_1(X_1) &= 2X_1 + X_1^2 \\ g_{\backslash 1}(\mathbf{X}_{\backslash 1}) &= X_2 + 3X_3 + X_2X_3 \\ g_{1,\backslash 1}(X_1, \mathbf{X}_{\backslash 1}) &= X_1X_2 + X_1X_3 + X_1X_2X_3. \end{aligned}$$

Using definitions (61) and (60) the total variance D can be given by:

$$D = D_i + D_{\backslash i} + D_{i,\backslash i}. \tag{65}$$

And a new sensitivity index S_{T_i} is introduced:

$$S_{T_i} \equiv S_i + S_{i,\backslash i} = 1 - S_{\backslash i}, \tag{66}$$

where $S_{\backslash i}$ equals the sum of all the $S_{i_1,\ldots,i_s}$ terms where the index i is excluded. Consequently, S_{T_i} denotes the 'total' effect of variable X_i, which includes the fraction of variance accounted for by variable X_i alone and the fraction accounted for by any combination of X_i with the remaining variables. $S_{\backslash i}$ can be estimated with one Monte Carlo integral (67) and this approach reduces the number of model evaluations to $N \times (k+1)$, where N the sample size and k the number of variables.

Comparing this measure with the Sobol indices we see that the total effect indices correspond to the sum of the Sobol indices that involve the factor in question. If there is no interaction between the variables the measure should be equal to the first order Sobol indices.

Homma and Saltelli propose the following Monte Carlo integral to estimate the $S_{\backslash i}$-term.

$$S_{\backslash i} = \frac{1}{D}\frac{1}{N}\sum_{m=1}^{N} Z_m Z_m^i - E(Z)^2 \tag{67}$$

where Z_m^i is the mth model evaluation of the all the variables, with X_i resampled.

3.6 Jansen's total effect measure

Jansen *et al* (cf. [23]) proposed a measure to compute the first order sensitivity of an input factor X_i, namely

$$D_i^J = V(Z) - \frac{1}{2}E[Z(X_i, \mathbf{X}_{\backslash i}) - Z(X_i, \mathbf{X}'_{\backslash i})]^2, \tag{68}$$

and the following to compute the total effect,

$$D_i^{J_{\text{tot}}} = \frac{1}{2}E[Z(X_i, \mathbf{X}_{\backslash i}) - Z(X_i, \mathbf{X}'_{\backslash i})]^2, \tag{69}$$

where $(X_i, \mathbf{X}'_{\backslash i})$ indicates that all variables are resampled, except for the ith one.

So, Jansen used the mean squared difference to compute the indices, whereas Sobol' indices use the product. It can be shown that

$$\frac{1}{2}E[Z(X_i,\mathbf{X}_{\backslash i})-Z(X_i,\mathbf{X}'_{\backslash i})]^2 = V(Z)-\mathrm{Cov}[Z(X_i,\mathbf{X}_{\backslash i})-Z(X_i,\mathbf{X}'_{\backslash i})] \tag{70}$$

and

$$\frac{1}{2}E[Z(X_i,\mathbf{X}_{\backslash i})-Z(X'_i,\mathbf{X}_{\backslash i})]^2 = V(Z)-\mathrm{Cov}[Z(X_i,\mathbf{X}_{\backslash i})-Z(X'_i,\mathbf{X}_{\backslash i})]. \tag{71}$$

These two covariance's (70), (71) are equivalent to the Sobol' first order and total effect indices (cf. [35]).

Jansen's total effect measure together with a Winding Stairs sampling scheme (cf. [35]) seems to be superior to the Sobol' total effect indices. However, this superiority is still subject of investigation.

The Sobol' first order indices equal the correlation ratio and can be estimated in the following way:

$$\hat{D_i} = \frac{1}{N}\sum_{m=1}^{N} Z(\mathbf{X_m},X_{im})Z(\mathbf{X_m}',X_{im}) - \hat{g_0}^2, \tag{72}$$

where $Z(\mathbf{X_m}',X_{im})$ is the mth model evaluation of Z with all variables, except X_i, resampled and $Z(\mathbf{X_m},X_{im})$ is the model evaluation with all the original values of the input factors. This estimator (72) is exactly the estimator for covariance (70), since

$$\begin{aligned}
\mathrm{Cov}[Z(X_i,\mathbf{X}_{\backslash i}),Z(X_i,\mathbf{X}'_{\backslash i})] &= -\frac{1}{2}E[Z(X_i,\mathbf{X}_{\backslash i})-Z(X_i,\mathbf{X}'_{\backslash i})]^2+V(Z)\\
&= -\frac{1}{2}E[Z(X_i,\mathbf{X}_{\backslash i})-Z(X_i,\mathbf{X}'_{\backslash i})]^2+EZ^2-(EZ)^2\\
&= -\frac{1}{2}[2EZ^2(\mathbf{X})-2EZ(X_i,\mathbf{X}_{\backslash \mathbf{i}})Z(X'_i,\mathbf{X}_{\backslash \mathbf{i}})]\\
&\quad +EZ^2-(EZ)^2\\
&= E(Z(X_i,\mathbf{X}_{\backslash \mathbf{i}})Z(X'_i,\mathbf{X}_{\backslash \mathbf{i}}))-(EZ)^2\\
&= \int Z(X_i,X_{\backslash i})Z(X_i,X'_{\backslash i})dX_i dX_{\backslash i}dX'_{\backslash i}-(EZ)^2,
\end{aligned}$$

which can be estimated by (72). This calculation method also equals the resampling-based method.

In the same way it can be shown that the total effect measure equals the total effect Sobol' indices.

The total effect Sobol' indices can be estimated by:

$$\hat{D_{\backslash i}} = \frac{1}{N}\sum_{m=1}^{N} Z(X_{im},\mathbf{X_m})Z(X'_{im},\mathbf{X_m}) - (EZ)^2. \tag{73}$$

And

$$
\begin{aligned}
\mathrm{Cov}[Z(X_i, \mathbf{X}_{\setminus i}), Z(X_i', \mathbf{X}_{\setminus i})] &= -\frac{1}{2}E[Z(X_i, \mathbf{X}_{\setminus i}) - Z(X_i', \mathbf{X}_{\setminus i})]^2 + V(Z) \\
&= -\frac{1}{2}[2E(Z^2) - 2E(Z(X_i, \mathbf{X}_{\setminus i})Z(X_i', \mathbf{X}_{\setminus i}))] \\
&\quad + E(Z^2) - (EZ)^2 \\
&= E(Z(X_i, \mathbf{X}_{\setminus i})Z(X_i', \mathbf{X}_{\setminus i})) - (EZ)^2 \\
&= \int Z(X_i, \mathbf{X}_{im})Z(X_i', \mathbf{X}_{im})dX_i dX_i' d\mathbf{X}_{\setminus i} - (EZ)^2,
\end{aligned}
$$

which can be estimated by the estimator given in (73). So, the Sobol total effect indices correspond to the Jansen total effect measure. The Jansen total effect measure is in fact an extension of the correlation ratio for correlated variables.

4 The FAST Method

This chapter describes the Fourier Amplitude Sensitivity Test (FAST), which is a method based on Fourier analysis, to calculate the correlation ratio. This method seems to be computationally efficient and accurate. This chapter also describes an extension of the classical FAST, the so-called Extended FAST.

4.1 Definition

Consider the model $Z = Z(\mathbf{X})$. The output of Z is related to a set of k input factors $\mathbf{X} = (X_1, X_2, ..., X_k)$ (cf. [35], [37]). Assume the domain of the input factors to be the unit hypercube

$$K^k = (\mathbf{X} | 0 \leq X_i \leq 1; i = 1, ..., k), \tag{74}$$

where $\mathbf{X}$ is a random vector with an assumed probability density function $f(\mathbf{X}) = f(X_1, X_2, ..., X_k)$.

The main idea of the Fourier Amplitude Sensitivity Test (FAST) is to convert the k-dimensional integral (which is needed to calculate the variance) into a one-dimensional integral. This is done by exploring the space K^k along a curve, defined by a set of parametric equations,

$$X_i(s) = G_i(\sin(\omega_i s)), \quad i = 1, 2, ..., k, \tag{75}$$

where s is a scalar variable varying over the range $-\infty...\infty$ and $\{\omega_i\}$ is a set of integer angular frequencies.

For properly chosen ω_i and G_i, the expectation of Z can be approximated by

$$E(Z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Z(s) ds, \tag{76}$$

where $Z(s) = Z(G_1(\sin(\omega_1 s)), ..., G_k(\sin(\omega_k s))) = Z(X_1, ..., X_k)$.

Due to the use of $\sin(\omega_i s)$ the output functions becomes periodic and can be Fourier analysed. By using the properties of Fourier series an approximation of the variance of Z is given by

$$\begin{aligned} V(Z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} Z(s)^2 ds - [E(Z)]^2 \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} Z(s)^2 ds - [\frac{1}{2\pi} \int_{-\pi}^{\pi} Z(s) ds]^2 \\ &\simeq \sum_{\omega=-\infty}^{\infty} (A_\omega^2 + B_\omega^2) - (A_0^2 + B_0^2) \\ &\simeq 2 \sum_{\omega=1}^{\infty} (A_\omega^2 + B_\omega^2) \stackrel{def}{=} \hat{D}^{\text{FAST}}, \end{aligned} \tag{77}$$

where A_ω and B_ω are the Fourier coefficients, defined by

$$A_\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} Z(s) \cos(\omega s) ds \tag{78}$$

and

$$B_\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} Z(s)\sin(\omega s)ds. \tag{79}$$

Thus, the expressions in (76) and (77) provide means to estimate the expected value and variance associated with Z.

4.2 Choice of the search curve

Cukier [13] proposes to choose a search curve $G_i(s)$ which is the solution of the following differential equation, with the boundary condition $G_i(0) = 0$:

$$\pi(1 - X_i^2)^{1/2} f_i(G_i)\frac{dG_i(X_i)}{dX_i} = 1 \tag{80}$$

This equation is based on the following. Cukier states: 'If we wish to obtain a specific (probability) distribution $f_i(X_i)$ in the ith direction, it is necessary to choose G_i such that the fraction of the total arc length of the curve which lies between the values X_i and $X_i + dX_i$ is equal to $f_i(X_i)dX_i$.' To meet this requirement $G_i(X_i)$ is taken to be the solution of (80). Cukier states that this equation can be solved by quadrature for *any* distribution function f_i. Weyl (cf. [50]) shows that the curve obtained by solving equations (75) and (80) is space filling if a set of incommensurably frequencies is used. A set of frequencies is called incommensurate if none of the frequencies may be obtained as a linear combination, with integer coefficients, of the other frequencies.

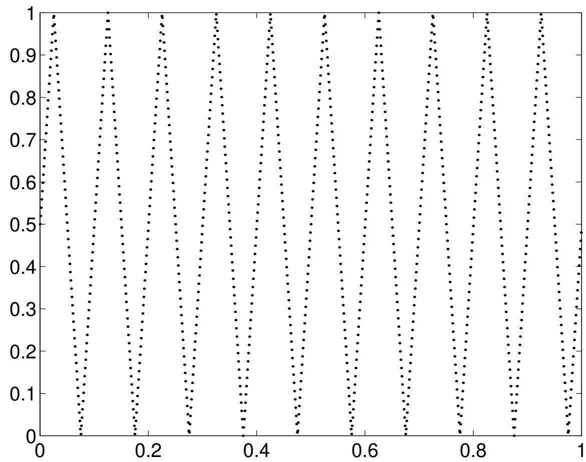

Figure 3: Sampling with transformation given in (81)

A solution for a uniform distribution is given by the following transformation [37]:

$$X_i(s) = \frac{1}{2} + \frac{1}{\pi}\arcsin(\sin\omega_i s). \tag{81}$$

By substitution of $f_i(X_i) = 1$ we obtain this solution. The term $\arcsin(\sin\omega_i s))$ can not be replaced by $\omega_i s$, since the sine maps ω_i into [-1,1] and the arcsine maps this into $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Dividing by π and adding $\frac{1}{2}$ results into points in [0,1], whereas $\omega_i s$ can result into values greater than $\frac{\pi}{2}$ (depending on ω_i) and sampling on [0,1] is then not guaranteed. Other transformations are also possible, cf. [11]. In this study transformation (81) has been used. Figure 3 and Figure 4 show how the points can be chosen if this transformation is used.

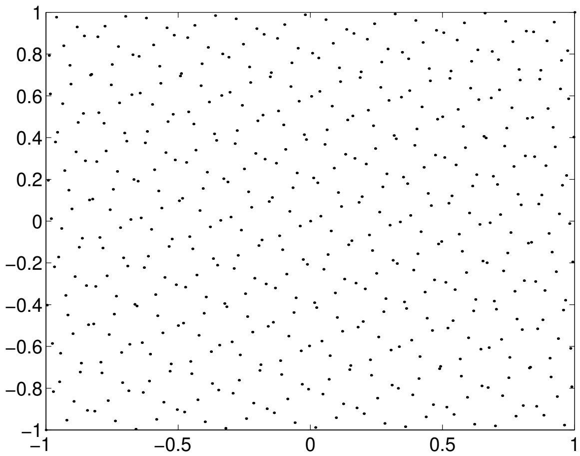

Figure 4: Sampling of U[-1,1] variables

Substituting a normal probability distribution into differential equation (80) leads to a non-linear, non-separable differential equation, which can not easily be solved, only approximated. However, having a solution for a uniform distribution, it is possible to transform this distribution to any desired distribution by taking the inverse cumulative distribution function, cf. Figure 5.

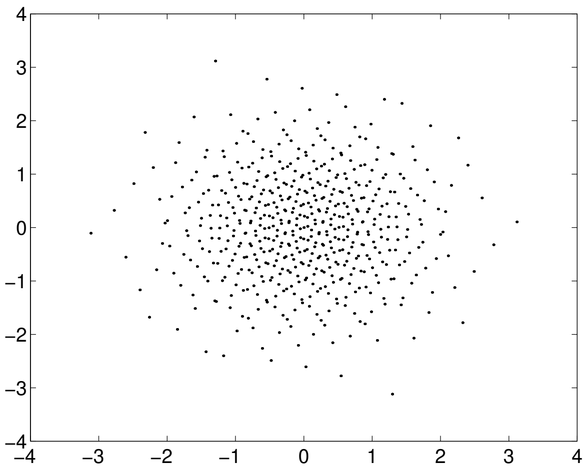

Figure 5: Normal distribution

4.3 First Order Indices (Classical FAST)

The first order indices are computed by evaluating A_j and B_j for the fundamental frequency ω_i, for $i = 1, 2, ..., k$ and its higher harmonics, denoted by $p\omega_i$ for $p = 1, 2, ...$ Provided that ω_i are integers, the contribution to total variance of Z by X_i can be approximated by

$$\hat{D}_{\omega_i} = 2\sum_{p=1}^{\infty}(A^2_{p\omega_i} + B^2_{p\omega_i}). \tag{82}$$

The Z-function becomes a function of the frequencies, since X_i are functions of the frequencies and as can be seen in Figure 6 the Fourier series has amplitudes at the assigned frequencies. If another set of frequencies had been chosen, the harmonics would have been at other points.

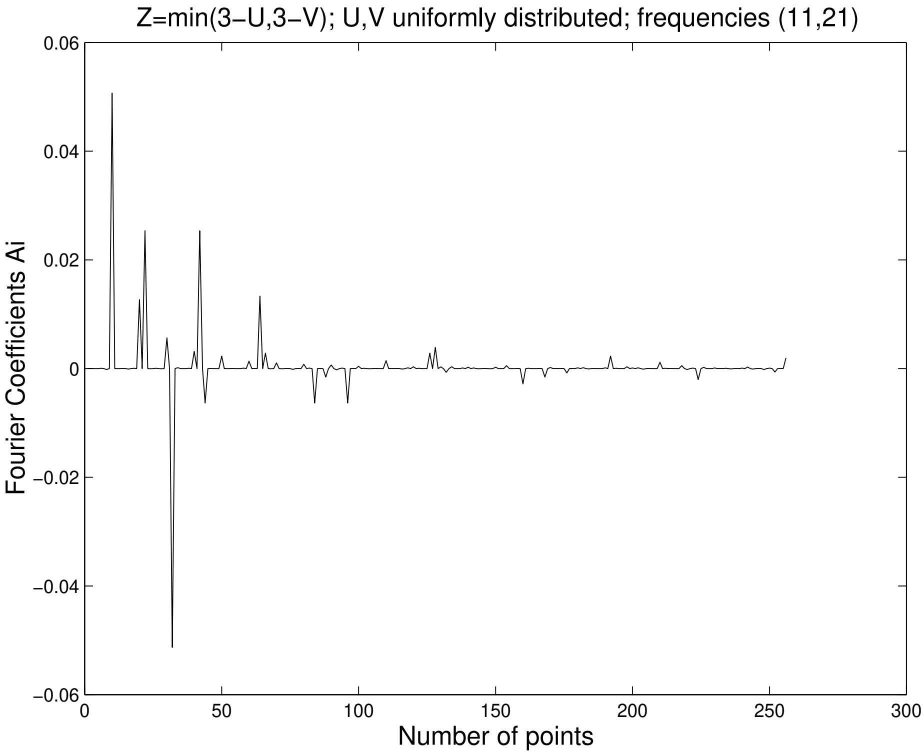

Figure 6: Plot of an Artificial Spectrum with $\omega_1 = 11$ and $\omega_2 = 21$

Furthermore, the Fourier amplitudes decrease as p increases, hence we may further approximate D_{ω_i} by

$$\hat{D}_{\omega_i} = 2\sum_{p=1}^{M}(A^2_{p\omega_i} + B^2_{p\omega_i}) \tag{83}$$

where M is the maximum harmonic we consider, which is usually taken to be 4 or 6. Figure 6 shows that indeed the amplitudes decrease rapidly. Taking only the first M harmonics decreases the problem of interference, which means that frequencies are multiples of each other. For instance, if we consider the frequencies 11 and 21, then the eleventh multiple of 21 is of course a multiple of 11. By considering only the first 4 or 6 multiples this problem does not occur.

