# MAINTENANCE STUDY FOR COMPONENTS UNDER COMPETING RISKS

*Cornel Bunea* and *Roger Cooke*
Delft University of Technology
Delft, The Netherlands
e-mail: c.bunea@its.tudelft.nl
e-mail: r.m.cooke@its.tudelft.nl

*Bo Lindqvist*
Norwegian University of Science and Technology
Trondheim, Norway
e-mail: bo@math.ntnu.no

**Summary**

The problem of competing risk data is considered with an application to Offshore Reliability Data Base (OREDA). Different models for preventive maintenance are discussed which make the failure rate identifiable, as there are more general bounding methods. A statistical test is used for the concordance of the results of the data interpretation and the theoretical models proposed. The results indicate the way to avoid an inappropriate model to fit data.

**Key words** - reliability data, competing risks, maintenance, Kolmogorov-Smirnov statistic, offshore installation

## 1 INTRODUCTION

Maintenance study requires the use of many modelling assumptions. We focus on issues related to reliability data bases (RDB's) interpretation, and in particular to dependent competing risk. Competing risk models are used to interpret data and a statistical test is used to find the appropriate competing risk model for RDB in question.

Modern RDB's may distinguish ten or more failure modes (ways of ending a service sojourn), often grouped in critical failures, degraded failures and incipient failures. The latter two are usually associated with preventive maintenance, whereas critical failures are of primary interest in risk and reliability calculations. A component exits a service sojourn due to the occurrence of one of its possible failure modes. The failure modes are competing each other to 'kill' the component, hence each failure mode censors the others.

Independent competing risks models have been studied for some time. By observing independent copies of competing risks we can estimate the subsurvival functions. Assuming independence of competing risks we can determine the underlying marginal distributions. In this case we have identifiability. The assumption of independence is questionable when failures are censored by preventive maintenance. The assumption of independence would imply that maintenance engineers take no account of the state of a component when taking the decision to preventively maintain. It is more reasonable to make a dependence assumption between the censoring processes.

A very good maintenance team will try to minimize the repair (replacement) cost over a long time interval. Since the repair (replacement) cost for a critical failure (corrective maintenance) is much higher than the cost for a degraded failure (preventive maintenance), the maintenance team will try to avoid a critical failure. Also, the maintenance team will try not to loose too much from the life time of the component because of the increase number of repairs (cost) over a long time interval. This entails that preventive maintenance should be highly correlated to failure. Ideally, the component is preventively maintained at time t if and only if it would have failed shortly after time t. This situation is captured in Random Signs Model developed by Cooke (1996) [3]: consider a component subject to right censoring, where X denotes the time at which a component would expire if not censored, then the event that the component's life be censored is independent of the age of X at which the component would expire, but given that the component is censored, the time at which it is censored may depend on X. Not every set of censored observation is consistent with a random signs model. Cooke (1996) proved that if the random signs model holds than the conditional subsurvival function for X dominates the conditional subsurvival function of preventive maintenance and they are equal for independent exponential model and conditionally independent model.

Cooke and Bedford [4] presented different dependent competing risk models with an application to pressure relief valves data from one Swedish nuclear station operating two identical reactors. Like most modern RDB's, this data base was designed to serve the interest of at least three type of engineers: the maintenance engineer interested in measuring and optimising maintenance performance, the design engineer interesting in optimising component performance and the risk analyst wishing to predict reliability of complex system in which the component operate. They showed that models for dependent competing risk enable the needs of these users to be better met. This involves selecting an appropriate competing risk model on the bases of empirical subsurvival functions. In [4] this selction was simply made graphically. An example is shown in Figure 1., where "alarm" and "unintended discovery" are events that maintenance personnel would try to avoid. There are 4 such events and 248 other events. The conditional subsurvival function of "alarm" and "unintended discovery" [CSSF1] dominates the conditional subsurvival function of "other" [CSSF2], hence a random signs model seems to describe data, but no evidence is given that this model or another fits data.

