

# Elliptical copulae

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**Abstract:** In this paper we construct a copula, that is, a distribution with uniform marginals. This copula is continuous and can realize any correlation value in  $(-1, 1)$ . It has linear regression and has the properties that partial correlation is equal to constant conditional correlation. This later property is important in Monte Carlo simulations. The new copula can be used in graphical models specifying dependence in high dimensional distributions such as Markov trees and vines.

**Keywords:** correlation, conditional correlation, conditional independence, partial correlations, tree dependence, copulae, vines.

## 1 Introduction

In modelling high dimensional distributions the problems encountered include

- (a) Determining whether a partially specified matrix can be extended to a correlation matrix;
- (b) Finding a convenient way of representing correlation matrices;
- (c) Choosing an unique joint distribution to realize a correlation matrix.

(a) is so called matrix completion which is receiving attention at the moment (Laurent [9]). To tackle these problems the graphical models called vines were introduced by (Cooke [3]). A vine is a set of trees such that the edges of the tree  $T_i$  are nodes of the tree  $T_{i+1}$  and all trees have the maximum number of edges. A vine is regular if two edges of  $T_i$  are joined by an edge of  $T_{i+1}$  only if these edges share a common node in  $T_i$ . Partial correlations, defined in (Yule and Kendall [13]), can be assigned to the edges of the regular vine such that conditioning and conditioned sets of the vine and partial correlations are equal ( for the details we refer readers to Bedford and Cooke [1]). There are  $\binom{n}{2}$  edges in the regular vine and there is a bijection from  $(-1, 1)^{\binom{n}{2}}$  to the set of full rank correlation matrices ([1]). Using regular vines with partial correlations we thus determine the entire correlation matrix in convenient way (b). Using regular vines with conditional correlations we can determine a convenient sampling routines (c). In general, however, partial and conditional correlations are not equal. For popular copulas such as the diagonal band (Cooke and Waij [4]) and the minimum information copulae with given correlation (Meeuwissen and Bedford [10]), when conditional rank correlation is held constant, the partial correlation and mean conditional product moment correlation are approximately equal (Kurowicka and Cooke [7]). This approximation, however, deteriorates as the correlations become more extreme. For the well known Fréchet copulae the partial and constant conditional correlations are equal but these copulae are not very useful from the application point of view ([7]). In (Kurowicka and Cooke [8]) it is shown how regular vines can be applied to the completion problem (a). For other

copulae and their properties we refer to (Dall'Agilo, Kotz and Salinetti [5]) and (Nelsen [12]). In this article we present the new copula for which partial and constant conditional correlations are equal. In constructing this new copula the properties of elliptically contoured and rotationally invariant random vectors were used (see Harding [2] and Misiewicz [11]). These copula present a striking companion with copulae previously used in Monte Carlo simulation codes (Unicorn and PREP/SPOP [6])

This paper is organized as follows. In Section 2 the uniform distribution on the sphere and its properties is presented. In Section 3 the copula is given. The properties of this function are shown. In Section 4 after introducing definitions of the partial and conditional correlations the equality of partial and constant conditional correlations for the new copula is proven. Section 5 contains conclusions.

## 2 Uniform distribution on the sphere and its properties

Let  $X = (X_1, X_2, X_3)$  have the uniform distribution on the sphere with the radius  $r$ ,  $S_2(r) \subseteq R^3$  where sphere in  $R^n$  is defined as

$$S^{n-1}(r) = \{x \in R^n \mid \sum_{k=1}^n x_k^2 = r^2\}.$$

We can see that for every  $t \in [-r, r]$

$$\begin{aligned} P(X_1 < t) = P(X_2 < t) = P(X_3 < t) &= \frac{2\pi}{4\pi r^2} \int_{-r}^t \sqrt{r^2 - x^2} \sqrt{1 + \left[\frac{d}{dx} \sqrt{r^2 - x^2}\right]^2} dx \\ &= \frac{1}{2} + \frac{t}{2r}, \end{aligned}$$

which means that each of the variables  $X_1, X_2$  and  $X_3$  has a uniform distribution on the interval  $[-r, r]$ .

Consider now a linear operator  $A : R^3 \rightarrow R^3$  represented by the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Since the random vector  $X$  is rotationally invariant, the random vector

$$W = (W_1, W_2, W_3) = AX^T$$

is elliptically contoured.