The ratios $\hat{D}_{\omega_i}/\hat{D}^{\mathrm{FAST}}$, denoted by S_i^{FAST}, provide a way to rank individual factors on the basis of their contribution to the variance of Z, D^{FAST}. It can be shown that S_i^{FAST} equals the Sobol' sensitivity indices of the first order, S_i, and therefore also the correlation ratio.

4.4 Choice of the frequencies

To compute the partial variances, D_{ω_i}, the frequencies ω_i must be chosen. The search curve drives arbitrarily close to any point X of the input domain *if and only if* the set of frequencies ω_i is incommensurate. The easiest way to obtain an incommensurably set of frequencies is by using irrational numbers. We will use integer frequencies, which is actually an approximation, due to limited computer precision. These frequencies cause a finite period for s, namely $(-\pi, \pi)$ or $(0, 2\pi)$, depending on the transformation function G_i, which can easily be handled on a computer. On this interval s is uniformly distributed, and the search curve traverses exactly once through the k-dimensional parameter space.

We won't further discuss how we can obtain such a set of frequencies. There exist numerical algorithms that provide a set of almost incommensurably frequencies. In the calculation of the first order indices an algorithm of Cukier [12] is used, providing us frequencies up to dimension 50.

The results of the FAST-analysis are independent of the set of frequencies used. This is due to the fact that the transformation depends on the assigned frequency, so the X_i's are functions of the frequencies.

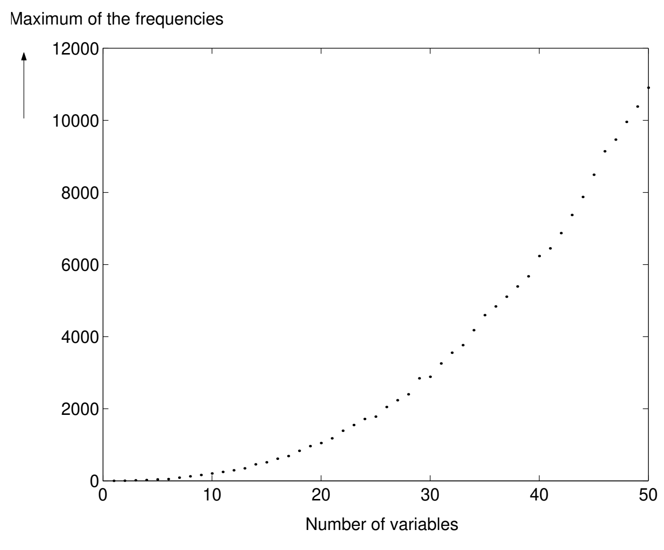

Figure 7: Maximum frequency as function of the number of variables

The minimum sample size needed to calculate the FAST measure depends on the maximum frequency and is given by

$$N_s = 2M\omega_{\max} + 1 \tag{84}$$

where M the interference factor (usually 4) and $\omega_{\max}$ the largest frequency among the set of ω_i frequencies. For the variable, to which the frequency $\omega_{\max}$ is assigned, at least $M\omega_{\max}$ points are needed, since the Mth multiple of the assigned frequency has to be added. Due to the Fourier transform, twice this number of points in needed, since the number of points will be halved because of symmetry of the Fourier spectrum. In [13] Cukier derives that, due to the Nyquist criterion, the sample size should at least be N_s.

The main advantage of FAST is that only one sample of size N_s is needed to compute all the first order indices. When the dimension of the problem becomes higher it becomes more difficult to find a set of incommensurably frequencies. Figure 7 shows that the maximum frequency grows quadratically as the number of variables increases. A least squares fit of a second order polynomial gives the following function $\omega_{\max} = 369.294 - 87.534k + 5.903k^2$. The minimum sample size is then given by $N_s = 8(369.294 - 87.534k + 5.903k^2)+1$. Leaving out the linear term we obtain $\omega_{\max} = -477.933 + 4.29k^2$ and $N_s = 8(-477.933 + 4.29) + 1$. With these formula the sample size for problems with more than ten variables can be predicted.

4.5 Total Effect Indices (Extended FAST)

Saltelli *et al* propose in [37] to calculate total effect indices. The computation of these indices does not give a complete characterisation of the system, but it allows a full quantification of the importance of each X_i.

Consider the frequencies that do not belong to the set $\{p_1\omega_1, p_2\omega_2, ..., p_k\omega_k\}$ (for $p_i = 1, 2, ..., \infty$ and $\forall i = 1, 2, ..., k$), which thus equals $\{\omega_1, 2\omega_1, ..., \omega_2, 2\omega_2, ..., \omega_k, 2\omega_k, ...\}$. These frequencies contain information about the residual variance,

$$D - \sum_{i}^{k} D_i, \tag{85}$$

that is not accounted for by the first order indices, that is, including the interactions between the factors at any order.

To calculate the total effect indices we must assign a new set of frequencies to the variables. If we are interested in the i-th factor we assign a high frequency to this variable and low ones to the other variables. Suppose we consider a function of two variables X_1 and X_2. By assigning $\omega_1 = 20$ and $\omega_2 = 1$, we can calculate the total effect index for X_1. Note that the components at $\{p\omega_1\} = \{20, 40, 60, ...\}$ contribute to D_1 and decrease rapidly in amplitude as p increases, so the variance $D_{(\backslash 1)}$, where $(\backslash 1)$ denotes 'all but 1', that is complementary to 1, of the $X_{(\backslash 1)}$ (set containing all the X_j with $j \neq i$, which is in this case only X_2) can be estimated from the first few spectral components. (The estimation from the first few spectral components is valid, since the higher harmonics of $\omega_{(\backslash 1)} = 1$ are 2,3,4,... and usually converge to zero after a few terms). Finally, the information about the term $D_{1(\backslash 1)}$, representing the interaction between X_1 and $\mathbf{X}_{\backslash 1}$, can be estimated from all the other frequencies in $[1, M\omega_{\max}]$. To avoid potential overlap we only add the first $(\omega_{\max}/2)$ components. We can now calculate $D_{(\backslash 1)}$ from

$$D_{(\backslash 1)} = \sum_{k=1}^{\omega_{\max}/2} (A_k^2 + B_k^2) = \sum_{k=1}^{10} (A_k^2 + B_k^2). \tag{86}$$

Suppose, for instance, we consider the following function $Z = X_1 + X_2 + X_3 + X_2X_3^2 + X_1X_3$ and we are interested in the first variable. Again, we assign X_1 frequency 20 and the others

1. Then information about the first order indices is contained in the first M multiples of 20. Information about X_2 and X_3 is contained in the first harmonics, $\{1, 2, 3, 4, ...\}$. These frequencies contain both the first order information about X_2, X_3 and the information about the interaction term $X_2X_3^2$. Then, the information about the interaction term X_1X_3 can be found in the other frequencies in $[1, M\omega_{\max}]$. Since we want to estimate $D_{(\backslash 1)}$ we only add the first $(\omega_{\max}/2)$ components, to make sure we don't add up frequencies which could contain information about $D_{1(\backslash 1)}$.

In general, $\hat{D}_{(\backslash i)}$ is calculated by adding all the spectral components in the frequency range $[1, \omega_i/2]$.

A measure $\hat{D}^{\text{FAST}}_{\backslash i}$ is defined by:

$$\hat{D}^{\text{FAST}}_{\backslash i} = 2\sum_{p=1}^{\infty}(A^2_{p\omega\backslash i} + B^2_{p\omega\backslash i}), \tag{87}$$

where $\hat{D}_{\backslash i}$ a measure including all the effects of any orders which do not involve the factor X_i. The total effect indices, $ST(i)$, are computed by using $ST(i) = 1 - \frac{\hat{D}^{\text{FAST}}_{\backslash i}}{\hat{D}^{\text{FAST}}}$.

4.6 Choice of frequencies for the total effect indices

The frequencies for the total effect indices are chosen according to an algorithm proposed by Saltelli *et al* [37]: First we set the maximum allowable frequency for the complementary factors $X_{\backslash i}$'s, i.e.

$$\max\{\omega_{\backslash i}\} = \frac{\omega_i}{2M} \tag{88}$$

where M is the number of terms considered in the partial variances summation and ω_i the frequency assigned to the factor of interest. The factor of interest gets a high frequency, for instance 1024, whereas the other factors get low frequencies. The frequencies for the complementary set are at most $\frac{\omega_i}{2M}$ to avoid overlap and they are chosen in such way that the entire range $[1, \max\{\omega_{\backslash i}\}]$ is covered, meeting the following constraints:

- The step between two consecutive frequencies must be as large as possible.
- The number of factors to which the same frequency is assigned must be as low as possible.

This algorithm is used in the calculation of the total effect indices in this study. The extended FAST method takes kN_s model evaluations. So, more information is obtained at a higher price.

4.7 An example

To illustrate how the FAST-method works we describe an example, based on [35] and [37]. Consider the *g function* of Sobol':

$$f = \prod_{i=1}^{k} g_i(X_i) \tag{89}$$

where k is the total number of input factors, X_i's, and $g_i(X_i)$ is given by

$$g_i(X_i) = \frac{|4X_i - 2| + a_i}{1 + a_i} \tag{90}$$

for $0 \leq X_i \leq 1$ and $a_i > 0$. We calculate both the first order indices and the total effect indices for the case with 8 input factors. The parameters a_i's are chosen to be $\{0, 1, 4.5, 9, 99, 99, 99, 99\}$ and the frequencies are taken to be $\{23, 55, 77, 97, 107, 113, 121, 125\}$ (based on table 2, [37]). By assigning 99 to the a_5 till a_8 we expect these parameters to be not important.

In our example the minimum sample size is $2 \times 4 \times 125 + 1 = 1001$. Since we use the fast Fourier transform, which requires the number of points to be a multiple of 2, we use 1024 as sample size.

With the assigned frequencies we can now calculate the first order indices, according to the procedure described in 4.3.

Dfast	0.9351
S1	0.7132
S2	0.1782
S3	0.0236
S4	0.0072
S5	7.8972e-005
S6	6.5251e-004
S7	1.8159e-005
S8	1.0196e-006

Table 2: First Order Indices of g-function of Sobol

The results (see Table 2) correspond to the results given in figure 7 of [35]. Repeating the procedure with another set of frequencies, namely $\{19, 59, 91, 113, 133, 143, 149, 157\}$, we get the results in Table 3.

Dfast	0.9328
S1	0.7132
S2	0.1785
S3	0.0236
S4	0.0072
S5	6.6674e-004
S6	6.6313e-004
S7	3.0194e-007
S8	4.9137e-005

Table 3: First Order Indices of g-function of Sobol with another set of frequencies

As expected, the results do not differ very much from the previous ones, except for the non-important variables. The main goal of a sensitivity analysis is to identify the important variables, whereas the quantitative importance of the unimportant variables plays a minor

ST1	0.8030
ST2	0.2597
ST3	0.0362
ST4	0.0106
ST5	1.1949e-004
ST6	9.1832e-004
ST7	3.6444e-005
ST8	3.6444e-005

Table 4: Total Effect Indices for the g-function of Sobol

part. The results should not be influenced by the choice of the frequencies, as long as these frequencies are nearly incommensurate.

If we want to calculate the total effect indices, we have to repeat the procedure with a different set of frequencies for each factor of interest. For the first variable we assign the frequencies $\{128, 3, 5, 7, 9, 11, 13, 15\}$. By using the algorithm defined in 4.5, we get $\max\{\omega_i\} = \frac{128}{8} = 16$. So, the range [1,16] must be covered. This is done by the following set $\{1, 3, 5, 7, 9, 11, 13, 15\}$. For the ith factor of interest the ith frequency is interchanged with 128.

The total effect index for the first factor can now be calculated by adding up the first 64 spectral components. If we repeat the procedure for all the variables, we get the results as shown in Table 4. These results correspond to the results in figure 8 of [35].

4.8 Final remarks

The FAST-method is a computationally efficient, robust and model-independent measure. Another advantage is that it allows a good visualisation of the portion of the total variance due to the various input factors. Disadvantages are that the underlying mathematics is complicated and that it is not possible to specify correlation's or other types of restrictions between variables (cf. [37]). It is also not completely clear how the differential equation (80) is obtained. Further research is needed to figure out the idea behind this equation.

The first order FAST indices equal the correlation ratio, since

$$V(Z) = 2\sum_{\omega=1}^{\infty}(A_\omega^2 + B_\omega^2) \tag{91}$$

and

$$V(E(Z|X_i)) = 2\sum_{p=1}^{\infty}(A_{p\omega_i}^2 + B_{p\omega_i}^2) \tag{92}$$

with ω_i the fundamental frequency of X_i (cf. [13]).

The extended FAST is an extension of the first order indices, which allows to take into account interaction effects between the different variables. This method takes much computational effort, as it requires kN_s model evaluations. However, it is possible to do a cheaper, but less accurate, extended FAST analysis by assigning a frequency of 8 to the factor of interest and 1's to the others. In this way the number of model evaluations is restricted to $65k$, where $N_s = 8 \cdot 8 + 1$.

5 Results of the sensitivity measures on the test cases

This chapter describes the results of the sensitivity measures on several test cases. First, it is interesting to know which sensitivity measures are suitable to use on the test cases. Conclusions and recommendations about the various sensitivity measures, which passed by in the previous chapters, are given in the first section. The results will be compared with a referee, which will be derived in the second section. The last section describes the test cases and the results of the calculation of the sensitivity indices on the test problems.

To calculate the FORM sensitivities two software packages, namely 'VaP' [45] and 'Comrel' [10] have been used. For the correlation ratio with Averaged Sampling 'Unicorn' [4] has been used to generate the samples and these samples have served as input to an Excel [31] sheet to calculate the correlation ratio.

5.1 Conclusions and recommendations about the various sensitivity measures

In the previous chapters we have seen that the correlation ratio corresponds to the linear regression sensitivity measure, the first order Sobol indices, Jansen's first order sensitivity measure and the first order FAST indices. The other measures are in fact extensions of the correlation ratio. Schematically, we can see the relationships in Figure 8.

A number of sensitivity measures have passed in review. Table 5 gives an overview of the sensitivity measures and an idea about the robustness of the different methods (RM denotes Resampling Based Method, AS denotes Averaged Sampling, WS denotes Winding Stairs and N the sample size). The FORM- and SORM-method take $l(k+1)$ model evaluations, where

Measure	**Introduced**	**Type Local/Global**	**Number of model evaluations**	**Accuracy**
LPSM	1998	Derivative/Local	Nk	
FORM	1982	Derivative/Local	$l(k+1)$	$-$
SORM	1982	Derivative/Local	$l(k+1)$	-
Correlation ratio	1995	Variance-based Global	LHS: $rm \times (k+1)$ RM: $N \times (k+1)$ AS: $100N$	$++$ $-$ $+$
FAST	1973	Variance-based Global	1^e order: $8\omega_{\max}+1$ total: $8k\omega_{\max}+1$ $\omega_{\max} \approx -478+4.29k^2$	$++$
HIM	1986	Variance-based Global	$N \times (k+1)$	$-$
Sobol' indices	1990	Variance-based Global	max $N \times 2^k$ 1^e order: $N \times (2k+1)$	$++$
Homma & Saltelli	1996	Variance-based Global	$N \times (k+1)$	$-$
Jansen's measure	1994	Variance-based Global	WS: $N \times k$	$-$

Table 5: Sensitivity measures

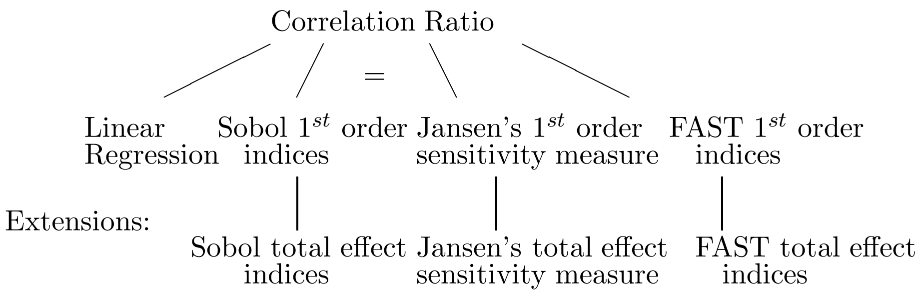

Figure 8: The global sensitivity measures and its relations

l depends on the linearity and continuity of the limit state function. The accuracy of the LPSM is not given, since it is not completely clear to what the results should be compared.

The following measures are interesting for further research:

- The Correlation Ratio

  The measure is given by a simple expression and with the computation scheme suggested by Ishigami and Homma it can be calculated taking $N \times (k+1)$ model evaluations, where N the sample size. A disadvantage is the lack of robustness.

- FAST

  The FAST is both computationally efficient and robust. Both the first order and the extended FAST are interesting, since the first is computationally very efficient, whereas the latter provides extra information about higher order terms.

- Homma and Saltelli Importance measure

  This measures provides a complete characterisation of the system, taking $N \times (k+1)$ model evaluations.

- FORM

  To be able to make a comparison, the FORM method will be calculated for the test cases.

The following measures will not be calculated for the test cases:

- LPSM

  The calculation of this measure is not straightforward.

- SORM

  SORM is only interesting for the failure probability and not for the sensitivities.

- Method of Sobol

  The first order Sobol indices correspond to the correlation ratio and can be calculated in the same way. The higher order Sobol indices take too much computational effort.