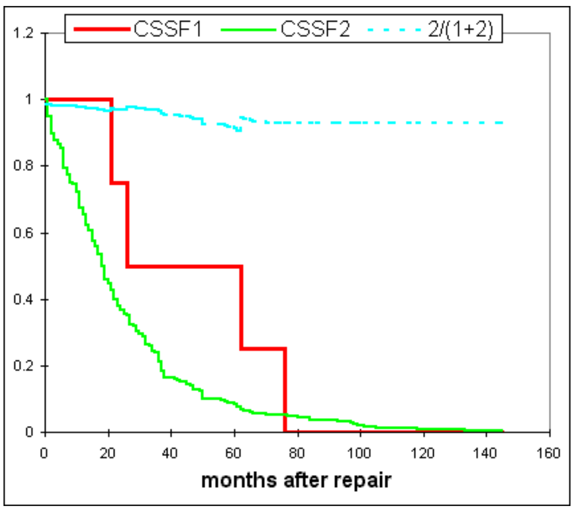


Figure 1. Graphical model interpretation of pressure relief valves data

In this paper, we will present the one sided Kolmogorov-Smirnov test for a two sample problem in order to test the exponential independent model against the alternative random signs model. Hence, we are going to test whether the conditional subsurvival function are coming from the same population against the alternative that the conditional subsurvival function of the censoring variable lies entirely below the conditional subsurvival of the censored variable. An algorithm how to calculate the one sided KS statistic is also given.

The performance of the probabilistic model we propose, is illustrated on the Gas Genenrator data used by Langseth [9]. This is a subset of Phase IV of the Gas Turbine dataset from the Offshore Reliability Database (OREDA [10]). We have 22 failures in this dataset, out of which 8 are classified as critical and 14 as degraded. The main results coming out from this dataset is that even for a small sample population we can say that the conditionally subsurvival functions are not from the same population and a random signs model is appropriate to interpret this data.

## 2 COMPETING RISK

In the competing risk approach we model the data as a renewal process, that is as a sequence of i.i.d. variables $Z_1$, $Z_2$,.... Each observable $Z$ is the minimum of two variables. The lifetime of the component is $X$: this is the lifetime that the component would reach if it were not preventively maintained. The preventive maintenance (PM) time of the component is $Y$: this is the time at which the component would be preventively maintained if it didn't fail first. Clearly,

$$Z \equiv [\min(X, Y), I(X < Y)].$$

(In fact, usually $X$ will be the minimum of several variables giving the time to failure by a particular failure mode: we shall just consider the case of one failure mode.)

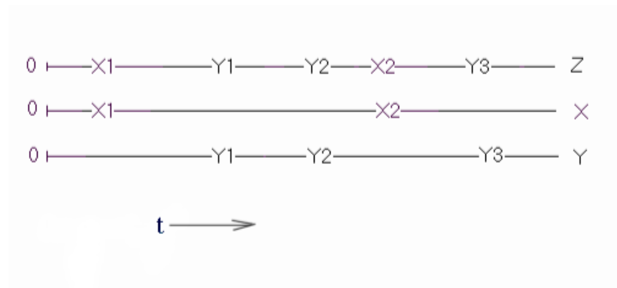


Figure 2. Graphical representation of competing risk data in calendar time

The observable data will allow us to estimate the *sub-survival functions*,

$$S_X^*(t) \equiv Pr\{X > t, X < Y\}$$

and

$$S_Y^*(t) \equiv Pr\{Y > t, Y < X\},$$

but not the true survivor functions of $X$ and $Y$. The conditional subsurvival function is the subsurvival function conditioned on the event that the failure mode in question is manifested. With continuity of $S_X^*(t)$ and $S_Y^*(t)$ at zero,

$$Pr\{X > t, X < Y | X < Y\} = S_X^*(t)/S_X^*(0),$$

$$Pr\{Y > t, Y < X | Y < X\} = S_Y^*(t)/S_Y^*(0).$$

Closely related to the notion of the subsurvival function is the probability of censoring beyond time $t$:

$$\begin{aligned} \Phi(t) &= Pr\{Y < X | X \wedge Y > t\} = \\ &= \frac{S_Y^*(t)}{S_Y^*(t) + S_X^*(t)}. \end{aligned}$$