Harding proved (see [2]) that every elliptically contoured<sup>1</sup> random vector on  $R^n, n \geq 2$  has the linear regression property if it has second moment. This means in particular that for every  $j \neq k, j, k \in \{1, 2, 3\}$  there exists  $a_{jk}$  such that

$$E(W_j | W_k) = a_{jk} W_k.$$

The numbers  $a_{jk}$  can be calculated directly:

$$W_j = a_{j1} X_1 + a_{j2} X_2 + a_{j3} X_3.$$

<sup>1</sup> A random vector  $X = (X_1, X_2, \dots, X_n)$  is *elliptically contoured* if it is pseudo isotropic with a function  $c : R^n \rightarrow [0, \infty)$  defined by an inner product on  $R^n$ ; i.e there exists a symmetric positive definite  $n \times n$  matrix  $\Sigma$  such that

$$c(\xi)^2 = \langle \xi, \Sigma \xi \rangle, \forall \xi \in R^n.$$

If  $\Sigma = I$  then the vector  $X$  is called *rotationally invariant*.

Since  $E(X_k) = 0$ ,  $E(X_j X_k) = 0$  for  $k \neq j$  and  $\text{Var}(X_k) = \frac{1}{2r} \int_{-r}^r x^2 dx = \frac{1}{3} r^2$  thus  $W_1, W_2, W_3$  have expectations 0 and

$$\text{Var}(W_j) = E((a_{j1}X_1 + a_{j2}X_2 + a_{j3}X_3)^2) = \frac{1}{3} r^2 (a_{j1}^2 + a_{j2}^2 + a_{j3}^2).$$

According to Harding's result the conditional expectation  $E(W_j|W_k)$  coincides with the orthogonal projection of vector  $W_j$  onto  $W_k$ . Then we can calculate for  $k \neq j$

$$\begin{aligned} E(W_j W_k) &= E((a_{j1}X_1 + a_{j2}X_2 + a_{j3}X_3)(a_{k1}X_1 + a_{k2}X_2 + a_{k3}X_3)) \\ &= \frac{1}{3} r^2 (a_{j1}a_{k1} + a_{j2}a_{k2} + a_{j3}a_{k3}). \end{aligned}$$

Finally we get

$$E(W_j|W_k) = \frac{E(W_j W_k)}{\text{Var}W_k} W_k = \frac{a_{j1}a_{k1} + a_{j2}a_{k2} + a_{j3}a_{k3}}{a_{k1}^2 + a_{k2}^2 + a_{k3}^2} W_k. \quad (1)$$

Notice now that random variables  $W_1, W_2, W_3$  have uniform distributions, which with appropriate choice of  $A$  will be uniform distributions on  $[-r, r]$ . We use a very helpful property of rotationally invariant random vectors, namely:

*if  $Y \in R^n$  is rotationally invariant and  $a \in R^n$  then the distribution of  $a_1 Y_1 + \dots + a_n Y_n$  is the same as the distribution of  $\|a\|_2 Y_1$ , where  $\|a\|_2$  is the Euclidean norm of the vector  $a$ .*

Now we can write the following:

$$\begin{aligned} P(W_k < t) &= P(a_{k1}X_1 + a_{k2}X_2 + a_{k3}X_3 < t) \\ &= P\left(X_1 < \frac{t}{\sqrt{a_{k1}^2 + a_{k2}^2 + a_{k3}^2}}\right) = \frac{1}{2} + \frac{t}{2r\sqrt{a_{k1}^2 + a_{k2}^2 + a_{k3}^2}} \end{aligned}$$

for  $t \in [-r\sqrt{a_{k1}^2 + a_{k2}^2 + a_{k3}^2}, r\sqrt{a_{k1}^2 + a_{k2}^2 + a_{k3}^2}]$ .

It is enough to assume that for all  $k = 1, 2, 3$

$$a_{k1}^2 + a_{k2}^2 + a_{k3}^2 = 1 \quad (2)$$

to have uniform distribution on  $[-r, r]$ .

### 3 The elliptical copulae

Taking the projection of the uniform distribution on the sphere  $S^2(\frac{1}{2})$  on a plane  $(X, Y)$  we can construct copulae.