- Hora and Iman Importance Measure

  The Hora and Iman Importance measure will not be mentioned separately, since it is not normalised and its calculation method is used in the calculation of the correlation ratio with resampling.

- Jansen's measure

  Both the Jansen's first order and total effect measures will not be calculated, since these measures equal the correlation ratio and the total effect indices of Homma and Saltelli. The first order measure can be computed with the same sampling scheme as developed by Ishigami and Homma to calculate the correlation ratio. The total effect measure can be computed with the sampling scheme derived by Homma and Saltelli to compute total effect indices.

  If the winding stairs sampling scheme is used, this method is computationally more efficient than the other methods. However, the accuracy of the results with the Winding Stairs is also reduced by the reduction of the number of model evaluations.

For the test cases the following measures have been determined: Correlation Ratio with Latin Hypercubic Sampling, Correlation Ratio with Resampling based method, Correlation Ratio with Averaged Sampling, FORM, First Order FAST, Total Effect FAST, Total Effect Indices Homma and Saltelli and the referee. The results of the test cases are described in section 5.3.

5.2 Referee

To define a measure as reference we consider a linear output function (Z) with standard normally distributed variables.

Suppose $Z = \sum_i c_i X_i$ where $X_i \sim N(\mu_i, \sigma_i)$. Z can be written as $Z = B + \sum_i A_i U_i$, where $\mu_Z = B = \sum_i c_i \mu_i$, $A_i = c_i \sigma_i$, $\sigma_Z = \sqrt{\sum_i A_i^2}$ and $U_i \sim N(0,1)$. Then

$$\beta = \frac{\mu_Z}{\sigma_Z} = \frac{B}{\sqrt{\sum_j A_j^2}} = \frac{\sum_j c_j \mu_j}{\sqrt{\sum_j c_j^2 \sigma_j^2}}.$$

Since the main goal is to develop a sensitivity measure for reliability problems, the output of the test model will be the reliability or the failure probability of the system. In a FORM-analysis the output is the reliability index β. In case of a linear model all methods coincide, so it is desirable that the referee equals α_i in the linear case. It can be advantageous to learn how the probability of failure changes if one of the parameters (expectation and standard deviation), that uniquely define the input variables, changes. It is mainly interesting to consider the standard deviations of the input variables, since these represent the uncertainty in the model and can be decreased by measuring or better modelling. Actually, we are interested in the sensitivities of the failure probability to the various input variables. We want to calculate a partial derivative of the failure probability to the input.

To maintain the FORM-definitions we use the reliability index β, which will be defined by the inverse cumulative normal of the failure probability $\beta = -\Phi^{-1}(P(Z < 0))$. We are interested how the reliability index (and also the failure probability) changes if the standard deviation of one of the input variables changes. In other words how does β change if the uncertainty of one of the input variables changes. To obtain this information we consider $\frac{\partial \beta}{\partial \sigma_i}$,

$$\frac{\partial \beta}{\partial \sigma_i} = -\frac{\sum_j c_j \mu_j c_i^2 \sigma_i}{(\sum_j c_j^2 \sigma_j^2)^{3/2}} = -\frac{\sum_j c_j \mu_j c_i}{\sum_j c_j^2 \sigma_j^2} \alpha_i = -\frac{\mu_Z c_i}{\sigma_Z^2} \alpha_i = -\frac{\beta c_i}{\sigma_Z} \alpha_i. \tag{93}$$

With substitution of

$$\alpha_i = \frac{c_i \sigma_i}{(\sum_j c_j^2 \sigma_j^2)^{1/2}}.$$

So,

$$M_i = -\frac{1}{\beta} \frac{\partial \beta}{\partial \sigma_i} \frac{\sigma_Z}{c_i},$$

equals α_i in the linear case. In the non-linear case, the M_i^2's do not add up to 1 and since σ_Z is a normalisation constant, the following measure is used as referee:

$$I_i = -\frac{1}{c_i \beta} \frac{\partial \beta}{\partial \sigma_i}. \tag{94}$$

In order to normalise the values of I_i are divided by $\sqrt{\sum_i I_i^2}$.

The coefficients c_i have to be taken into account, since they also represent the influence of the variation of the standard deviation of the different variables. In the linear case the defined measure works well, but in the non-linear case a problem arises in the calculation of the coefficients c_i. The coefficients c_i equal the derivative of the limit state function to the different variables $\partial Z/\partial X_i$. In the non-linear case these derivatives can also contain the variables themselves. For instance, consider a simple parabola, given by:

$$Z = 3 - V + U^2$$

Suppose we want to calculate the importance of U. With the measure defined above, we need to calculate $\partial Z/\partial U$ which equals $2U$. We need to substitute a value for U, but we don't know what U is. In this way we also need some kind of a design point, which is exactly the thing we want to avoid. We want to take into account the whole contour of Z and not just some point. If we substitute $U = 0$ we get the FORM results, stating $I_U = 0$ and $I_V = 1$, whereas we want to identify U as the most important variable, since the square blows up the function. To be able to meet this demand we have to drop the demand that the importance measure should equal α_i. One way to avoid the described problem is to define new variables for the non-linear terms. In our example we could replace U^2 with a new variable W with corresponding distribution function and then apply the above theorem by varying the standard deviation of the new variable. This method becomes very impractical for problems with many variables and many non-linear or interaction terms. Furthermore, it is questionable if this method provides good results.

We will derive a more general formulation for $\partial\beta/\partial\sigma_i$. Consider $z_0 = (Z = 0) = Z(X_1^0, ..., X_k^0)$, which defines the contour of $Z = 0$. We want to know how the probability of failure $P(Z < 0)$ changes if the standard deviation of one the variables changes.

$$\begin{aligned}
P(Z<0) &= F_Z(0) \\
\beta &= \Phi^{-1}(F_Z(0)) \qquad (95) \\
\Phi(\beta) &= F_Z(0) \\
\frac{\partial\Phi}{\partial\beta}\cdot\frac{\partial\beta}{\partial\sigma_i} &= \frac{\partial F_Z}{\partial Z}\Big|_{Z=0}\cdot\frac{\partial Z}{\partial\sigma_i}\Big|_{Z=0} \\
\frac{\partial\beta}{\partial\sigma_i} &= \frac{\frac{\partial F_Z}{\partial Z}\Big|_{Z=0}\frac{\partial Z}{\partial\sigma_i}\Big|_{Z=0}}{\frac{\partial\Phi}{\partial\beta}} \\
&= e^{\beta^2}\sqrt{2\pi}f_Z(0)\frac{\partial Z}{\partial\sigma_i}\Big|_{Z=0}. \qquad (96)
\end{aligned}$$

In this study equation (96) has not been worked out. This equation should also be equal to the FORM-α in the linear case. Further research is needed to develop a meaningful referee. In most of the test cases of the coefficients c_i were equal for all variables and the referee has been calculated as $-\frac{1}{\beta}\frac{\partial\beta}{\partial\sigma_i}$, which can always be calculated. The referee has been calculated with crude Monte Carlo simulation. First, the reliability index (β) has been calculated with 1.000.000 samples. The number of samples needed for sufficient accuracy depends on the failure probability. The minimum required number of samples in Monte Carlo sampling, given a target for the coefficient of variation of the reliability index $VC(P_f)$ is given by (cf. [48]):

$$N > \frac{1}{VC(P_f)^2}\left(\frac{1}{P_f} - 1\right), \qquad (97)$$

where $VC(P_f)$ is for instance 0.05. Subsequently, the standard deviation of one of the variables has been multiplied by 1.1 or 0.9, so the standard deviation has been changed relatively 10 %. With this change β is calculated again and the difference between the beta's is obtained. $\frac{\partial\beta}{\partial\sigma}$ is obtained by dividing the difference in beta's by 0.1 σ_i (the difference in sigma's).

5.3 Description and results of the test cases

The next sections describe the test cases in arbitrary order and the results of these tests. CR-LHS denotes Correlation Ratio computed with Latin Hypercubic Sampling, CR-RM denotes Correlation Ratio computed with the Resampling Based method of Ishigami and Homma, CR-AS denotes Correlation Ratio computed with Averaged Sampling, TEI-HS denotes Total Effect Indices of Homma and Saltelli, FAST 1^{st} denotes First Order FAST, FAST-TE denotes total effect FAST and FORM denotes First Order Reliability Method. In some cases the referee could be calculated analytically, this is denoted by referee-AN. For the total effect FAST the expectation and variance of Z are not given, since the approximations may be inaccurate, due to the use of commensurably frequencies. I_U denotes the importance of variable U and I_V denotes the importance of variable V. For the CR-LHS and the CR-RM the averages of 10 simulations are taken as measure. The simulation results can be found in appendix B.2. The involved table is given in brackets. The tables contain also values of the variance and expectation of the Z-function, to have other possibilities to compare the methods. The expectation has not been calculated for all methods.

The number of model evaluations of the various methods is also given. As indication for the computational effort of the Latin Hypercubic Sampling scheme the number of model evaluations without averaging is given. For the Latin-Hypercubic sampling the number of replicates r is taken to be 100 and the number of intervals m is also taken to be 100 to obtain sufficient accuracy. It is also possible to take smaller values for m and r, for instance 10 and 50 respectively, however in the test cases smaller values of m and r resulted in low accuracy. In the LHS-calculations the variables are assumed to be independent and since m is taken to be 100 the introduced correlation's of the (artificial) sample are assumed to be negligible. Furthermore, the LHS-estimator slowly converges to the analytical values by increasing r. According to [20] the number of intervals should at least be $\frac{4}{3}k$, with k the number of input variables.

The results of the Correlation Ratio with the Resampling Based Method are obtained with 8.000 resamples. First, the resampling based method has been tested with 1.000 resamples; these results are given in appendix B.2. Due to low accuracy with 1.000 resamples, the calculations have been repeated with 8.000 resamples. These calculations have been carried out with a student edition of Matlab, which can handle at most about 16.000 elements, resulting in 8.000 resamples for problems with two variables.

For the Averaged Sampling algorithm 100 Unicorn [4] runs are used, which equals 10.000 samples. For the FAST total effect indices the frequency assigned to the factor of interest is chosen to be 1024, which leads to a sample size of $N_s = k \cdot 2M(1024 + 1)$. For the first order FAST M is taken to be 4. The algorithm of Cukier provides incommensurably frequencies for at least three variables. In test cases with two variables, the frequencies (27,35) are used. It is possible to do a cheaper first order FAST-analysis, with for instance the frequencies (5,11), taking $8 \cdot 11 + 1 = 89$ model evaluations. It is possible to obtain less accurate results with the total effect FAST demanding less model evaluations by assigning a lower frequency for the factor of interest.

To be able to make a comparison between the various methods the square root of the Correlation Ratio, the total effect indices of Homma and Saltelli, the first order FAST and the total effect FAST has been taken. The FAST method has also been normalised in some cases. In all the test cases of this chapter the variables are assumed to be independent. It should be noted that the variance-based methods always give positive results, since a division of variances is considered. By taking the square root the results remain positive, whereas the FORM- and referee-results may also be negative. Information about the sign of the sensitivities is lost by the use of variance-based methods.

Almost all the test cases have been calculated with normally distributed variables, since reliability problems are in general formulated in terms of normally distributed variables. Most of the test cases have also been calculated with uniformly distributed variables, since the FAST-method has been derived initially for uniformly distributed variables and sometimes the use of uniformly distributed variables is more straightforward.

5.3.1 Linear function

The linear function (98) has been chosen since the sensitivity measures should certainly have to be able to handle a simple linear function. Furthermore, this case should show that all the methods coincide with each other if linear models are considered.

The linear function is formulated as:

$$Z = a - \sum_{i=1}^{n} c_i X_i, \tag{98}$$

where the X_i are normally or uniformly distributed variables for all i, $a = 3$ and the c_i are 1 in the tests for all i.

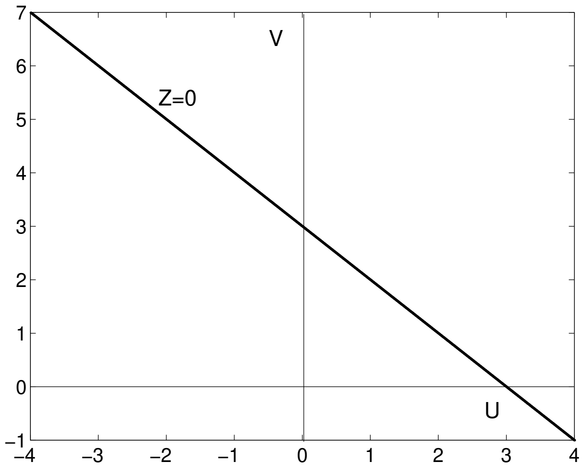

Figure 9: Linear Function

methods	I_U	I_V	$E(Z)$	σ_Z	# Model Evaluations
FORM	0.707	0.707	3	1.4142	14
CR-LHS (B.16)	0.7115	0.7143	3.0002	1.4119	30.000
CR-AS	0.7148	0.7071	3.00	1.4071	10.000
CR-RM (B.17)	0.6961	0.6746	-	-	16.002
CR-AN	0.7071	0.7071	3.000	1.4142	
TEI-HS (B.18)	0.7100	0.7203	-	-	24.000
FAST 1^{st}	0.6940	0.6940	3.0059	1.3983	281
FAST-TE	0.7133	0.7072	-	-	16.386
Referee	0.7134	0.7007	3.0003	1.4132	3.000.000
Referee-AN	0.7071	0.7071			

Table 6: Results of linear function with two standard normally distributed variables

Table 6 shows the results of this example with two standard normally distributed variables. As expected all models give about the same results. Leaving FORM out of consideration, the FAST first order method takes by far the least model evaluations, compared to the other methods.

The results of uniformly distributed variables can be found in appendix B. As expected both variables are again of equal importance.

methods	I_{X_1}	I_{X_2}	I_{X_3}	I_{X_4}	I_{X_5}	$E(Z)$	# Model Evaluations
FORM	0.316	0.316	0.316	0.316	0.316	3.00	90
CR-LHS (B.22)	0.3260	0.3153	0.3317	0.3297	0.3234	-	110.000
CR-RM (B.23)	0.2798	0.2768	0.2631	0.2877	0.2561	-	15.010
CR-AS	0.3271	0.3256	0.3302	0.3391	0.3225	3.02	10.000
CR-AN	0.3162	0.3162	0.3162	0.3162	0.3162	3.00	
FAST 1^{st}	0.3081	0.3059	0.2987	0.3197	0.3100	3.00	1.641
FAST-TE	0.3407	0.3378	0.3369	0.3362	0.3353	-	81.930
Referee	0.2863	0.2577	0.3865	0.3114	0.2593	3.0003	11.000.000
Referee-AN	0.3162	0.3162	0.3162	0.3162	0.3162		
methods	I_{X_6}	I_{X_7}	I_{X_8}	I_{X_9}	$I_{X_{10}}$	$\sigma(Z)$	
FORM	0.316	0.316	0.316	0.316	0.316	3.1623	
CR-LHS	0.3285	0.3270	0.3342	0.3276	0.3273	3.1529	
CR-RM	0.2768	0.2769	0.2850	0.2702	0.2793	-	
CR-AS	0.3209	0.3194	0.3271	0.3347	0.3464	3.1289	
CR-AN	0.3162	0.3162	0.3162	0.3162	0.3162	3.1623	
FAST 1^{st}	0.3157	0.3043	0.3089	0.3150	0.3095	3.1678	
FAST-TE	0.3345	0.3435	0.3333	0.3333	0.3308	-	
Referee	0.2577	0.3400	0.3400	0.3150	0.3758	3.1639	
Referee-AN	0.3162	0.3162	0.3162	0.3162	0.3162		

Table 7: Results of linear function with 10 standard normally distributed variables

Table 7 shows the results of this example with ten standard normally distributed variables. These ten variables should all be equally important, and all methods give more or less this importance ranking. The referee (with crude Monte Carlo simulation) is quite noisy in this case. It should be noticed that, in general, Monte Carlo derivatives are not stable. Table 7 also shows that the number of model evaluations of the first order FAST is increasing rapidly as compared to the situation where two variables are considered.

5.3.2 Series system

The series system (99) has been chosen as test case since the commonly used FORM-method finds one of the points (0,3) or (3,0) as design-point and gives as corresponding sensitivities U (or V) is very important and V (or U) not, whereas both U and V should be of equal importance. It is also very difficult to calculate the LPSM for this example, cf. [6]. Consider

$$Z = \min(a - U, a - V), \tag{99}$$

where U and V are normally or uniformly distributed variables and $a = 3$ in the tests.

Table 8 shows the results of this example with standard normally distributed variables. The results of uniformly distributed variables can be found in appendix B.1. For all methods except the FORM-method, the ranking is the same, namely $I_U = I_V$. The square root of the correlation ratio corresponds to the alpha's given by a FORM-analysis in the linear case. To

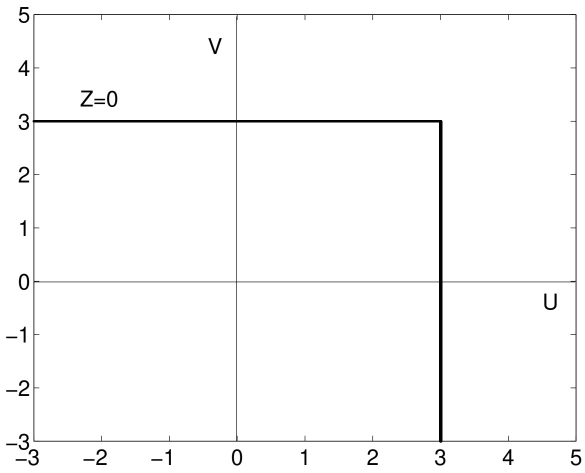

Figure 10: Series system

methods	I_U	I_V	$E(Z)$	σ_Z	# Model Evaluations
FORM	$-0.4385 \cdot 10^{-9}$	-1.0000	3	1	8
CR-LHS (B.24)	0.6574	0.6593	2.4356	0.8278	30.000
CR-RM (B.26)	0.6592	0.6492	-	-	16.002
CR-AS	0.6618	0.6504	2.4300	0.8258	10.000
CR-AN	0.6325	0.6325	2.4350	0.8246	
TEI-HS (B.27)	0.7508	0.7660	-	-	24.000
FAST 1^{st}	0.6325	0.6275	2.4431	0.8221	281
FAST-TE	0.7071	0.7071	-	-	16.286
Referee	0.7075	0.7067	2.4350	0.8246	3.000.000
Referee-AN	0.7071	0.7071			

Table 8: Results of series system with standard normally distributed variables

compare the other sensitivity measures with the referee it is necessary to take the square root and normalise.