From data we can only estimate the subsurvival functions. However, we want to have an estimate for the marginal survival function $S_X(t)$. Without additional assumptions on the joint distribution of $X$ and $Y$, assumptions about the interrelation of corrective and preventive maintenance, it is impossible to estimate the marginal. By making extra assumptions, it is possible to restrict oneself to a subclass of models in which this marginal is identifiable [1].

## 2.1 Independent Exponential Competing Risk

By observing independent copies of $Z \equiv [\min(X,Y), I(X < Y)]$ we can estimate the subsurvival functions. Assuming independence of $X$ and $Y$ we can determine uniquely the survival functions of $X$ and $Y$; in this case the survival functions of $X$ and $Y$ are said to be identifiable from censored data. A continuous subsurvival pair is always consistent with an independent model, but is not always consistent with an independent exponential model. We can derive a very sharp criterion for independence and exponentiality in terms of the subsurvival functions:

**Theorem 1** *Let $X$ and $Y$ be independent life variables, then any two of the following imply the others:*

- $S_X(t) = \exp(-\lambda t)$
- $S_Y(t) = \exp(-\gamma t)$
- $S_X^*(t) = \frac{\lambda}{\lambda+\gamma}\exp(-(\lambda+\gamma)t)$
- $S_Y^*(t) = \frac{\gamma}{\lambda+\gamma}\exp(-(\lambda+\gamma)t)$

*Remark:* If $X$ and $Y$ are independent exponential life variables with failure rates $\lambda$ and $\gamma$, then the conditional subsurvival functions of $X$ and $Y$ are equal and exponential distributed with failure rate $\lambda+\gamma$ and the probability of censoring beyond time $t$ is constant:

$$\begin{aligned} S_X^*(t)/S_X^*(0) &= S_Y^*(t)/S_Y^*(0) = \\ &= \exp(-(\lambda+\gamma)t) \end{aligned}$$

$$\Phi(t) = \frac{\gamma}{\lambda+\gamma}.$$

## 2.2 Random Signs Censoring

Perhaps the simplest dependent competing risk model which leads to identifiable marginal distributions without restricting their form is random sign censoring. Consider a component subject to right censoring, where $X$ denotes the time at which a component would expire if not censored. Suppose that the event that the life of the component be censored is independent of the age $X$ at which the component would expire, but given that the component is censored, the time at which it is censored may depend on $X$. This might arise, if a component emits warning before expiring; if the warning is seen then the component is taken out, thus censoring its life, otherwise it fails. The random signs model assumes that the probability of seeing the warning is independent of the component's age. This situation is captured in the following definition:

**Definition 1** *Let $X$ and $Y$ be life variables with $Y = X - W\delta$, where $0 < W < X$ is a random variable and $\delta$ is a random variable taking values $\{1, -1\}$, with $X$ and $\delta$ independent. The variable $Z \equiv [\min(X,Y), I(X < Y)]$ is called a random sign censoring of $X$ by $Y$.*

Note that

$$\begin{aligned} S_X^*(t) &= Pr\{X > t, \delta = -1\} = \\ &= Pr\{X > t\}Pr\{\delta = -1\} = \\ &= S_X(t)Pr\{Y > X\} = \\ &= S_X(t)S_X^*(0). \end{aligned}$$

Note also that $Pr\{Y > X\}$ and $S_X^*(t)$ can be estimated from observing independent copies of $Z$ and that under random signs censoring $S_X(t)$ is equal to the conditional subsurvival function of $X$.