The area of surface given in functional form  $z = g(x, y)$  above area  $D \subseteq R^2$  can be calculated as:

$$\int \int_D \sqrt{1 + \left(\frac{d}{dx}g(x, y)\right)^2 + \left(\frac{d}{dy}g(x, y)\right)^2} dx dy. \quad (3)$$

Using (3) for the sphere with radius  $\frac{1}{2}$ , hence for function  $g(x, y) = 2\sqrt{\frac{1}{4} - x^2 - y^2}$ , and dividing by the whole area of surface of  $S^2(\frac{1}{2})$ , which is equal to  $\pi$ , we obtain

$$\int \int_{x^2 + y^2 < \frac{1}{4}} \frac{1}{\pi \sqrt{\frac{1}{4} - x^2 - y^2}} dx dy = 1.$$

Hence function

$$f(x, y) = \begin{cases} \frac{1}{\pi\sqrt{\frac{1}{4}-x^2-y^2}} & (x, y) \in B \\ 0 & (x, y) \notin B \end{cases} \quad (4)$$

where  $B = \{(x, y) | x^2 + y^2 < \frac{1}{4}\}$  is a density function in  $R^2$ .

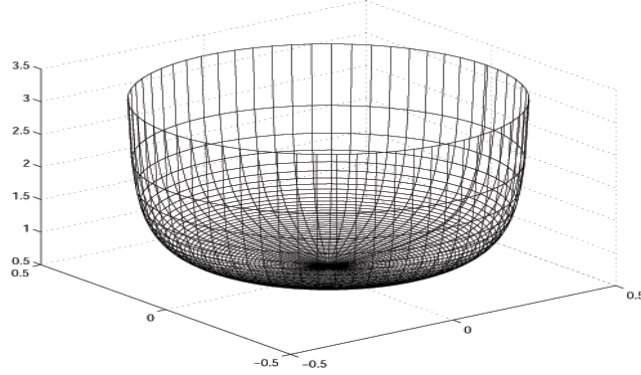


Figure 5.1. The density function  $f$ .

We can easily check that the function  $f$  has uniform marginals.

For  $x \in [-\frac{1}{2}, \frac{1}{2}]$

$$f_X(x) = \int_{-\sqrt{\frac{1}{4}-x^2}}^{\sqrt{\frac{1}{4}-x^2}} \frac{1}{\pi\sqrt{\frac{1}{4}-x^2-y^2}} dy = \frac{1}{\pi} \arcsin \frac{y}{\sqrt{\frac{1}{4}-x^2}} \Bigg|_{-\sqrt{\frac{1}{4}-x^2}}^{\sqrt{\frac{1}{4}-x^2}} = 1.$$

To construct family of copulae which can represent all correlations  $\rho \in (-1, 1)$  we consider linear transformation represented by a matrix

$$A = \begin{bmatrix} \cos \varphi & \sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5)$$

where

$$\varphi \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right).$$

This transformation satisfies condition (2).

Let  $(x', y', z') \in S^2(\frac{1}{2})$ . Applying transformation (5) we get points  $(x, y, z)$  from ellipsoid

$$\begin{aligned} x &= \cos(\varphi)x' + \sin(\varphi)y' \\ y &= \sin(\varphi)x' + \cos(\varphi)y' \\ z &= z'. \end{aligned}$$

We now find the equation of this ellipsoid. Since

$$x'^2 + y'^2 + z'^2 = \frac{1}{4}$$

and

$$\begin{aligned} x' &= \frac{\cos(\varphi)x - \sin(\varphi)y}{\cos(2\varphi)} \\ y' &= \frac{-\sin(\varphi)x + \cos(\varphi)y}{\cos(2\varphi)} \\ z' &= z. \end{aligned}$$

then the ellipsoid is given by

$$x^2 + y^2 - 2 \sin(2\varphi)xy + (\cos^2(2\varphi))z^2 = \frac{\cos^2(2\varphi)}{4}.$$

This can be also written as

$$x^2 + \left( \frac{y - \sin(2\varphi)x}{\cos(2\varphi)} \right)^2 + z^2 = \frac{1}{4}.$$