In this case the FORM-method may be quick, but gives a completely wrong answer. The FORM-results are obtained with a random start point in the search for the design point. Normally the origin is taken as start point, but this leads to an error. The first order FAST gives reasonable results in a relatively small number of function evaluations, namely 281. This problem represents a series system and since the FORM-method can not handle series systems, the method of Hohenbichler [47] should be used to combine the two functions, instead of applying FORM on the total function.

Appendix B.2 shows the results of all the simulations used to calculate the averages given in the table. The Latin Hypercubic sampling method appears to be very sensitive to a long-tailed distribution. The tables show that the results with uniformly distributed variables are much more stable than the results with normally distributed variables. The tables (see B.25, B.31, B.33) also show that the results obtained with the resampling based method can be negative or greater than one if the number of resamples is not big enough (cf. [35]). This phenomenon is not desirable and the resampling based method does not seem to be very accurate. By increasing the sample size more accurate results can be obtained. However, the method has been developed to limit the number of model evaluations.

The correlation ratio's have also been calculated for a series system of 10 elements with Latin Hypercubic sampling and the resampling based method. The results can be found in Appendix B.2. These results are still very unstable and there is a lot of variance between the runs.

5.3.3 Parabola

This parabola problem (100) has been chosen since the sensitivity measure should be able to handle a simple parabola. Furthermore, it is interesting to see whether the intuitive idea that U will be much more important than V, due to the square, will be confirmed.

The parabola problem is formulated as:

$$Z = a - V + (U - b)^2, \tag{100}$$

where U and V are normally or uniformly distributed variables, $a = 3$ and $b = 0$ in the tests.

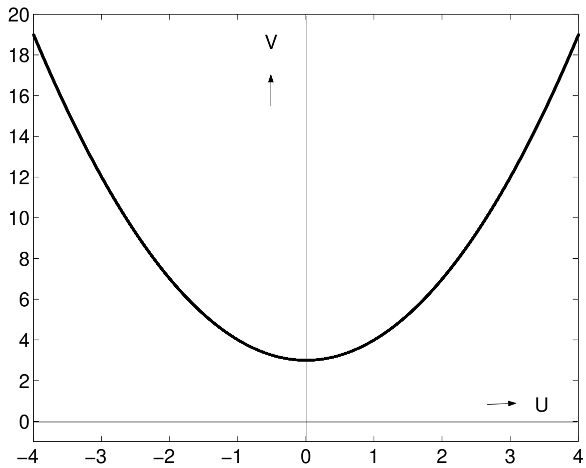

Figure 11: Parabola

Table 9 shows the results of this example with standard normally distributed variables. The results of uniformly distributed variables can be found in appendix B. The FORM-method performs badly, by identifying the variable V as the most important and U non-important. The FORM-method finds the point (0,3) as design-point and linearizes in that point, which leads to the given sensitivities. The global sensitivity measures confirm our intuition, assigning U as the most important variable. The referee has not been calculated, since it is not completely clear how the term $\partial Z/\partial U = 2U$ must be treated, as mentioned in section 6.2. $-\frac{1}{\beta}\frac{\partial \beta}{\partial \sigma_i}$ can be calculated, but does not give correct answers in this case, namely $I_U = -0.0591, I_V = 0.9983$.

Due to bad performance of the resampling based method this measure has not been computed for the other examples. Appendix B shows that there is very much variation in the

methods	I_U	I_V	$E(Z)$	σ_Z	# Model Evaluations
FORM	-0.0008	1.0000	4.00	1.79165	8
CR-LHS (B.39)	0.8103	0.5919	4.0032	1.7256	30.000
CR-RM (B.41)	0.8479	0.5885	-	-	8.001
CR-AS	0.8068	0.5925	4.00	1.6941	10.000
CR-AN	0.8044	0.5941	4.00	1.6832	-
TEI-HS (B.42)	0.8113	0.5400	-	-	24.000
FAST 1^{st}	0.6843	0.5690	3.9952	1.7259	281
FAST-TE	0.8414	0.9991	-	-	16.386

Table 9: Results of parabola with standard normally distributed variables

results, even for a high value of N. It is even possible that negative values for the correlation ratio are obtained (see B.40, B.44), which is not possible by definition. The total effect indices of Homma and Saltelli use a similar sampling scheme, and have consequently the same bad performance. For this reason, these sensitivity indices have also not been computed for the other examples.

5.3.4 Curve

The curve problem (101) has been chosen since this problem is even more difficult than the series system for the commonly used FORM-method. All the points on the curve have the same distance to the origin and can be chosen as design-point. It depends on the optimisation routine whether the FORM results are correct, which is in this case that U and V have both the same importance. Again, it is very difficult to calculate the LPSM for this example, see [6].

The curve problem is formulated as:

$$\begin{aligned} Z &= a - U, & v < 0 \\ Z &= a^2 - U^2 - V^2, & v > 0, u > 0, \\ Z &= a - V, & v > 0, u < 0, \end{aligned} \tag{101}$$

where U and V are normally or uniformly distributed variables and $a = 3$ in the tests.

methods	I_U	I_V	$E(Z)$	σ_Z	# Model Evaluations
FORM	$-0.7974 \cdot 10^{-1}$	-0.9968	-	-	8
CR-LHS (B.48)	0.4882	0.4770	3.7952	2.2464	30.000
CR-AS	0.4940	0.4785	3.8300	2.2782	10.000
FAST 1^{st}	0.4127	0.4161	3.8032	2.2823	281
FAST-TE	0.8888	0.8885	-	-	16.286
Referee	0.7186	0.6954	-	-	3.000.000

Table 10: Results of curve with standard normally distributed variables

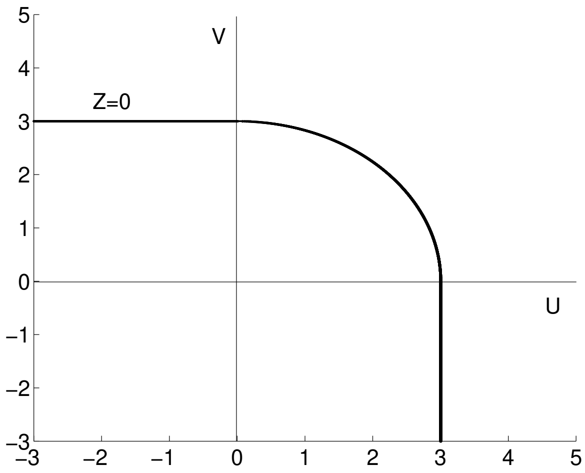

Figure 12: Curve

Table 10 shows the results of this example with standard normally distributed variables. The results of uniformly distributed variables can be found in appendix B. The referee has been calculated with 'Comrel' (with Monte Carlo simulation), which does not give the expectation and the variance of the limit state function. Since this problem is symmetrical in U and V the referee can be calculated by $-\frac{1}{\beta}\frac{\partial\beta}{\partial\sigma_i}$ and there is no problem with the coefficients.

The importance ranking based on the table is $I_U = I_V$ for all methods, except for the FORM-method. In this case the FORM-method is again very quick but completely wrong. The first order FAST gives good results in a relatively small number of function evaluations. It should be noticed that variables with sensitivities, which are about the same (e.g. 0.4940 and 0.4785 for CR-AS) should be ranked at the same place by the construction of the importance ranking. The results given in the table are according our expectation, both U and V are of equal importance. The FAST first order indices should equal the correlation ratio, but there is a big difference between them, whereas the variance is much alike. The normalisation and the square root again cause the difference between the various methods and the referee.

5.3.5 V-curve

The V-curve problem (102) has been selected since the FORM-method has convergence problems with this case, due to the design-point constantly moving between the two branches.

The V-curve problem is formulated as:

$$Z = \max(aU - V + b, -aU - V + b), \tag{102}$$

where U and V are normally or uniformly distributed variables and $a = b = 1$ in the tests. This example is taken from [48].

Table 11 shows the results of this example with standard normally distributed variables. The results of the global sensitivity measures do not differ very much, whereas the referee gives other results. The importance ranking is in all cases the same: V is assigned to be the most important. The referee provides us the information that the failure probability decreases if the variance of U is increased, whereas the failure probability increases if the

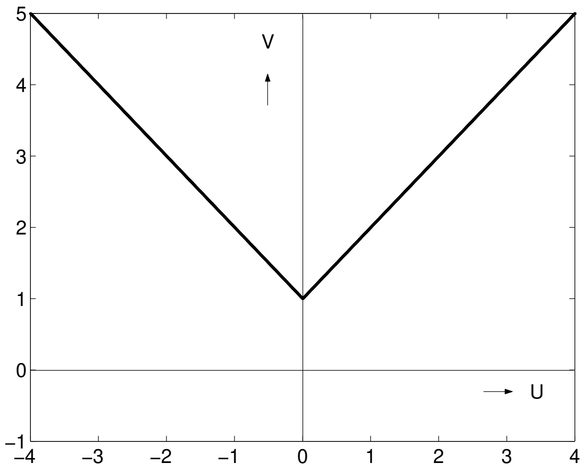

Figure 13: V-Curve

methods	I_U	I_V	$E(Z)$	σ_Z	# Model Evaluations
FORM	error	error	-	-	792
CR-LHS (B.51)	0.5218	0.8642	1.7980	1.1646	30.000
CR-AS	0.5329	0.8614	1.81	1.1662	10.000
CR-AN	0.5134	0.8581			-
TEI-HS (B.52)	0.5426	0.8717	-	-	24.000
FAST 1^{st}	0.4897	0.8377	1.7987	1.1654	281
FAST-TE	0.5341	0.8642	-	-	16.386
Referee	-0.3372	0.9414	1.7976	1.1684	3.000.000

Table 11: Results of V-curve with standard normally distributed variables

variance of V is increased. Unfortunately, the other methods do not give us this information. The FORM-method does not find an answer at all, whereas the first order FAST gives a good ranking of the variables taking a smaller number of model evaluations. Compared to the referee the results are not very good, but they are certainly better than FORM and the referee is suspicious in this case, since the problem is not symmetrical in U and V resulting in different coefficients. Z can be rewritten as $Z = |U| - V + 1$. Then $\partial Z/\partial V = -1$, whereas $\partial Z/\partial U = -1$, if $U < 0$ and $\partial Z/\partial U = +1$, if $U > 0$. So, the coefficients of U and V are not equal on the entire input space of Z.

The results of uniformly distributed variables can be found in appendix B.

5.3.6 Hyperbola

The hyperbola problem (103) has been selected, since it is interesting to consider a function with so-called 'interaction'-terms, which means products and other composite functions of the various input factors.

The hyperbola problem is formulated as:

$$Z = UV - k \tag{103}$$

where U and V are normally or uniformly distributed variables and $k = 1$ in the tests.

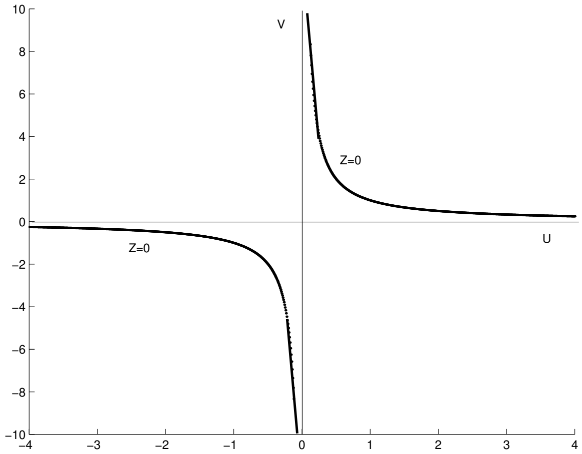

Figure 14: Hyperbola

methods	I_U	I_V	$E(Z)$	σ_Z	# Model Evaluations
FORM	0.7071	0.7071	*)	*)	74
CR-LHS (B.56)	0.0106	0.0090	-0.9966	0.9898	30.000
CR-AS	0.0086	0.0083	-1.01	1.0198	10.000
CR-AN	0	0	-1.0	1.0	
FAST 1^{st}	0.0006	0.0006	-1.0146	1.0717	281
FAST-TE	0.7071	0.7071	-	-	16.386
Referee	0.6952	0.7188	-1.0000	1.0011	3.000.000
Referee-AN	0.7071	0.7071			

Table 12: Results of hyperbola with standard normally distributed variables

Table 12 shows the results of this example with standard normally distributed variables. In this case the table gives the original correlation ratio and first order FAST and not the square root of these values, since the results should be 0, corresponding to the analytical values. The results of uniformly distributed variables can be found in appendix B. The correlation ratio performs badly at this case, as both factors are identified as not important, whereas we expect them both to be of the same importance. This problem can be solved by defining a new variable which is actually UV. The first order FAST gives the same bad performance, since the first order FAST indices correspond to the correlation ratio's. The FAST total effect method both assigns the interaction effect UV completely to U and to V. So adding them up leads to 2 instead of 1. The results are normalised to correct this.

5.3.7 Linear function with noise

The linear function with noise (104) has been chosen to identify how sensitive the various methods are to noise.

The linear function with noise is formulated as:

$$Z = c + aU + bV + 0.001\sin(100U) + 0.001\sin(100V), \tag{104}$$

where U and V are normally or uniformly distributed variables, $a = b = 1$ and $c = 3$ in the tests.

methods	I_U	I_V	$E(Z)$	σ_Z	# Model Evaluations
FORM	0.7071	0.7071	3	1.4142	10
CR-LHS (B.59)	0.7113	0.7129	3.0060	1.4168	30.000
CR-AS	0.7148	0.7071	2.9996	1.4071	10.000
CR-AN	0.7071	0.7071	3	1.4142	
FAST 1^{st}	0.6914	0.6898	2.9941	1.4183	281
FAST-TE	0.7071	0.7071	-	-	16.385
Referee	0.7068	0.7075	3.0003	1.4138	3.000.000
Referee-AN	0.7071	0.7071			

Table 13: Results of linear function with noise with standard normally distributed variables

Table 13 shows the results of this example with standard normally distributed variables. All methods give as result that U and V are equally important. The methods don't seem to be sensitive to noise. Compared to the referee the results are very good. FORM uses the least number of model evaluations, followed by First order FAST.

The results of uniformly distributed variables can be found in Appendix B.

5.3.8 Noisy limit state function

The noisy limit state function (105) has been chosen to determine the sensitivity of the methods to noisy limit state functions with variables, which are not normally distributed.

$$Z = X_1 + 2X_2 + 2X_3 + X_4 - 5X_5 - 5X_6 + 0.001\sum_{i=1}^{6}\sin(100X_i), \tag{105}$$

where X_1, X_2, X_3, X_4 are lognormally distributed with mean 120 and variance 12, X_5 is lognormally distributed with mean 50 and variance 15 and X_6 is lognormally distributed with mean 40 and variance 12. This example is taken from [15].

Table 14 shows the results of this example. In the calculation of the referee the coefficients of X_i have been taken into account, except for the coefficients of the noise. The noise terms have the same coefficients for each variable and have been neglected. The FORM-method and the referee provide us information about the failure probability increasing or decreasing, whereas the global methods do not give this information. The ranking of all methods is about the same. This time the first order FAST gives good results taking even less computational effort than the FORM-method.

methods	I_{X_1}	I_{X_2}	I_{X_3}	I_{X_4}	I_{X_5}	I_{X_6}	$E(Z)$	σ_Z	# Model Evaluations
FORM	0.077	0.152	0.152	0.077	-0.808	-0.537	270.005	103.271	479
CR-LHS	0.1581	0.2494	0.2551	0.1526	0.7340	0.5821	269.734	104.2689	70.000
CR-AS	0.1600	0.2594	0.2608	0.1581	0.7362	0.5788	269	101.98	10.000
FAST 1^{st}	0.1162	0.2540	0.2263	0.1456	0.7065	0.5550	270.128	103.4698	393
FAST-TE	0.2425	0.3146	0.3143	0.2419	0.7405	0.6004	-	-	49.158
Referee	0.1615	0.2201	0.1212	0.1131	-0.7728	-0.5484	269.828	103.207	7.000.000

Table 14: Results of noisy limit state function

5.3.9 Oblate spheroid

The oblate spheroid (106) has been selected since this problem contains quadratic terms, and is therefore non-linear. The FORM-method linearizes the function in the design-point, and since this function is non-linear an error is made.