Cooke [3] proved that the random signs model is consistent given subsurvival functions if and only if the conditional subsurvival function of $X$ is greater than the conditional subsurvival function of $Y$ for all $t > 0$. In this case the probability of censoring beyond time $t$ is maximum at the origin. This results suggests that if the random signs model holds then the independent exponential model is difficult to characterize data.

# 3 Gas Turbine Data - OREDA

A number of offshore platforms had been in operating in Europe for a significant length of time, and the Offshore Reliability Data (OREDA) handbook project was established to compile a comprehensive basis of reliability information from failure and repair records already existing in company files and records.

For the purpose of our study, we consider a subset of Phase IV of the Gas Turbine dataset from the Offshore Reliability Database (OREDA [10]), as used by Langseth [9]. These data involve 22 failures, out of which 8 are classified as critical and 14 as degraded. Degraded failures can be associated to a preventive maintenance action and critical failure to a corrective maintenance action.

The empirical subsurvival functions and conditional subsurvival functions are defined as [5]:

$$\widehat{S}_X^*(t) = \frac{number\ \ of\ \ X\ \ events\ \ after\ \ time\ \ t}{total\ \ number\ \ of\ \ events}$$

$$\widehat{S}_Y^*(t) = \frac{number\ \ of\ \ Y\ \ events\ \ after\ \ time\ \ t}{total\ \ number\ \ of\ \ events}$$

$$\widehat{CS}_X^*(t) = \frac{number\ \ of\ \ X\ \ events\ \ after\ \ time\ \ t}{total\ \ number\ \ of\ \ X\ \ events}$$

$$\widehat{CS}_Y^*(t) = \frac{number\ \ of\ \ Y\ \ events\ \ after\ \ time\ \ t}{total\ \ number\ \ of\ \ Y\ \ events}$$

which are estimators for subsurvival functions and conditional subsurvival functions.

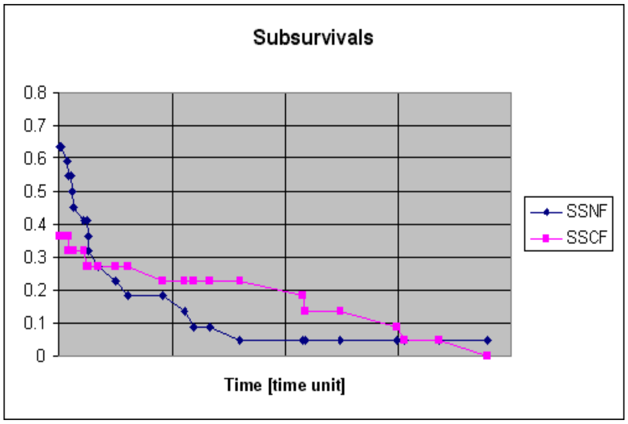


Figure 3. Empirical subsurvival functions

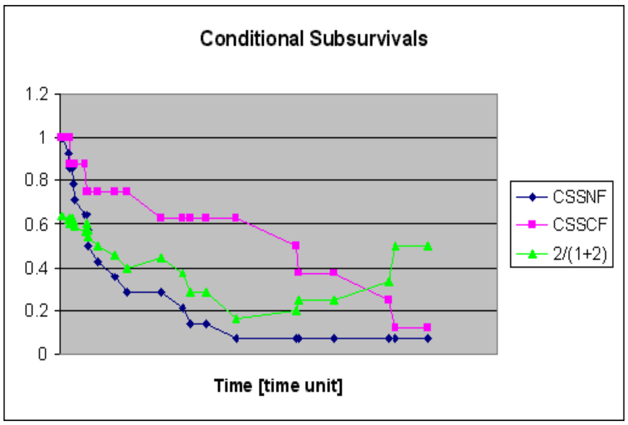


Figure 4. Conditional subsurvival functions and $\Phi(t)$ function

Figure 4 shows that the function $\Phi(t)$ $[\frac{2}{1+2}]$ is minimum at the origin and the conditional subsurvival function of critical failure dominates the conditional subsurvival function of degraded failure, as predicted by random signs model. Hence, an independent exponential model is not appropriate for this data. Further we will present a statistical test for testing if the independent exponential model fits this data.