For all points from ellipse

$$x^2 + \left( \frac{y - \sin(2\varphi)x}{\cos(2\varphi)} \right)^2 < \frac{1}{4}$$

density function is given by following formula

$$f_\varphi(x, y) = \frac{1}{\pi \cos(2\varphi)} \frac{1}{\sqrt{\frac{1}{4} - x^2 - \left( \frac{y - 2 \sin(2\varphi)x}{\cos(2\varphi)} \right)^2}}. \quad (6)$$

The distribution with density function given by formula (6) has uniform marginals so this is a copula. This copula depends on parameter  $\varphi$ . We will write  $C_\varphi$ . For two variables joined by copula  $C_\varphi$  on  $[-\frac{1}{2}, \frac{1}{2}]^2$  the following holds:

**Proposition 3.1** *If  $X, Y$  are joined by the copula  $C_\varphi$ , then*

$$\rho_{XY} = \sin(2\varphi). \quad (7)$$

**Proof.** We get

$$\rho_{XY} = \frac{E(XY)}{\sigma_X \sigma_Y} = \frac{E(XE(Y|X))}{\sigma_X^2}.$$

By (1)

$$\begin{aligned} E(Y|X) &= 2 \cos(\varphi) \sin(\varphi)X \\ &= \sin(2\varphi)X \end{aligned}$$

hence

$$\rho_{XY} = \frac{\sin(2\varphi)E(X^2)}{\sigma_X^2} = \sin(2\varphi)$$

which concludes the proof.  $\square$

We can see that the function  $f$  given by (4) and presented on the Figure 5.1 is a density function of the copula  $C_0$ . Correlation between variables  $X$  and  $Y$  joined by the copula  $C_0$  is equal to 0.

It is more convenient to start with the assumption that elliptical copula depends on correlation  $\rho \in (-1, 1)$ . The parameter  $\varphi$  can be recovered as follows

$$\varphi = \frac{\arcsin(\rho)}{2}.$$

We will consider from now on the copula  $C$  with given correlation  $\rho$  and write  $C_\rho$ .

The density function of the elliptical copulae with given correlation  $\rho \in (-1, 1)$  is

$$f_\rho(x, y) = \begin{cases} \frac{1}{\pi \sqrt{1-\rho^2}} \frac{1}{\sqrt{\frac{1}{4} - \frac{x^2+y^2-2\rho xy}{1-\rho^2}}} & (x, y) \in B \\ 0 & (x, y) \notin B \end{cases}$$

where

$$B = \left\{ (x, y) \mid x^2 + \left( \frac{y - \rho x}{\sqrt{1 - \rho^2}} \right)^2 < \frac{1}{4} \right\}$$

The figures below show graphs of density function of the copula  $C$  with correlation  $\rho = 0.8$  and projection of this density on the plane.

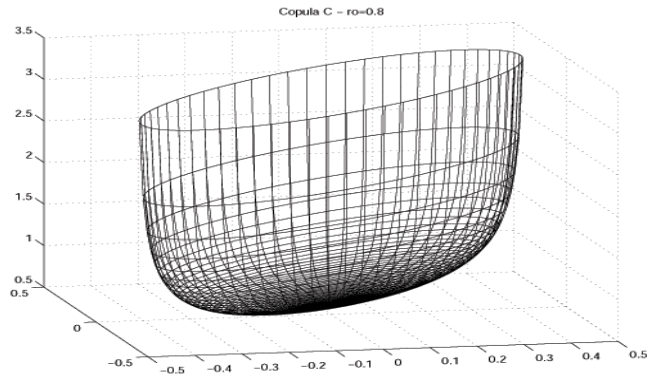


Figure 5.2. A density function of the copula  $C$  with correlation  $\rho = 0.8$ .

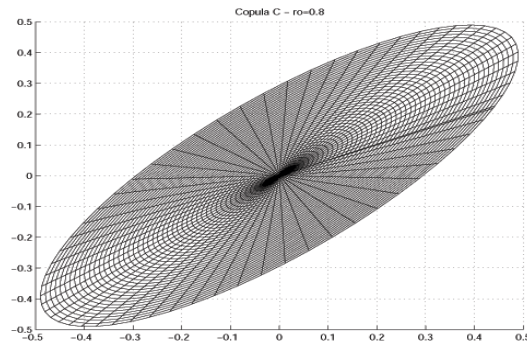


Figure 5.3. Projecting of density function of the copula  $C$  with correlation  $\rho = 0.8$  on the plane.