The oblate spheroid is formulated as:

$$Z = R - \sum_{i=1}^{10} \frac{S_i^2}{1 + \frac{i}{10}}, \tag{106}$$

where S_i standard normally distributed for all i and $R = 10$.

methods	I_{S_1}	I_{S_2}	I_{S_3}	I_{S_4}	I_{S_5}	$E(Z)$	# Model Evaluations
FORM	1.000	$3.0 \cdot 10^{-3}$	$4.3 \cdot 10^{-6}$	$-1.3 \cdot 10^{-7}$	$-3.5 \cdot 10^{-9}$	-	804
CR-LHS (B.62)	0.4260	0.4040	0.3873	0.3159	0.3051	3.3459	110.000
CR-AS	0.4266	0.3924	0.3521	0.3406	0.3240	3.40	10.000
FAST 1^{st}	0.3583	0.3165	0.3063	0.2970	0.2665	3.3218	1.641
FAST-TE	0.3804	0.3558	0.3353	0.3221	0.3118	-	81.930
Referee	0.5334	0.4581	0.3678	0.3283	0.2840	3.3064	11.000.000
methods	I_{S_6}	I_{S_7}	I_{S_8}	I_{S_9}	$I_{S_{10}}$	σ_Z	
FORM	$-3.3 \cdot 10^{-9}$	$-3.1 \cdot 10^{-9}$	$-2.9 \cdot 10^{-9}$	0	0		
CR-LHS	0.2897	0.2423	0.2613	0.2884	0.2653	8.9039	
CR-AS	0.3053	0.2848	0.2812	0.2687	0.2492	8.34	
FAST 1^{st}	0.2615	0.2551	0.2406	0.2154	0.1972	9.0486	
FAST-TE	0.3022	0.2937	0.2868	0.2806	0.2768	-	
Referee	0.2404	0.2203	0.1810	0.1528	0.1397	3.0477	

Table 15: Results of oblate spheroid with standard normally distributed variables

Table 15 shows the results of this example. In the calculation of the referee ($-\frac{1}{c_i \beta}\frac{\partial \beta}{\partial \sigma_i}$) terms like $\frac{2S_i}{1+\frac{i}{10}}$ arise. All variables are distributed equally and as an approximation only the terms $\frac{1}{1+\frac{i}{10}}$ have been taken into account. Not taking the coefficients into account is not justified in this case, since these are not equal for all variables.

As expected, the importance of the variables decreases with almost all methods. FORM performs badly, assigning the first variable as the most important, and the others as non-important. The referee shows bigger differences between the various sensitivities than the other methods. First order FAST performs rather well, taking twice the number of model evaluations FORM needed and giving correct results.

5.4 Conclusions

The referee has not been calculated for the series system, the curve, the parabola, the V-curve, the linear function and the oblate spheroid with uniform variables, since these functions do not become zero with the uniform distributions. The FORM-results of the series system, the curve, the hyperbola and oblate spheroid have been obtained with a random start point. In general, the origin is taken as start point, but this resulted in an error in these cases.

If we consider the results of the test cases and compare the different sensitivity measures, the following conclusions can be drawn.

- FORM
  In some test cases the FORM-method does not find an answer and in other cases the results are not correct. The method is fast, but inaccurate if the limit state function is non-linear or discontinuous.

- Correlation Ratio with Latin Hypercubic Sampling
  The method gives accurate results, which resemble the analytical values. However, the method is sensitive to long-tailed distributions and it takes considerable computational effort to obtain accurate results. Furthermore, interaction effects are not taken into account.

- Correlation Ratio with the Resampling based method
  The results obtained with the resampling based method are very unstable and can be negative or greater than 1. This method lacks reproducibility and does not take interaction effects into account.

- Correlation Ratio with Averaged Sampling
  The method is very accurate, but interaction effects are not taken into account and a considerable number of model evaluations are needed.

- Total Effect Indices Homma and Saltelli
  The results obtained with this sampling scheme are very unstable. The method takes interaction effects into account, but needs quite a lot of model evaluations to obtain stable results.

- First Order FAST
  The first order FAST method is computationally efficient and the results are reproducible and accurate. However, the computational effort grows quadratically as the number of variables grows and interaction effects are not taken into account.

- Total Effect FAST
  The methods takes interaction effects into account and the results are reproducible.

More information is obtained at a higher price, namely more model evaluations. However, it is possible to do a less accurate, but quicker, total effect FAST analysis.

These conclusions are summarised in Table 16.

	Computational Efficiency	**Reprodu-cibility**	**Accuracy**	**Model Independent**	**Robustness**
FORM	++	+	2) +/-	no	−
CR-LHS	—	+/-	+	yes	+/-
CR-RM	+/-	—	−	yes	+/-
CR-AS	+/-	1)	+	yes	+
TEI-HS	-	−	-	yes	+/-
FAST 1^{st}	+/-	++	+	yes	+
FAST-TE	+/-	++	3) +/-	yes	+

Table 16: Comparison of the various sensitivity methods

1. The Correlation Ratio with Averaged Sampling has not been tested on reproducibility.
2. The accuracy of the FORM-method depends on the type of limit state function.
3. The computational effort for the total effect FAST depends on the desired accuracy.

Model independence means the extent in which the level of additivity of non-linearity of the model influences the correctness of the results. With reproducibility is meant in what extent repetition of the calculation procedure leads to the same results. Accuracy means the extent in which the calculated values resemble the analytical values of the involved methods. The accuracy of the Correlation Ratio with Latin Hypercubic Sampling and with the Resampling-based Method is obtained by taking averages. In this case it is possible to have a low reproducibility and a higher accuracy. Otherwise, low reproducibility would imply low accuracy.

The first order FAST is computationally efficient when problems with less than 25 variables are considered. With problems with more variables the computational cost grows quadratically. That is why this method scores +/- at computational efficiency.

From Table 16 it follows that the first order FAST and Correlation ratio give best overall performance. However, both methods do not take into account interaction terms, whereas the total effect FAST actually does. So, both the first order FAST and the total effect FAST sensitivity indices as well as the Correlation Ratio with Averaged Sampling are interesting for further research. The next Chapter describes the results of these measures on more complicated problems. Of course, the FORM influence factors will also be calculated for these problems.

6 FAST and Correlation Ratio on dike-ring reliability problems

6.1 Overtopping of a dike-ring

The general description over overtopping is given in Appendix A.2. This section describes a simplified model of the failure mechanism overtopping.

6.1.1 Problem description

Consider a dike ring existing of m exterior identical dike sections. Let the simplified reliability function for the failure mechanism 'overtopping' be given by:

$$Z_i = Z_i(H, V, q), \tag{107}$$

where H the water level, V the wind velocity and q the critical discharge. For each dike section these are modelled as independent stochastic variables. Appendix A.2 describes the original reliability function. Furthermore, suppose that each dike section has the same limit state function and that the results of a FORM-analysis are given by:

$$P(Z_i < 0) = 0.32 \cdot 10^{-4}; \quad \beta_i = 4,$$
$$\alpha_{iH} = 0.79 \quad \alpha_{iV} = 0.44 \quad \alpha_{iq} = 0.44,$$

where β_i the reliability index for dike section i and α_{ij} the sensitivities for dike section i. Furthermore, suppose that the water level H for two dike sections is completely correlated, that the wind velocity V is also completely correlated for two dike sections, whereas the critical discharge of dike section i is independent of the critical discharge of dike section j. Thus, for the correlation coefficients we get:

$$\rho(H_iH_j) = \rho(V_iV_j) = 1; \quad \rho(q_iq_j) = 0$$

It follows that

$$\rho(Z_iZ_j) = \sum_k \alpha_{ik}\alpha_{jk}\rho_{ijk} = 0.81$$

where ρ_{ijk} the correlation between the kth variable of dike section i and the kth variable of dike section j.

Finally, suppose that the dike sections are all correlated equally. The goal is to determine the FAST-sensitivities and the correlation ratio's and to compare these with de FORM-sensitivities obtained with the method of Hohenbichler [47] for various numbers of dike sections.

6.1.2 Approach

First, consider the case that Z is a linear function and all the q's are independent. It follows that:

$$Z_i = 4 + 0.44q_i + 0.79H_i + 0.44V_i.$$

Since $\rho(H_iH_j) = \rho(V_iV_j) = 1$, we can define $H = H_i \quad \forall i$ and $V = V_i \quad \forall i$.
With $Y \overset{def}{=} 4 + 0.79H + 0.44V$ and $\rho(q_iq_j) = 0$ we obtain

$$\begin{aligned}
Z_i &= Y + 0.44q_i, \qquad q_i \perp q_j \\
Z &= \min_i(Z_i) = \min_i(0.44q_i) + Y = 0.44\min_i(q_i) + Y.
\end{aligned}$$

The sensitivities have been calculated using FORM with Hohenbichler [47], FORM with PERT-approximation [39], FORM-SERIE (both will be explained hereafter), First Order FAST, Total Effect FAST and the Correlation Ratio with Averaged Sampling for 1, 2, 8, 16, 24 and 48 dike sections. The results are given in the next subsection.

The PERT-approximation approximates the minimum of normally distributed variables by one single normally distributed variable. In [39] is explained how this can be achieved. For $T = \max(N_1, N_2) + N_3$, with N_i normally distributed with mean μ_i and variance σ_i^2, and

$$\begin{aligned}
a^2 &= \sigma_1^2 + \sigma_2^2 \\
\alpha &= (\mu_1 - \mu_2)/a
\end{aligned}$$

we have

$$\begin{aligned}
\mu_T &= \mu_1\Phi(\alpha) + \mu_2\Phi(-\alpha) + a\Psi(\alpha) + \mu_3 \\
\sigma_T^2 &= (\mu_1^2 + \sigma_1^2)\Phi(\alpha) + (\mu_1^2 + \sigma_2^2)\Phi(\alpha) + (\mu_1 + \mu_2)a\Psi(\alpha) + \sigma_3^2 - (\mu_T - \mu_3)^2
\end{aligned}$$

where $\Phi(\alpha)$ denotes the cumulative distribution function of the standard normal distribution, evaluated in α; and $\Psi(\alpha)$ denotes the probability density function of the standard normal distribution evaluated in α. The mean and variance can be calculated with the above expression using that $\min(N_1, N_2) = \max(-N_1, -N_2)$ and $N_3 = 0$.

The FORM sensitivities α_i have also been calculated with use of the following: If q_1 and q_2 are independent and normally distributed and $Q = \min(q_1, q_2)$, then

$$\begin{aligned}
F_Q = P(Q < 0) &= 1 - P(Q > 0) = 1 - P(q_1 > 0 \wedge q_2 > 0) = 1 - P(q_1 > 0)P(q_2 > 0) \\
\Rightarrow P(Q < 0) &= 1 - (1 - P(q < 0))^2 \quad (\text{since} \quad q_1 \sim q_2 \sim q) \\
&= 1 - (1 - \Phi(q))^2.
\end{aligned}$$

where $\sim$ denotes equal in distribution. In general $Q = \min_i q_i, i = 1, ..., n$ and it follows that $F_Q = q - (q - \Phi(q))^n$. Since the distribution of F is unknown, the equation is solved by $\Phi(U) = F(Q)$, which leads to $Q = \Phi^{-1}(1 - (1 - \Phi(U))^{1/n})$. The FORM-method is applied to problems where this expression for Q is substituted. This approach is referred to as FORM-SERIE.

To be able to compare the different methods, the square root of the first order FAST, total effect FAST and the correlation ratio has been taken. Tables 17-22 show the results for various numbers of dike sections. The First Order FAST, total effect FAST and correlation ratio calculate sensitivities for all the q_i's. These can be found in appendix C. It is not completely clear what the relationship is between the sensitivities for the q_i's and the sensitivity for $Q = \min_i(q_i)$. In these tables *) indicates that there are separate sensitivities for the q_i's. It is possible to calculate the expectation and variance of Z with the first order FAST method and the Averaged Sampling algorithm. The reliability index β is calculated as $E(Z)/\sigma_Z$ and put in the table for comparison.

From the results the following conclusions can be drawn. The sensitivities found with FORM with Hohenbichler, FORM with PERT-approximation, FORM-SERIE, First Order

	FORM	FAST 1^{st}	FAST-TE	CR-AS
I_q	0.438	0.4174	0.4503	0.4472
I_H	0.786	0.7801	0.7879	0.7899
I_V	0.438	0.4342	0.4465	0.4528
β	3.98	3.91	-	4.01

Table 17: Sensitivities for one dike section

	FORM HOH	FORM PERT	FORM SERIE	FAST 1^{st}	FAST-TE	CR-AS
I_q	0.3271	0.372	0.3974	*)	*)	*)
I_H	0.8256	0.811	0.8017	0.7942	0.8141	0.8136
I_V	0.4598	0.452	0.4465	0.4419	0.4645	0.4626
β	3.8532	3.85	3.84	3.85	-	3.87

Table 18: Sensitivities for 2 dike sections

	FORM HOH	FORM PERT	FORM SERIE	FAST 1^{st}	FAST-TE	CR-AS
I_q	0.0913	0.262	0.3242	*)	*)	*)
I_H	0.8699	0.843	0.8265	0.8200	0.8379	0.8479
I_V	0.4846	0.470	0.4603	0.4623	0.4868	0.4690
β	3.61	3.60	3.58	3.67	-	3.85

Table 19: Sensitivities for 8 dike sections

	FORM HOH	FORM PERT	FORM SERIE	FAST 1^{st}	FAST-TE	CR-AS
I_q	0.0754	0.0480	0.2912	*)	*)	*)
I_H	0.8711	0.7259	0.8356	0.8273	0.8535	0.8485
I_V	0.4853	0.2256	0.4645	0.4572	0.5129	0.4796
β	3.49	3.49	3.45	3.42	-	3.51

Table 20: Sensitivities for 16 dike sections

	FORM HOH	FORM PERT	FORM SERIE	FAST 1^{st}	FAST-TE	CR-AS
I_q	0.0663	0.197	0.2768	*)	*)	*)
I_H	0.8717	0.857	0.8395	0.8325	0.8550	0.8544
I_V	0.4856	0.477	0.4676	0.4609	0.5071	0.4889
β	3.42	3.44	3.38	3.37	-	3.39

Table 21: Sensitivities for 24 dike sections

	FORM HOH	FORM PERT	FORM SERIE	FAST 1^{st}	FAST-TE	CR-AS
I_q	0.0518	0.163	0.2517	*)	*)	*)
I_H	0.8724	0.862	0.8455	0.8364	0.8631	0.8660
I_V	0.4860	0.480	0.4709	0.4596	0.5123	0.4817
β	3.32	3.35	3.27	3.26	-	3.26

Table 22: Sensitivities for 48 dike sections

FAST, Total Effect FAST and Correlation Ratio with Averaged Sampling do not differ very much. The sensitivities found with Hohenbichler are in general higher than the other sensitivities, whereas first order FAST gives the lowest sensitivities.

6.1.3 Correlation between the dike sections of 0.95

Since the results obtained with the different methods did not differ very much in the previous case, we consider the same problem again, but with a higher correlation between the dike sections. To obtain a correlation between the dike sections of 0.95 a correlation between the q's must be induced. If we choose $\rho(Z_iZ_j) = \sum_k \alpha_{ik}\alpha_{jk}\rho_{ijk} = 0.95$ and $\alpha = [0.44, 0.79, 0.44]$ it follows that $\rho(q_iq_j) = 0.683$. From the formula [24] (which is only valid for the bivariate normal distribution):

$$\rho_r = \frac{6}{\pi}\arcsin\left(\frac{\rho}{2}\right) \tag{108}$$

it follows that the rank correlation ρ_r should equal 0.666. To calculate the correlation ratio with Averaged Sampling, a correlation structure with a latent variable is designed according to Figure 15.

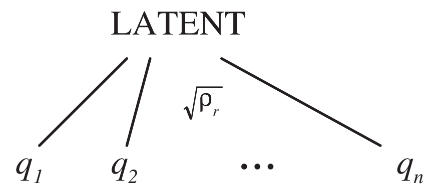

Figure 15: Latent variable structure

To calculate the FAST-indices another approach has been followed. For two dike sections it is possible to sample on the diagonal band [9]. Table 18 shows the results of the sampling on the diagonal band. For more dike sections another calculation method has to be used. It is possible to transform correlated variables to uncorrelated variables. If the variables are normally distributed the following procedure can be used.

The correlation's between a set of variables can be represented in a covariance matrix [29]:

$$\mathbf{C_Y} = \begin{pmatrix} \sigma^2_{Y_1} & \mathrm{Cov}(Y_1, Y_2) & \dots & \mathrm{Cov}(Y_1, Y_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(Y_n, Y_1) & \mathrm{Cov}(Y_2, Y_1) & \dots & \sigma^2_{Y_n} \end{pmatrix}.$$

If $(Y_1, Y_2, \ldots, Y_n)$ represents a set of mutually correlated variables, then a set uncorrelated variables $(X_1, X_2, \ldots, X_n)$ can be obtained by

$$X = A^T Y, \tag{109}$$

where the columns of A^T consist of the orthonormal eigenvectors of C_Y. The obtained relationships for $Y_i = Y_i(X_1, X_2, \ldots, X_n)$ have to be substituted in the original Z-function.

In this case it is not possible to use the SERIE-approximation to calculate FORM-sensitivities, since this approximation uses the assumption that the q_i's are independent, which is not the case. The PERT-approximation also assumes independence of the q_i's, and has therefore not been calculated. The sensitivities of one dike section are given in Table 17. Table 23 till Table 27 show the results for 2, 8, 16, 24 and 48 dike sections. The sensitivities of the q_i's can be found in appendix C.2. In this case, the sensitivities obtained with the different methods are still very similar.