## 4 Two-sample Kolmogorov-Smirnov Test

If an independent exponential model holds then the conditional subsurvival functions are equal and the probability of censoring beyond time $t$ is constant. Hence, we want to test if the empirical estimation of the conditional subsurvival functions are from the same population.

Our data consists of two independent random samples drawn independently from each of two population. Let $U$ and $V$ be random variables with survival functions $S_U(t)$ and $S_V(t)$ equal to the condtional subsurvial functions of $X$ respectively $Y$. From Gas Turbine Data we have two samples $U_1, U_2, ..., U_8$ of size $m = 8$, drawn from the $U$ population and $V_1, V_2, ..., V_{14}$ of size $n = 14$, drawn from the $V$ population.

The hypothesis of interest in the two-sample problem is that the two-samples are drawn from identical populations [8],

$$H_0 : S_U(t) = S_V(t) \quad for \quad all \quad t.$$

The one-sided Kolmogorov-Smirnov two sample test criteria, denoted by $D^+_{m,n}$ is the maximum difference between the empirical functions of $S_U(t)$ and $S_V(t)$:

$$D^+_{m,n} = \max[S^m_U(t) - S^n_V(t)].$$

Since here the directional differences are considered, $D^+_{m,n}$ is appropriate for a general one-sided alternative:

$$H_1 : S_U(t) \geq S_V(t) \quad for \quad all \quad t.$$

The null hypothesis $H_0$ is rejected at the significance level $\alpha$ if

$$D^+_{m,n} > d_\alpha,$$

where

$$Pr\{D^+_{m,n} > d_\alpha\} = \alpha.$$

The asymptotic distribution of $\sqrt{\frac{mn}{m+n}} D^+_{m,n}$ is [7]:

$$\lim_{m,n\to\infty} Pr\{D^+_{m,n} \leq d_\alpha\} = 1 - e^{-2d^2_\alpha}.$$

If the size of the samples are bigger than 50 then the asymptotic formula can be used to determine the significance level at which the null hypothesis is rejected, otherwise tables should be use. Further we will present an algorithm how to calculate the tail probability for small samples, necessary for programing implementation [6].

Let $U_{(1)}, U_{(2)}, ..., U_{(m)}$ and $V_{(1)}, V_{(2)}, ..., V_{(n)}$ be the order statics of the two samples of size $m = 8$ and $n = 14$ from continuous populations $S_U(t)$ and $S_V(t)$. To compute $Pr\{D^+_{m,n} \leq d_\alpha\}$, where $D^+_{m,n} = \max[S^m_U(t) - S^n_V(t)]$, we first arrange the combined sample of $m + n$ observation in increasing order of magnitude (Table 1).

| Sample | v | v | v | v | u |
|---|---|---|---|---|---|
| v | v | v | u | u | u |
| v | v | v | v | u | v |
| u | u | v | v | u | |

Table 1. Failure times of Gas Turbine data set

The arrangement can be plotted on a Cartesian coordinate system by a path which stars at the origin and moves one step up for a $u$ observation and one step to the right for a $v$ observation, ending at $(n, m)$. The observed values of $mS_U(t)^m$ and $nS_V(t)^n$ are the coordinates of all points $(i, j)$ on the path, where $i$ and $j$ are integers. The number $d_\alpha$ is the largest of the difference

$$\frac{i}{m} - \frac{j}{n} = \frac{ni - mj}{mn}.$$

The vertical distance from any point $(i, j)$ on the path to the line $nu - mv = 0$ and situated below it is $\max[j - \frac{ni}{m}]$. Hence, $nd_\alpha$ for the observed sample is the distance from the diagonal to that point on the path which is farthest from the diagonal line and is situated below it.