For comparison we present below graphs of the density functions for diagonal band and minimum information copulae with correlation 0.8.

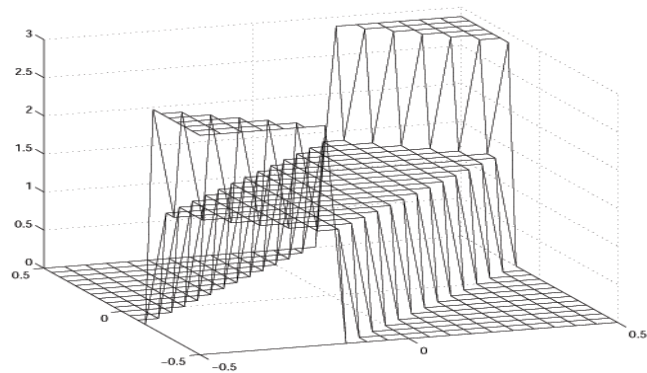


Figure 5.4. The diagonal band distribution with correlation 0.8.

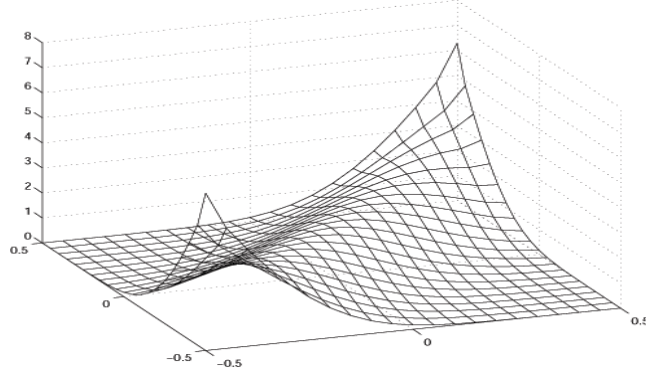


Figure 5.5. The minimum information distribution with correlation 0.8.

We show now some properties of the copula  $C_\rho$ .

**Theorem 3.1** *If  $X, Y$  joined by the copula  $C_\rho$  then*

- (a)  $E(Y|X) = \rho X$ ,
- (b)  $\text{Var}(Y|X) = \frac{1}{2}(1 - \rho^2) \left(\frac{1}{4} - X^2\right)$ .

**Proof.** By (1) and (7) the copula  $C_\rho$  has linear regression with coefficient equal to correlation hence (a) holds. We verify condition (b)

$$\begin{aligned}
 \text{Var}(Y|X) &= \int_{\rho X - \sqrt{1-\rho^2}\sqrt{\frac{1}{4}-X^2}}^{\rho X + \sqrt{1-\rho^2}\sqrt{\frac{1}{4}-X^2}} (y - \rho X)^2 \frac{1}{\pi\sqrt{1-\rho^2}} \frac{1}{\sqrt{\frac{1}{4} - \frac{X^2+y^2-2\rho Xy}{1-\rho^2}}} dy \\
 &= \frac{1}{\pi\sqrt{1-\rho^2}} \int_{\rho X - \sqrt{1-\rho^2}\sqrt{\frac{1}{4}-X^2}}^{\rho X + \sqrt{1-\rho^2}\sqrt{\frac{1}{4}-X^2}} (y - \rho X)^2 \\
 &\quad \frac{1}{\sqrt{\frac{1}{4} - X^2}} \frac{1}{\sqrt{1 - \left(\frac{y-\rho X}{\sqrt{1-\rho^2}\sqrt{\frac{1}{4}-X^2}}\right)^2}} \\
 &= \frac{1}{\pi}(1-\rho^2)\left(\frac{1}{4} - X^2\right) \int_{-1}^1 t^2 \frac{1}{\sqrt{1-t^2}} dt \\
 &= \frac{1}{2}(1-\rho^2)\left(\frac{1}{4} - X^2\right)
 \end{aligned}$$

which concludes the proof.  $\square$

## 4 Partial and conditional correlations

Let us consider variables  $X_i$  with zero mean and standard deviations  $\sigma_i$ ,  $i = 1, \dots, n$ . Let the numbers  $b_{12;3,\dots,n}, \dots, b_{1n;2,\dots,n-1}$  minimize

$$E\left((X_1 - b_{12;3,\dots,n}X_2 - \dots - b_{1n;2,\dots,n-1}X_n)^2\right);$$

then the *partial correlations* are defined as (Yule and Kendall [13]):

$$\rho_{12;3,\dots,n} = \text{sgn}(b_{12;3,\dots,n}) (b_{12;3,\dots,n} b_{21;3,\dots,n})^{\frac{1}{2}}, \text{ etc.}$$