	FORM HOH	FAST 1^{st} Diag	FAST 1^{st}	FAST-TE	CR-AS
I_q	0.4244	*)	*)	*)	*)
I_H	0.7910	0.7973	0.8004	0.8142	0.8006
I_V	0.4406	0.4419	0.4417	0.4647	0.4517
β	3.89	3.96	3.96	-	3.90

Table 23: Sensitivities for 2 dike sections with correlation between the dike section of 0.95

	FORM HOH	FAST 1^{st}	FAST-TE	CR-AS
I_q	0.2926	*)	*)	*)
I_H	0.8354	0.8230	0.8533	0.8031
I_V	0.4653	0.4230	0.4947	0.4593
β	3.6806	3.57	-	3.60

Table 24: Sensitivities for 8 dike sections with correlation between the dike section of 0.95

	FORM HOH	FAST 1^{st}	FAST-TE	CR-AS
I_q	0.2040	*)	*)	*)
I_H	0.8552	0.8311	0.8552	0.7944
I_V	0.4764	0.4585	0.5035	0.4494
β	3.5820	3.45	-	3.56

Table 25: Sensitivities for 16 dike sections with correlation between the dike section of 0.95

Appendix C.3 gives the results of the Hohenbichler procedure for 1 till 48 dike sections. The β's do not decrease monotonically, which is not desirable. This is due to the fact that dike sections with highest correlation are combined first. A picture that explains this phenomenon can be found in Appendix C.4.

	FORM HOH	FAST 1^{st}	FAST-TE	CR-AS
I_q	0.1784	*)	*)	*)
I_H	0.8596	0.8332	0.8517	0.7956
I_V	0.4789	0.4598	0.5057	0.4561
β	3.5216	-	-	3.45

Table 26: Sensitivities for 24 dike sections with correlation between the dike section of 0.95

	FORM HOH	FAST 1^{st}	FAST-TE	CR-AS
I_q	0.2053	*)	*)	*)
I_H	0.8550	0.8370	0.8531	0.7956
I_V	0.4763	0.4628	0.4971	0.4561
β	3.4236	3.26	-	3.33

Table 27: Sensitivities for 48 dike sections with correlation between the dike section of 0.95

Appendix C.2. shows that the sensitivities of the last q's tend to be higher than the others. This phenomenon only appears with correlated variables. A Spearman rank correlation test can be used to test if these higher values are just random fluctuations or not. If these higher values are not random fluctuations it would be recommendable to figure out the cause of this phenomenon.

We can conclude that the sensitivities found with the various methods are still very similar. The I_V's are all very similar, whereas the I_H's found with FAST are somewhat greater than the I_H's found with Averaged Sampling. With 16 or more dike sections the sensitivities found with Hohenbichler are greater than those found with other methods, but the differences are small.

6.1.4 Conclusions and remarks

The first order FAST-method is an appropriate sensitivity measure for problems with less than 25 variables. When more variables are involved the computational costs grows quadratically. Models with more than 25 variables could be solved with a less accurate, but quicker total effect FAST. This can be achieved by assigning a not too big frequency to the factor of interest, for instance 64. The model evaluations are then limited and grow linear with the number of variables.

The correlation ratio with Averaged Sampling is appropriate for more complicated problems, as long as the limit-state-function is not too expensive. The differences between the various methods are very small and we can conclude that the method of Hohenbichler works pretty well. The combination of linear limit-state functions goes well. However, if non-linear limit-state functions are considered, the disadvantages of FORM arise again. Furthermore, the Hohenbichler method does not work well if the correlation between the limit-state functions is higher than 0.95 and in some cases the β's do not decrease monotonically.

6.2 Overtopping of the Hoeksche Waard dike-ring

The reliability study of the Hoeksche Waard carried out by TNO Building and Construction Research in co-operation with The Dutch Ministry of Transport, Public Works and Water Management and Delft University of Technology showed corresponding failure probabilities, but disagreeing sensitivities (cf. [7], [8]). Therefore, we choose this case as a test case for the new sensitivity methods.

6.2.1 Problem description and approach

The problem description can be found in Appendix A.2. This problem description is the general description of overtopping of a dike section. A specific formulation of the model has been used to calculate the sensitivities of dike section 1 of the Hoeksche Waard dike-ring. Only one wind direction is considered, namely North West wind. For more details on this model we refer to [7]. The random variable descriptions and the specific values that have been used are given in Appendix A.3.

From the reliability study of the Hoeksche Waard carried out by TNO Building and Construction Research in co-operation with The Dutch Ministry of Transport, Public Works and Water Management and Delft University of Technology, it followed that dike section 11 was the critical dike section. That is why we are especially interested in the sensitivities of dike section 11 (cf. [8]). These sensitivities have been calculated with first order FAST, total effect FAST and Correlation Ratio with Averaged Sampling. For the total effect FAST the frequency assigned to the variable of interest has been taken 1024 and the other frequencies have been obtained with the algorithm of Saltelli et al. The referee has not been calculated, as it is not clear how the derivatives of Z to X_i can be calculated, as mentioned in section 6.2. Since the calculation of the FORM-sensitivities was complicated, due to discontinuities in the limit state function, these have not been calculated for dike section 11. To be able to compare results with FORM, some simplified models based on dike section 1 have served as test case. Dike section 1 is chosen, because complete data of all variables of this dike section is available. A Unicorn-model of this dike section was also available and could be used to calculate the Correlation Ratio with Averaged Sampling. The FORM influence factors have been calculated with the professional version of 'Comrel'. In this simplified models distributions are assigned to variables, which are actually functions of the input variables.

This chapter also describes some of the problems that came across in the use of FORM to calculate the sensitivities of one dike sections.

6.2.2 Results of dike section 11

Table 28 shows the results of the first order FAST, total effect FAST, Correlation Ratio with Averaged Sampling and the referee of dike section 11 of the Hoeksche Waard. This table shows the original values of the correlation ratio and the FAST idices, and not the square root. Only the total effect indices have been normalised. Both the First Order FAST and the correlation ratio identify mqc11 (model-factor critical discharge for dike section 11) as the most important variable. From the definition of the limit-state function $Z = m_{qc11}q_c - m_{qo11}q_o$ this result is maybe not very surprising. Normally, the occurring discharge is a lot smaller than the critical discharge. Only a very small model factor for the critical discharge or a very big model factor for the occurring discharge will bring the limit-state function to zero. The critical discharge is followed by the storm duration (te) and the roughness of the inner talud

	FAST 1^{st}	FAST-TE	CR-AS
$E(Z)$	$2.81 \cdot 10^{-2}$	-	$2.83 \cdot 10^{-2}$
σ_Z	$1.49 \cdot 10^{-2}$	-	$1.48 \cdot 10^{-2}$
ur	0.000	0.020	0.009
uw	0.000	0.192	0.012
uq1	0.000	0.190	0.012
uq2	0.000	0.190	0.013
uu	0.000	0.183	0.009
gg	0.000	0.112	0.011
ss	0.000	0.177	0.012
aq1	0.000	0.191	0.010
bq1	0.000	0.191	0.009
aq2	0.000	0.191	0.009
bq2	0.000	0.191	0.011
aq3	0.000	0.197	0.010
bq3	0.000	0.194	0.011
beta	0.000	0.191	0.010
krw	0.013	0.191	0.024
fb	0.000	0.191	0.009
fn	0.000	0.191	0.010
te	0.057	0.191	0.060
afi	0.000	0.190	0.009
bfi	0.000	0.187	0.007
tgo11	0.000	0.156	0.009
tgi11	0.005	0.191	0.013
hk11	0.000	0.201	0.010
ht11	0.002	0.191	0.010
hb1	0.000	0.191	0.013
mzw11	0.000	0.183	0.011
mgh11	0.000	0.166	0.011
mgt11	0.000	0.180	0.008
mqc11	0.806	0.189	0.904
mqo11	0.000	0.191	0.011

Table 28: Results of Hoeksche Waard dike section 11

(krw). The zero's in the table indicate very small values of the FAST first order indices, these are about order 10^{-7}.

If there are no interaction terms the first order FAST indices add to 1. In this case the sensitivities add to 0.87, so there are indeed interaction terms and therefore the results should not be attached great value. The correlation ratio's add to 1.27 in this case, which should not be possible by definition. This is probably due to the calculation method.

The values for the non-important variables obtained with the FAST-method are about an order 5 smaller than the values obtained with Averaged Sampling. FAST seems to have trouble with the non-influential factors.

The FAST total effect indices give another impression. Almost all variables are identified as being important. In this case the crown height of the dike (hk11) is a bit more important than the other variables, followed by aq3, which influences the discharge of the Rijn at Lobith Q. The total effect indices indicate that there are a lot of interaction terms. The overtopping model is very complex and the problem description in Appendix A.2 shows that there are indeed a lot of interaction terms. To get a picture, a first order FAST can be carried out with interaction terms defined as new variables. However, this is impractical for complex problems. The total effect FAST result may be suspected, since the frequency assigned to the factor of interest is taken to be 1024, which may be too low for a problem with 30 or more variables.

The referee has not been calculated for this example as written in the first section. Not taking the derivatives of Z to the different variables into account resulted in strange results.

6.2.3 Results of dike section 1

Table 29 shows the results of the first order FAST, total effect FAST, Correlation Ratio with Averaged Sampling and the referee of dike section 1 of the Hoeksche Waard. The table shows the original values of the correlation ratio and the FAST indices, and not the square root.

With both FAST first order and Correlation Ratio with Averaged Sampling the model factor for the critical discharge (mqc1) is again the most important variable, followed by the storm length (te) and the roughness of the inner talud (krw). Again, the total effect FAST gives another impression: Almost all variables are of about the same importance, the modelfactor for the Bretschneider period (mgt1) being somewhat more important than the others. Is should be noted that if problems with many variables are considered, it happens often that all variables are considered to be of the same importance. Again, the first order FAST indices do not add to 1, implying the presence of interaction terms. Therefore the first order FAST results should not be attached much value. The correlation ratio's obtained with averaged sampling add to more than 1, and should therefore not be trusted either.

To be able to compare with the FORM-method two simplified models of dike section 1 have been considered. In one model only breaking waves are taken into account, and in the other only non-breaking waves. Furthermore, the wind velocity, the North sea level and the discharge have not been calculated as a function of the input variables, but were treated as input variables. Appendix A.3 shows the assigned distributions, based on [7].

Table 30 shows the results for breaking waves. The square root of the FAST indices have been taken to be able to compare with the FORM-alpha's. Again, both the first order FAST and the total effect FAST assign the model factor for the critical discharge (mqc1) as the most important variable. Followed by the storm duration (te) and the roughness of the inner talud (krw).

The correlation ratio gives another picture. The modelfactor for the critical discharge is indeed assigned to be the most important, followed by the storm duration (te). But then the variables wave angle variation (beta), roughness inner talud (krw), crown height (hk1), berm height (hb1), modelfactor Bretschneider waves (mgh1), modelfactor occurring discharge (mqo1) and the discharge (q) are all of the about the same importance. In theory, the correlation ratio and the first order FAST should be the same. It is not clear why a difference arises in this case.

The FORM-method also gives another impression. This method assigns the wind velocity as the most important variable, followed by the modelfactor Bretschneider period (mgt1), the factor for breaking waves (fb), the North sea level (hvh), the modelfactors for the occurring and the critical discharge (mqc1, mqo1), and the modelfactor Bretschneider waves (mgh1),

	FAST 1^{st}	FAST-TE	CR-AS
$E(Z)$	$2.06 \cdot 10^{-2}$	-	$2.07 \cdot 10^{-2}$
σ_Z	$1.08 \cdot 10^{-2}$	-	$1.08 \cdot 10^{-2}$
ur1	0.000	0.021	0.014
ur2	0.000	-	0.009
uw1	0.000	0.184	-
uw2	0.000	-	0.012
uq1	0.000	0.184	0.012
uq2	0.000	0.184	0.013
uu	0.000	0.186	0.009
gg	0.000	0.082	0.011
ss	0.000	0.185	0.012
aq1	0.000	0.184	0.009
bq1	0.000	0.184	0.010
aq2	0.000	0.184	0.009
bq2	0.000	0.184	0.011
aq3	0.000	0.189	0.010
bq3	0.000	0.185	0.011
beta	0.000	0.184	0.010
krw	0.013	0.184	0.024
fb	0.000	0.177	0.009
fn	0.000	0.196	0.010
te	0.057	0.184	0.060
afi	0.000	0.184	0.009
bfi	0.000	0.163	0.008
d1	0.000	0.185	0.010
tgo1	0.000	0.187	0.009
tgi1	0.005	0.184	0.013
hk1	0.000	0.181	0.010
ht1	0.001	0.183	0.010
hb1	0.000	0.184	0.013
mzw1	0.000	0.189	0.011
mgh1	0.000	0.184	0.011
mgt1	0.000	0.200	0.008
mqc1	0.808	0.193	0.905
mqo1	0.000	0.184	0.011

Table 29: Results of Hoeksche Waard dike section 1

modelfactor Zwendel (mzw1) and the crown height (hk1). This case is non-linear and contains a lot of discontinuities and therefore it is not surprising that the FORM-results differ from the results obtained with the other methods.

The simplified model has been calculated by replacing $q_{o1} = \min(q_{b1}, q_{n1})$ with $q_{o1} = q_{b1}$. The calculation of q_{n1} is not removed. In the FORM-method the factor for non-breaking waves (fn) is assigned as not important, whereas both the FAST methods and the correlation ratio do assign some importance to fn. The methods should not assign importance to a

	FORM	FAST 1^{st}	FAST-TE	CR-AS
beta	0.0000	0.0026	0.0715	0.1030
krw	0.0318	0.1101	0.0912	0.1439
fb	0.2771	0.0097	0.0744	0.0995
fn	0.0000	0.0067	0.0715	0.0968
te	-0.0643	0.2340	0.1492	0.2668
d1	-0.0052	0.0064	0.0715	0.0985
tgo1	-0.0120	0.0098	0.0717	0.0899
tgi1	-0.0193	0.0716	0.0778	0.1245
hk1	-0.1413	0.0069	0.0730	0.1025
ht1	0.0000	0.0059	0.0716	0.0976
hb1	0.0000	0.0051	0.0726	0.1082
mzw1	-0.2120	0.0134	0.0726	0.0920
mgh1	-0.2436	0.0105	0.0715	0.1072
mgt1	-0.4096	0.0177	0.0718	0.0927
mqc1	0.2441	0.8637	0.9775	0.9518
mqo1	-0.2441	0.0147	0.0715	0.1049
q	-0.0818	0.0191	0.0710	0.1091
v	-0.6396	0.0203	0.0716	0.0988
hvh	-0.2764	0.0212	0.0746	0.0971

Table 30: Results of Hoeksche Waard dike section 1, only breaking waves

variable, which does not influence the output at all. For the averaged sampling method this can be explained by the fact that fn is really sampled and ordered, with the corresponding Z-values in the same way. Then averages are taken and finally the variance of these averages. If there are variables in the model which are not important at all this is obviously no good method to calculate the correlation ratio. The reason that the FAST-method (both first order and total effect) also assigns fn as a bit important is probably due to the use of frequencies. A frequency is also assigned to this variable.

As expected the models disagree with FORM. The referee defined by $\frac{1}{\beta}\frac{\partial\beta}{\partial\sigma_i}$ has been calculated, but didn't give good results, due to the ignorance of the coefficients c_i, see also section 6.2. Further research is needed to judge the results.

Table 31 shows the results for non-breaking waves. The table shows the square roots of the first order FAST indices and normalised total effect FAST indices.

Both first order FAST and total effect FAST assign the North sea level (hvh) as the most important variable. First order FAST gives the following ranking: hvh > mzw1 > fn, whereas total effect FAST interchanges fn and mzw1, namely hvh > fn > mzw1. FORM-method gives other results, namely v > mgh1 > fn. The correlation ratio with averaged sampling gives other results by again assigning the model factor for the critical discharge as the most important. The fact that the correlation ratio disagrees with the first order fast method, is due to the calculation method. The occurring debit in case of breaking waves is about $3.5 \cdot 10^{-8}$, whereas in case of non-breaking waves it is about $5.4 \cdot 10^{-8}$. However, the distribution of Z remains about the same and just the values of Z are used in the calculations.

If the first order FAST indices are calculated for models where respectively the factor

	FORM	FAST 1^{st}	FAST-TE	CR-AS
beta	0.0000	0.0093	0.1010	0.1030
krw	0.0313	0.0373	0.1060	0.1439
fb	0.0000	0.0127	0.1010	0.0995
fn	0.3751	0.2940	0.5067	0.0968
te	-0.0633	0.0814	0.1171	0.2668
d1	-0.0040	0.0274	0.1068	0.0985
tgo1	0.0000	0.0236	0.1070	0.0899
tgi1	-0.1898	0.0319	0.1019	0.1245
hk1	0.1255	0.0991	0.1664	0.1025
ht1	0.0000	0.0699	0.1656	0.0976
hb1	0.0000	0.0118	0.1765	0.1082
mzw1	-0.1883	0.3162	0.4755	0.0919
mgh1	-0.4345	0.1433	0.2231	0.1072
mgt1	0.0000	0.0239	0.1010	0.0927
mqc1	0.2401	0.0253	0.3033	0.9518
mqo1	-0.2401	0.0318	0.1010	0.1049
q	-0.0650	0.1098	0.1401	0.1091
v	-0.0664	0.1264	0.1854	0.0988
hvh	-0.2323	0.3787	0.6687	0.0971

Table 31: Results of Hoeksche Waard dike section 1, only non-breaking waves

breaking waves and non-breaking waves are omitted, the results are slightly different. In case of only breaking waves, both fb (factor breaking waves) and hk1 (the crown height) are a little more important.

It is evident that more research is needed to figure out the problems with the used method and to elaborate the referee, so that it can also be calculated for complex problems, to judge the results.