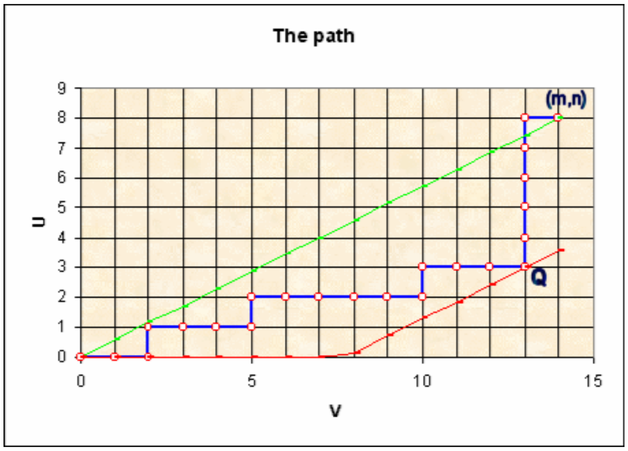


Figure 5. Cartesian representation of the combine sample

For our case the farthest point is $Q$ and $d_\alpha = 0.554$. The total number of arrangements of $mU$ and $nV$ r.v. is $C^m_{m+n}$, and under $H_0$ each of the corresponding paths is equally likely. The probability of an observable value of $D_{m,n}$ not less than $d_\alpha$ then is the number of paths which have points at a below distance from the diagonal not less than $nd_\alpha$, divided by $C^m_{m+n}$. We mark off a line at vertical distance $nd_\alpha$ from the diagonal, below it, as in Figure 5. Denote by $A(m,n)$ the number of paths from $(0,0)$ to $(m,n)$ which lie entirely above this line. Then

$$Pr\{D^+_{m,n} \leq d_\alpha\} = \frac{A(m,n)}{C^m_{m+n}}.$$

$A(i,j)$ at any intersection $(i,j)$ satisfies the recursion relation:

$$A(i,j) = A(i-1,j) + A(i,j-1),$$

with boundary conditions $A(0,j) = A(i,0) = 1$. Thus $A(i,j)$ is the sum of the numbers at the intersection where the previous point on the path could have been while it still was within the boundaries. Since $A(8,14) = 310,751$, we have:

$$Pr\{D^+_{8,14} \leq d_\alpha\} = 0.0282$$

Hence we reject the null hypothesis that the conditional subsurvival functions are coming from the same population at the significant level $\alpha = 0.0282$.

## 5 Conclusions

Figure 4 shows that the conditional subsurvival function for critical failures dominates the conditional subsurvival function for degraded failures, as predicted by random signs model.

The statistical test rejected the null hypothesis that the exponential independent model is appropriate for this data at the significance level $\alpha = 0.0282$, indicating that the random signs model might be indicated to interpret data. The algorithm of calculating the tail probability for small samples can be implemented also to other sets of data with a small number of events. We mention also the example of the pressure relief valves data from a Swedish nuclear station [4]. The Kolmogorov-Smirnov-test can be applied to test if the exponential independent model describes data. The null hypothesis (exponential independent model) is rejected at the significance level $\alpha = 0.0209$. It is worth mentioning that this test can yield important conclusions even if one or both of the competing risks are scarce. The test is powerful in spite of having only 4 "alarm" and "unintended discovery" events, because there are then a large number of events of competing risk. Thus, although the estimate of the conditinal subsurvival function of "alarm" and "unintended discovery" is very noisy, the estimate of the competing conditional subsurvival function is not.

Given the Glivenko-Cantelli theorem that the estimated distribution function converges with probability one to the real distribution function when the number of observation increase, this mean that the quality of estimating the parameters of the distribution is increasing but the population from which the sample is drawn remains the same. Hence, we can conclude that the conditional subsurvival functions are not coming from the same population and even for a small number of samples, the exponential independent model is not appropriate for this data.

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