6.2.4 Problems with FORM

The critical velocity v_c (114) depends on the storm length. Actually, the tenth logarithm of the storm length is taken. The storm length is lognormally distributed and with very small probability it is possible to obtain a value smaller than 1. This leads to a negative critical velocity, which is not desirable. To prevent this situation a truncated normal distributed distribution can be assigned to a secondary variable log_te, truncated from zero, with the corresponding expectation and standard deviation, and the exponential of this variable can be taken. However, the use of truncated variables is not recommendable, since the global optimisation algorithm, which looks for the design point, gets stuck with the upper and lower bounds.

Similar problems occurred in the formulas (116) and (123), where the square root of $\tan\alpha_i$ and $\tan\alpha_{repr}$, respectively, is taken. Since both slopes are normally distributed, it is possible that the tangent of the slopes takes negative values. Complex values appear when taking the square root of these values. Assigning other distributions to the variables can prevent these problems.

One of the main problems of the FORM-method is the search for the design point. To find the design point, a global optimiser is used and this optimiser has trouble to find the design point if there are discontinuities in the limit state function. It is difficult to control the points chosen by the optimiser we have used. A possible solution is to use a gradient-based algorithm and control the search points. This has not been investigated. Another problem is the fact that discontinuities in the limit state function can not be handled properly.

6.2.5 Final Remarks

We have tried to calculate a sort of referee for this case, but some problems occurred. The referee cannot be calculated for every arbitrary problem. If the variance of an input factor is increased or decreased it is possible that logarithms or square roots of negative values are taken.

From this test case we can conclude that the FORM-method cannot easily be applied to problems with a lot of discontinuities and non-linearity's. The FAST-method can be applied straightforward. However, the results should be validated, since some of the results given in this section seem to be strange. The calculation with the averaged sampling scheme can go wrong in some cases, for instance if some variables are non-important. In further research emphasis should be put on the elaboration of the referee, to be able to judge the results.

7 Conclusions and recommendations

7.1 Conclusions

The main goal of this research was to find a sensitivity measure for cases in which the common method FORM does not give good results. These cases mainly involve non-linear and discontinuous output functions. To achieve this goal the existing sensitivity measures have been investigated and a referee has been developed to judge the correctness of the results. The various methods have been tested on several test cases and from these results the following conclusions can be drawn.

It can be shown that in the linear case the correlation ratio is equal to the linear regression coefficient, the first order Sobol indices, the Jansen's first order sensitivity measure, the first order FAST indices and the FORM influence coefficient. The Sobol total effect indices, the Jansen's total effect indices and the FAST total effect indices are extensions of the correlation ratio.

The following measures have been tested on several cases: FORM, correlation ratio, total effect indices of Homma and Saltelli, first order FAST, total effect FAST and the referee. The following conclusions can be drawn.

- FORM
  In some test cases the FORM-method does not find an answer and in other cases the results are not correct. The method is fast, but inaccurate if the limit state function is non-linear or discontinuous.

- Correlation Ratio with Latin Hypercubic Sampling
  The method gives accurate results, which resemble the analytical values. However, the method is sensitive to long-tailed distributions and it takes much computational effort to obtain accurate results. Furthermore, interaction effects are not taken into account.

- Correlation Ratio with the Resampling based method
  Despite the computational efficiency, the resampling based method performs badly. The results obtained with this method are very unstable and can be negative or greater than 1, which is not possible by definition. This method lacks reproducibility and does not take interaction effects into account.

- Correlation Ratio with Averaged Sampling
  The method is very accurate, but interaction effects are not taken into account and a considerable number of model evaluations are needed. Furthermore, in some cases the averaged sampling scheme can lead to wrong results.

- Total Effect Indices Homma and Saltelli
  The results obtained with this sampling scheme are very unstable. The method takes interaction effects into account, but needs quite a lot of model evaluations to obtain stable results.

- First Order FAST
  The first order FAST method is computationally efficient and the results are reproducible and accurate. However, the computational effort grows quadratically as the number of variables grows and interaction effects are not taken into account.

- Total Effect FAST
  The methods takes interaction effects into account and the results are reproducible. More insight is obtained at a higher price, namely more model evaluations. However, it is possible to do a less accurate, but quicker, total effect FAST analysis.

- Referee
  The referee can not be calculated for every arbitrary problem. The formulation needs to be worked out.

The first order FAST method seems to be the best alternative for FORM, since with a relatively small number of model evaluations good results are obtained. However, if problems with more than 25 variables are considered, the computational cost grows quadratically and first order FAST is no longer attractive. Another disadvantage is the fact that interaction effects are not taken into account. To take these effects into account an extended FAST analysis, which gives more inside at a higher cost, can be carried out. It is also possible to do a less expensive, but also less accurate, total effect FAST analysis. The Correlation Ratio with Averaged Sampling also seems to be a good alternative for FORM. However, in some cases wrong results can be obtained. Both the FAST-method (first order and extended) and the correlation ratio have been applied to more complex problems. Since both the first order FAST and the correlation ratio only calculate the main effects, these methods are not very suitable for complex models containing a lot of interaction terms.

The foundation for a referee, based on the FORM-method, has been made. A simplified form of this referee has been calculated for several test cases. In most of the test cases the sensitivities obtained with the referee and the other methods agreed very well. One disadvantage of the global methods is the fact that these methods give no information about the sign of the sensitivities. It should always be kept in mind which method is used and if that method really provides us the sensitivities we are looking for.

Furthermore, sensitivities obtained with method of Hohenbichler for system analysis, have been compared with the correlation ratio and FAST indices. In the test cases the sensitivities agreed very well and in these cases the method of Hohenbichler worked pretty well. However, if the correlation between different output functions is higher than 0.95 the method does not work well and in same cases the β's do not increase monotonically.

In terms of computational efficiency, the FORM-method is by far the quickest method, but sometimes the results are not correct or the method does not even give results. In these cases the investigated measures are good alternatives.

Summarising we conclude that both the correlation ratio and the FAST-method gave good results on the test cases and can be a valuable supplement to the existing techniques. The first order FAST method is especially suitable for problems with less than 25 variables, whereas the correlation ratio with averaged sampling can be used on bigger problems.

7.2 Recommendations for further research

Conclusions drawn in the previous section are mainly based on problems with uniformly and normally distributed variables. It is recommendable to test the sensitivity measures on problems with other probability distributions, such as an exponential distribution.

Furthermore, it is interesting to figure out whether it is possible to combine FAST and FORM or if the methods can be used to supplement each other.

It would also be interesting to calculate the sensitivities of problems with correlated variables, which can be modelled with a tree-structure, with FAST-sampling on the diagonal band.

With unimportant variables great fluctuations in the sensitivities of both the correlation ratio as the FAST-method may arise. It would be recommendable to do a Spearman rank correlation coefficient test to check if these fluctuations are just random fluctuations.

It would also be recommendable to look more thoroughly at the calculation of the correlation ratio and figure out if there are smarter ways to calculate this measure. The Winding Stairs scheme used by Jansen should also be investigated, since with this method the number of model evaluations can be reduced considerably.

The differential equation, given in the section about FAST, to obtain a transformation function, deserves more attention. It would be useful to know how this equation is derived.

Last but not least, in further research emphasis should be put on the elaboration of the referee, since it is important to judge the results. In this report a simplified form of the referee, based on a specific formulation of the output function, is used. It would be very useful to figure out what form the theory will take if a general description of the limit state function is used.

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List of symbols and abbreviations

Symbols

Symbol	Description
$Z(\mathbf{X})$	limit state function
X_i	stochastic variable
$\mathbf{x}^*$	design point
μ_Z	expectation of Z
σ_Z	standard deviation of Z
α_i	influence factor of variable X_i
β	reliability index
U_i	standard normally distributed variable
Φ^{-1}	inverse normal cumulative distribution
F_X	cumulative distribution of X
γ	vector containing sensitivities
Γ_0	transformation
$E(X)$	expectation of X
$\rho(X,Y)$	Pearson product moment correlation between X and Y
ρ_r	Spearman rank correlation
$\mathbf{J}(t)$	Jacobian
$\mathbf{a}$	vector of system parameters
$\mathbf{K}(t,\tau)$	$n \times n$ matrix called the Green's function
$\delta_{a_j y_{0i}}$	n vector with all components being 0, except
	the ith component being 1 if $a_j = y_{0i}$
ν	performance measure
x	phase-space position vector
$u(x)$	state vector
$\mathbf{Q}$	vector containing inhomogeneous source terms
$\mathbf{N}$	vector containing nonlinear operators
A	operator representing all inhomogeneous boundary terms
B	operator
$\partial\Omega$	boundary of Ω, subset of $\mathbb{R}^J$
R_i^2	linear regression correlation coefficient
$V(X)$	variance of X
ϵ	error
$f_j(X_j)$	probability density of variable X_i
$V(Z\|X_i)$	variance of Z given X_i
k	number of variables
η_i^2	correlation ratio
n	sample size
r	number of replicates in Latin Hypercubic Sampling
m	size of Latin Hypercubic Sampling/Number of intervals
X^s	subset of variables
D	design matrix for LHS-sampling
HI_i	Hora and Iman importance measure

List of symbols and abbreviations

Symbols

Symbol	Description
K^k	unit k-dimensional hypercube
$G_i(s)$	search curve
A_j, B_j	Fourier coefficients
ω_i	frequency
D_{ω_i}	contribution of the variance of Z by X_i
M	maximum harmonic taken into account
D^{FAST}	variance of Z
N_s	minimum sample size in a FAST-analysis
S_i	first order Sobol sensitivity index
S_{ij}	second order Sobol sensitivity index
$(\backslash i)$	'all but i'; complementary to i
$\mathbf{X}_{\backslash i}$	set containing all the X_j with $j \neq i$
$D_{\backslash i}$	variance of the model where the variable X_i is omitted
$g_{i,\backslash i}(X_i, \mathbf{X}_{\backslash i})$	all terms of g representing interaction between X_i and $\mathbf{X}_{\backslash i}$
Z_m^i	mth evaluation of Z with X_i resampled
$(X_i, \mathbf{X}'_{\backslash i})$	all variables of $\mathbf{X}$ resampled, except for the ith one
$Z(\mathbf{X_m}', X_{im})$	mth model evaluation with all variables resampled, except for the ith one
I_U	importance of variable U
$\mathbf{C_Y}$	covariance matrix
$\text{Cov}(X, Y)$	covariance between X and Y
A	matrix containing the orthonormal eigenvectors of $\mathbf{C_Y}$

Symbols of overtopping

Symbol	Description
q_c	critical discharge
q_o	occurring wave discharge
m_{qc}	modelfactor describing the uncertainty in the model which determines q_c
m_{qo}	modelfactor describing the uncertainty in the model which determines q_o
v_c	critical velocity
α_i	slope of the inner talud
C	roughness factor of Chezy
f_g	deterministic factor giving the quality of the grass
t_e	storm duration
k	roughness inner talud
ξ_{eq}	equivalent index of breaking
ξ_{op}	index of breaking
γ_b	reduction factor of the berm

Symbols of overtopping

Symbol	Description
α_{repr}	characteristic slope of the outer talud
α_u	slope outerdyke
S_{op}	wave steepness
H_s	significant wave steepness
T_p	wave velocity
g	acceleration of gravitation
F	strike length
d	water depth
T_p	wave period
T_s	significant wave period
m_{gH}	model factor for H_s
m_{gT}	model factor for T_s
h_d	depth of the water at the toe of the dike
h	local water level
h_t	height of the toe of the dike with reference to NAP
Q_b	dimensionless discharge for breaking waves
f_b	fitparameter
R_b	dimensionless crown height for breaking waves
γ_b	reduction factor which represents the influence of the berm
γ_f	reduction factor which represents the influence of the roughness
γ_β	reduction factor which represents the influence of the angle of wave attack
γ_v	reduction factor which represents the influence of the presence of shallow land
Q_n	dimensionless discharge for non-breaking waves
R_n	dimensionless crown height for breaking waves
r_B	factor which represents the influence of the width of the berm
r_{db}	factor which represents the influence of the location of the berm with reference to the waterside
d_B	height of the berm
d_h	difference between the local water level and the height of the berm

Abbreviations

Abbreviation	Complete Description
FORM	First Order Reliability Method
SORM	Second Order Reliability Method
MC	Crude Monte Carlo Simulation
LPSM	Local Probabilistic Sensitivity Measure
SA	Sensitivity Analysis
LSF	Limit State Function
VCE	Variance Conditional Expectation
HIM	Hora and Iman Importance Measure
CR-LHS	Correlation Ratio computed with Latin Hypercubic Sampling
CR-RM	Correlation Ratio computed with the Resampling Based method
CR-AS	Correlation Ratio computed with Averaged Sampling
TEI-HS	Total Effect Indices of Homma and Saltelli
FAST	Fourier Amplitude Sensitivity Test
FAST 1^{st}	First Order FAST
FAST-TE	Total Effect FAST

List of software

This section gives a brief description of the m-files that were written to calculate the sensitivities with the different methods.

FAST

- **fast1order('lsf',StochSpec)**
  Calculates the FAST first order indices. Input: file with a description of the limit state function and the stochast specifications of the input variables. Uses the function 'frecuency.m' and 'RdDistInfo.m', which reads the variables and their distribution information from the stochast specification. Assumes in general uniform distributions, other distributions have to be given manually. To be improved: number of variables in header of function.

- **frequency(k)**
  Calculates a set of k incommensurate frequencies, where k the number of variables, with k at least 3 and at most 50.

- **fasttoteff('lsf')**
  Calculates the FAST total effect indices. Input: file with a description of the limit state function and k the number of variables. Uses the function 'frequencytot.m'. Assumes in general uniform distributions, other distributions have to be given manually. To be improved: stochast specifications as inputfile and number of variables k in header of function.

- **frequencytot(k,M)**
  Calculates a set of k frequencies according the algorithm of Saltelli *et al*, where k the number of variables and M the number of term taken into account in the calculation of the total effect indices. The frequency of the ith variable will be set on 2^n in the program 'fasttoteff.m', with $n \leq k$.

- **diagband**
  Calculates first order indices with the variables sampled on the diagonal band. The number of correlated variables should be two and these variables should be uniformly distributed.

Correlation Ratio

- **mckay**
  Calculates the correlation ratio's for all input variables with the Latin Hypercubic sampling plan of McKay. Limit state function has to be given manually. Assumes in general uniform distributions, other distributions have to be given manually. To be improved: stochast specifications and limit state function as inputfile.

- **ishhom**
  Calculates the correlation ratio's for all input variables with the resampling based method of Ishigami and Homma. Limit state function has to be given manually. Assumes in general uniform distributions, other distributions have to be given manually. To be improved: stochast specifications and limit state function as inputfile.

Homma and Saltelli

- **homsal**
  Calculates the Homma and Saltelli total effect indices. Limit state function has to be given manually. To be improved: limit state function as inputfile.

Hohenbichler

- **nserie(n)**
  Program to combine n elements in a series system with the method of Hohenbichler. Uses seriefu.m.

- **seriefu(beta1,beta2,alpha1,alpha2,rho1)**
  Combines two elements and calculates the beta and alpha's of the corresponding element. Input the beta's and alpha's of the original elements and de correlation between these elements. Alpha1 and alpha2 are vectors. Output are the new beta and alpha's. Uses equibeta.m, linregsrch.m.

- **linregsrch('gXfuns','numXgrad',EX,CovXX,Params,Xstart)**
  Finds a candidate for the locally most central limit state point of given limit-states. The search algorithm is the iterative linear regression scheme based on theorem 5.2 in [14]. Input gXfuns (Cell array of strings containing names of X-space state functions), gradXfuns (Cell array of strings containing names of X-space gradient evaluation function), EX (Mean vector of input; size(EX) = [n,1]), CovXX (Covariance matrix of input; size(CovXX) = [n,n]), Params (Column vector of distribution and limit-state parameters) and Xstart (Initial guess vector for beta-point search). Sten de Wit (TNO Bouw) has written this program.

- **equibeta(beta1,beta2,betan)**
  Calculates equivalent beta according to (4.3.4), p. 27 of [47]. Input original beta's of the elements and betan the result of the linear regression search. Output is the equivalent beta.

- **z2prime(X,par)**
  This file gives the z2 prime function as described on p. 25 of [47]. This file is used as input for linregsrch.m.

Appendix A: Mathematical Derivations

Appendix A.1 Proof of Lemma

In this section we give a proof of a lemma which states that the maximum of the correlation ratio between the function $Z(\mathbf{X})$, where $\mathbf{X} = [X_1, ..., X_k]$ and some function of the variables is taken by $E(Z|X_i)$. In the rest of this chapter we assume that $Z = Z(X, Y)$ and we consider $E(Z|X)$. The proof uses the following fact.

Fact

$$Cov(Z, E(Z|X)) = Cov(E(Z|X), E(Z|X)) = Var(E(Z|X)) \tag{110}$$

Proof

$$\begin{aligned} Cov(Z, E(Z|X)) &= E(Z \cdot E(Z|X)) - E(Z) \cdot E(E(Z|X)) \\ &= E(E(Z \cdot E(Z|X)|) - E(E(Z|X)) \cdot E(E(Z|X)) \\ &= E(E(Z|X) \cdot E(Z|X)) - E^2(E(Z|X)) \\ &= E(E^2(Z|X)) - E^2(E(Z|X)) = Var(E(Z|X)). \qquad \square \end{aligned}$$

Lemma (Cooke, 2000)

$$\max_{f(x), \sigma^2_{f(x)} < \infty} \rho^2(Z, f(x)) = \frac{\sigma^2_{E(Z|X)}}{\sigma^2_Z}, \tag{111}$$

Proof
Suppose $f(X) = E(Z|X) + \delta(X)$, with $\delta(X)$ a function with finite variance, so $\sigma^2_{\delta(X)} < \infty$.

$$\begin{aligned} \rho^2(Z, E(Z|X) + \delta(X)) &= \frac{Cov^2(Z, E(Z|X) + \delta(X))}{\sigma^2_Z \sigma^2_{E(Z|X)+\delta(X)}} \\ &= \frac{(Cov(Z, E(Z|X)) + Cov(Z, \delta(X)))^2}{\sigma^2_Z \cdot (\sigma^2_{E(Z|X)} + \sigma^2_{\delta(X)} + 2Cov(E(Z|X), \delta(X)))} \\ &= \frac{(\sigma^2_{E(Z|X)} + Cov(Z, \delta(X)))^2}{\sigma^2_Z \cdot (\sigma^2_{E(Z|X)} + \sigma^2_{\delta(X)} + 2Cov(E(Z|X), \delta(X)))}. \quad (*) \end{aligned}$$

Let

$$\begin{aligned} A &= \sigma^2_{E(Z|X)} \\ B &= Cov(E(Z|X), \delta(X)) \\ C &= \sigma^2_Z \\ D &= \sigma^2_{\delta(X)}. \end{aligned}$$

We will use the Cauchy Schwarz inequality, which states that

$$\begin{aligned} & Cov^2(E(Z|X), \delta(X)) \leq \sigma^2_{E(Z|X)} \cdot \sigma^2_{\delta(X)} \\ \Leftrightarrow \quad & B^2 \leq A \cdot D \end{aligned}$$

Then:

$$\begin{aligned}\rho^2(Z, E(Z|X)) + \delta(X)) &= \frac{(A+B)^2}{C(A+D+2B)} = \frac{A^2+2AB+B^2}{C(A+2B+D)} \\ &\leq \frac{A(A+2\sqrt{AD}+D}{C(A+2\sqrt{AD}+D)} = \frac{A}{C} = \frac{\sigma^2_{E(Z|X)}}{\sigma^2_Z} = \rho^2(Z, E(Z|X)) = \eta_i^2.\end{aligned}$$

From the lemma it follows that the correlation ratio is a good indicator to measure the correlation between $Z(\mathbf{X})$ and its variables X_i. A similar proof is given in [26].

Appendix A.2 Problem description overtopping

This paragraph describes the mathematical model of the failure mechanism overtopping. Local overtopping or overflow of a dike shall not immediately result into failure of the dike, but causes erosion of the inner talud, which will indeed in the end result into failure of the dike.

The stresses on the dike sections are due to the occurring discharge [40]. The strength of the dike is given by the maximum critical discharge, which the dike can resist. The reliability function is given by:

$$Z = m_{qc}q_c - m_{qo}q_o \tag{112}$$

where q_c the critical discharge, q_o the occurring discharge, m_{qc} and m_{qo} modelfactors describing the uncertainty in the models which determine q_c and q_o.

Strength

The critical discharge q_c is determined from

$$q_c = \frac{v_c^3}{\tan\alpha_i C^2} \tag{113}$$

where C the roughness factor of Chezy and α_i the slope of the inner talud. The critical velocity v_c is given by

$$v_c = f_g \frac{3.8}{1 + 0.8^{10}\log t_e} \tag{114}$$

where f_g a deterministic factor giving the quality of the grass, t_e the storm duration and m_{qc} takes the uncertainty about f_g into account.

The roughness factor of Chezy is determined according to Strickler:

$$C = 25 \cdot \left(\frac{q_c}{kv_c}\right)^{1/6} \tag{115}$$

where k the roughness of the inner talud. Substitution of (115) into (113) gives

$$q_c = \frac{v_c^3}{\tan\alpha_i C^2} = \left(\frac{v_c^{10/3}k^{1/3}}{\tan\alpha_i 25^2}\right)^{3/4} = \frac{v_c^{10/4}k^{1/4}}{125\tan\alpha_i^{3/4}}. \tag{116}$$

Stress

To determine the occurring discharge a distinction between breaking and non-breaking waves has to be made. To determine if the waves break or not the equivalent index of breaking ξ_{eq} is used.

$$\xi_{eq} = \gamma_b\xi_{op} \tag{117}$$

where γ_b a reduction factor for the berm. ξ_{op} is the index of breaking defined by

$$\xi_{op} = \frac{\tan\alpha_{repr}}{\sqrt{S_{op}}} \tag{118}$$

where α_{repr} the characteristic slope of the outer talud. In this analysis we assume the slope of the outer talud to be equal to the slope outerdyke, so $\alpha_{repr} = \alpha_u$. The wave steepness S_{op} is given by

$$S_{op} = \frac{2\pi H_s}{gT_p^2}. \tag{119}$$

The wave velocity is calculated from the significant wave steepness H_s (significant wave height), the wave period T_p and the acceleration of gravitation g. H_s follows from

$$H_s = m_{gH} \cdot \frac{V^2}{g} 0.283 \tanh\left(0.530 (\frac{gd}{V^2})^{0.750}\right) \tanh\left(\frac{0.0125 (\frac{gF}{V^2})^{0.42}}{\tanh\left(0.530 (\frac{gd}{V^2})^{0.750}\right)}\right) \tag{120}$$

and T_p follows from

$$T_s = m_{gT} \cdot 2\pi \frac{V}{g} 1.2 \tanh\left(0.833 (\frac{gd}{V^2})^{0.375}\right) \tanh\left(\frac{0.077 (\frac{gF}{V^2})^{0.25}}{\tanh\left(0.0833 (\frac{gd}{V^2})^{0.375}\right)}\right) \tag{121}$$

and

$$T_p \approx T_s. \tag{122}$$

With V the wind velocity, F is the strike length, d the water depth and g acceleration of gravitation. For the calculation of V I refer to [40]. H_s and T_s are determined according to Bretschneider [44]. To correct H_s and T_s for the error introduced by the model of Bretschneider the results are multiplied with a model factor m_{gH} and m_{gT} respectively, both with mean 1.

The significant wave height is limited in the following way:

$$\text{if} \quad H_s > 0.4h_d, \quad \text{then} \quad H_s = 0.4h_d$$

where h_d the depth of the water at the toe of the dike. h_d is calculated as the difference between the local water level h and the height of the toe with reference to NAP (Nieuw Amsterdams Peil), h_t: $h_d = h - h_t$.

The passage from breaking waves to non-breaking waves is found at $\xi_{eq} = 2$.

q_o for breaking waves

For the case of breaking waves the occurring wave discharge q_o [m^2/s] is given by

$$q_o = Q_b \sqrt{gH_s^3} \sqrt{\frac{\tan \alpha_{repr}}{S_{op}}} \tag{123}$$

where Q_b the dimensionless discharge, given by

$$Q_b = 0.06 \cdot \exp^{(-f_b R_b)} \tag{124}$$

where f_b a fitparameter and R_b the dimensionless crown height, which is determined by

$$R_b = \frac{h_k - h}{H_s} \cdot \frac{\sqrt{S_{op}}}{\tan \alpha_{repr}} \cdot \frac{1}{\gamma_b \gamma_f \gamma_\beta \gamma_v} \tag{125}$$

where h_k the crown height of the dike, h the local water level and $\gamma_b, \gamma_f, \gamma_\beta, \gamma_v$ reduction factors which represent the influence of berm, roughness, angle of wave attack and the presence of shallow land respectively. The local water level h depends on the discharge Q, the wind velocity V and the Northsea level hvh. For the determination of Q, V and hvh I refer to [40].

q_o for non-breaking waves
In the case of non-breaking waves q_o is determined by

$$q_o = Q_n \sqrt{g H_s^3} \tag{126}$$

where Q_n follows from

$$Q_n = 0.2 \exp^{(-f_n R_n)} \tag{127}$$

where R_n is given by

$$R_n = \frac{h_k - h}{H_s} \frac{1}{\gamma_b \gamma_f \gamma_\beta \gamma_v} \tag{128}$$

Reductionfactor γ_b
The reduction factor γ_b consists of two other factors, one to determine the influence of the width of the berm (r_B) and one to determine the influence of the location of the berm with reference to the waterside r_{dh}.

$$\gamma_b = 1 - r_B(1 - r_{dh}) \quad \text{with} \quad 0.6 \leq \gamma_b \leq 1.0 \tag{129}$$

where

$$r_B = 1 - \frac{\tan \alpha_{eq}}{\tan \alpha_{repr}}. \tag{130}$$

If the slope of the talud above the berm equals the slope under the berm equation (129) can be replaced by

$$r_B = \frac{B/H_s}{2 \cot \alpha_u + B/H_s} = \frac{1}{1 + 2H_s/(B \tan \alpha_u)}, \tag{131}$$

where r_{dh} is determined by

$$r_{dh} = 0.5 \left(\frac{d_h}{H_s} \right)^2 \quad \text{with} 0 \leq r_{dh} \leq 1, \tag{132}$$

where $d_h = h - h_B$ the difference between the local water level and the height of the berm.

The value determined for γ_b is valid for breaking waves if $\xi_{op} < 2/\gamma_b$. For non-breaking waves the value is corrected as follows:
If $\xi_{op} > 4$ then $\gamma_b = 1$.
If $2/\gamma_b < \xi_{op} < 4$ then

$$\gamma_b = \gamma_b + \frac{(\xi_{op} - 2/\gamma_b) \cdot (1 - \gamma_b)}{(4 - 2/\gamma_b}. \tag{133}$$

Reductionfactor γ_β
The angle of wave attack β is given by the difference between the direction of the waves and the direction of the perpendicular (normal) of the dike ψ. On the average, the waves will have the same direction as the wind. To be able to consider the difference between the direction of the waves and the direction of the wind, a stochastic variable β^* is added up to the wind direction φ. The angle of wave attack can be determined according

$$\beta = |(\varphi + \beta^*) - \psi|. \tag{134}$$

If $\beta > 180°$ then $\beta = 360° - \beta$. And if there are onshore waves, $0° \leq \beta \leq 90°$, then

$$\gamma_\beta = 1 - 0.0033\beta. \tag{135}$$

With offshore waves, if $90° \leq \beta \leq 180°$, there is no overtopping and only overflow is considered.

Reductionfactor γ_f
Table 32 gives the values of the reductionfactors for the roughness of the various taluds. The factors are valid for $\xi_{op} < 3$. From $\xi_{op} = 3$ the reduction factor linearly grows till 1 for $\xi_{op} = 5$. For big values the reductionfactor remains 1.

Type of talud	**Reductionfactor γ_f**
smooth concrete	1.0
thick smooth bricklaying	1.0
grass (3 cm)	0.90-1.0
fixed stone	0.85-0.90
one layer pour stone	0.55-0.60
1/4 v setting	0.75-0.80

Table 32: Reductionfactors for the roughness of various taluds

Reductionfactor γ_h
In shallow land the waves will break and the wave distribution will differ from the Raleigh-distribution, which is valid for deep water. This causes the discharge to decrease and the reductionfactors are given by

$$\begin{aligned} \gamma_h &= \frac{H_{2\%}}{1.4H_s} = 1 - 0.03\left(4 - \frac{h_d}{H_s}\right)^2, \quad \text{for} \quad \frac{h_d}{H_s} < 4 \\ \gamma_h &= 1, \qquad \text{for} \quad \frac{h_d}{H_s} \geq 4 \end{aligned}$$

where h_d the depth of the water at the toe of the dike with reference to the motionless water level. The water depth is again the difference of the local water level h and the height of the toe with reference to NAP, h_t, so

$$h_d = h - h_t.$$

Appendix A.3 Random Variable Descriptions

The following table shows the numerical values of the parameters of the random variables of dike section 1. For normally and lognormally distributed variables par1 is equal to the expectation μ and par2 is equal to the standard deviation σ. For uniform variables par1 gives the lower bound and par2 gives the upper bound. For constants par1 gives the numerical value of the constant.

Random variable descriptions for dike section 1

Variable	Symbol	Description	Type of Distribution	par1	par2
ur1		cdf Pareto water hvh given wind	uniform	0	1
ur2			uniform	0	1
uw	W	cdf wind independent of water	uniform	0	1
uq1			uniform	0	$2 \cdot 10^{-3}$
uq2			uniform	0	1
uu	u	location Pareto hvh	normal	2.53	$1.73 \cdot 10^{-1}$
gg	h	shape Pareto	normal	$3.64 \cdot 10^{-2}$	$6.00 \cdot 10^{-2}$
ss	s		normal	$2.474 \cdot 10^{-1}$	$1.24 \cdot 10^{-2}$
aq1	A_{Q1}		normal	$5.727 \cdot 10^{3}$	$1.718 \cdot 10^{2}$
bq1	B_{Q1}		normal	$1.699 \cdot 10^{3}$	$8.495 \cdot 10^{1}$
aq2	A_{Q2}		normal	$7.020 \cdot 10^{3}$	$2.106 \cdot 10^{2}$
bq2	B_{Q2}		normal	$1.150 \cdot 10^{3}$	$5.750 \cdot 10^{1}$
aq3	A_{Q3}		normal	$7.900 \cdot 10^{3}$	$2.370 \cdot 10^{2}$
bq3	B_{Q3}		normal	$9.950 \cdot 10^{2}$	$4.975 \cdot 10^{1}$
beta	β	wave angle variation	normal	0	$3.750 \cdot 10^{-3}$
krw	k	roughness inner talud	lognormal	$1.546 \cdot 10^{2}$	$3.862 \cdot 10^{3}$
fb	f_b	factor breaking waves	normal	5.20	$5.50 \cdot 10^{-1}$
fn	f_n	factor non-breaking waves	normal	2.60	$3.50 \cdot 10^{-1}$
te	t_e	storm duration	lognormal	7.450	1.873
afi	$A_{h\varphi_i}$	location Gumbel hvh given wind	normal	1.05	$3.15 \cdot 10^{-2}$
bfi	$B_{h\varphi_i}$	scale Gumbel hvh given wind	normal	$3.26 \cdot 10^{-1}$	$1.63 \cdot 10^{-2}$
av	a	location indep wind component	constant	1.16	
bv	b	scale, indep wind component	constant	$3.00 \cdot 10^{-1}$	
rv	ρ	rho indep wind component	constant	$6.80 \cdot 10^{-1}$	
mv	M	factor indep wind component	constant	$6.70 \cdot 10^{-1}$	
f1	F	fetch	constant	$4.00 \cdot 10^{2}$	
d1	d	depth	normal	8.00	$3.00 \cdot 10^{-1}$
tgo1	$\tan_{\alpha_{repr}}$	slope outerdyke	normal	$4.00 \cdot 10^{-1}$	$2.00 \cdot 10^{-2}$
tgi1	$\tan_{\alpha_i}$	slope innerdyke	normal	$5.00 \cdot 10^{-1}$	$2.50 \cdot 10^{-2}$
hk1		crown height of the dike	normal	5.00	$1.00 \cdot 10^{-1}$
ht1	h_t	bermheight front	normal	-1.00	$2.00 \cdot 10^{-1}$
hb1	h	bermheigth	normal	1.800	$2.00 \cdot 10^{-1}$

Variable	Symbol	Description	Type of Distribution	par1	par2
fg1	f_g	grass factor	constant	$8.00 \cdot 10^{-1}$	
b1	B	berm width	constant	6.00	
mzw1	h_{zw}	modelfactor Zwendel	normal	0.00	$1.50 \cdot 10^{-1}$
mgh1	m_{gH}	modelfactor Bretschneider waves	lognormal	1.011	$15.255 \cdot 10^{-2}$
mgt1	m_{gT}	modelfactor Bretschneider period	lognormal	1.011	$15.255 \cdot 10^{-2}$
mqc1	m_{qc}	modelfactor critical discharge	lognormal	1.0	0.5
mqo1	m_{qo}	modelfactor critical occurring discharge	lognormal	1.0	0.5

Random variable descriptions for dike section 11

The variables ur1, ur2, uw, uq1, uq2, uu, gg, ss, aq1, bq1, aq2, bq2, aq3, bq3, beta, krw, fb, fn, te, afi, bfi, av, bv, rv, mv have the same distributions as for dike section 1.

Variable	Symbol	Description	Type of Distribution	par1	par2
f11	F	fetch	constant	$1.104 \cdot 10^{3}$	
d11	d	depth	constant	$2.00 \cdot 10^{-1}$	
tgo11	$\tan_{\alpha_{repr}}$	slope outerdyke	normal	$3.330 \cdot 10^{-1}$	$1.70 \cdot 10^{-2}$
tgi11	$\tan_{\alpha_i}$	slope innerdyke	normal	$3.300 \cdot 10^{-1}$	$1.700 \cdot 10^{-2}$
hk11		crown height of the dike	normal	4.300	$1.000 \cdot 10^{-1}$
ht11	h_t	bermheight front	normal	$-2.00 \cdot 10^{-1}$	$2.00 \cdot 10^{-1}$
hb11	h	bermheigth	normal	$4.00 \cdot 10^{-1}$	$2.000 \cdot 10^{-1}$
fg11	f_g	grass factor	constant	$8.00 \cdot 10^{-1}$	
b11	B	berm width	constant	5.00	
mzw11	h_{zw}	modelfactor Zwendel	normal	0.00	$1.50 \cdot 10^{-1}$
mgh11	m_{gH}	modelfactor Bretschneider waves	lognormal	1.011	$15.255 \cdot 10^{-2}$
mgt11	m_{gT}	modelfactor Bretschneider period	lognormal	1.011	$15.255 \cdot 10^{-2}$
mqc11	m_{qc}	modelfactor critical discharge	lognormal	1.0	0.5
mqo11	m_{qo}	modelfactor critical occurring discharge	lognormal	1.0	0.5

Assigned distributions

Variable	Description	Type of Distribution	expectation	standard deviation
q	discharge	normal	$1.510 \cdot 10^{4}$	$1.068 \cdot 10^{3}$
v	wind velocity	lognormal	16.60	3.406
hvh	North sea level	lognormal	2.70	$3.138 \cdot 10^{-1}$