Partial correlations can be computed from correlations with the following recursive formula:

$$\rho_{12;3,\dots,n} = \frac{\rho_{12;3,\dots,n-1} - \rho_{1n;2,\dots,n-1} \cdot \rho_{2n;1,3,\dots,n-1}}{\sqrt{1 - \rho_{1n;2,\dots,n-1}^2} \sqrt{1 - \rho_{2n;1,3,\dots,n-1}^2}}. \quad (8)$$

The *conditional correlation* of  $Z$  and  $Y$  given  $X$

$$\rho_{YZ|X} = \rho(Y|X, Z|X)$$

is the product moment correlation computed with the conditional distribution given  $X$ . In general this depends on the value of  $X$ , but it may be constant.

We are interested in finding the relationship between partial  $\rho_{YZ;X}$  and conditional correlations  $\rho_{ZY|X}$  if variables  $X$  and  $Y$  are joined by the copula  $C_{\rho_{XY}}$  and  $X$ ,  $Z$  are joined by the copula  $C_{\rho_{XZ}}$ .

It is shown in (Kurowicka and Cooke [7]) that the linear regression property leads to equality of partial and conditional correlations in the case of conditional independence. We present now some numerical results prepared in Matlab 5.3. We assume variables  $X$  and  $Y$  are joined by the copula  $C_{\rho_{XY}}$  and  $X$ ,  $Z$  are joined by the copula  $C_{\rho_{XZ}}$  and  $Y$  and  $Z$  conditionally independent given  $X$ .

Stipulated		Computed	
$\rho_{XY}$	$\rho_{XZ}$	$\rho_{YZ}$	$\rho_{YZ;X}$
0	0	-7.74e-19	-7.74e-19
0.4	0.8	0.3199	-0.0002
0.2	-0.9	-0.1799	0.0002
-0.8	0.7	-0.5598	0.0005
0.9	-0.9	-0.8097	0.0017

Table 1: Numerical results for conditional independence.

**Theorem 4.1** *Let  $X, Y, Z$  be uniform on  $[-\frac{1}{2}, \frac{1}{2}]$  and suppose*

(a)  $X, Y$  are joined by  $C_{\rho_{XY}}$ ,

(b)  $X, Z$  are joined by  $C_{\rho_{XZ}}$ ,

(c)  $\rho_{YZ|X} = \rho$

then

$$\rho_{YZ;X} = \rho.$$

**Proof.** By Theorem 3.1

$$\begin{aligned} E(Y|X) &= \rho_{XY}X, \\ E(Z|X) &= \rho_{XZ}X. \end{aligned}$$

The partial correlation  $\rho_{YZ;X}$  can be calculated in the following way

$$\rho_{YZ;X} = \frac{\rho_{ZY} - \rho_{XY}\rho_{XZ}}{\sqrt{(1 - \rho_{XY}^2)(1 - \rho_{XZ}^2)}}.$$



We also get

$$\rho = \rho_{YZ|X} = \frac{E(YZ|X) - E(Y|X)E(Z|X)}{\sigma_{Y|X}\sigma_{Z|X}} = \frac{E(YZ|X) - \rho_{XY}\rho_{XZ}X^2}{\sigma_{Y|X}\sigma_{Z|X}}.$$

Hence

$$E(YZ|X) = \rho\sigma_{Y|X}\sigma_{Z|X} + \rho_{XY}\rho_{XZ}X^2.$$

Since

$$\rho_{ZY} = \frac{E(E(YZ|X))}{\sigma_X^2}$$

then

$$\rho_{YZ;X} = \frac{\rho E(\sigma_{Y|X}\sigma_{Z|X})}{\sigma_X^2 \sqrt{(1 - \rho_{XY}^2)(1 - \rho_{XZ}^2)}}.$$

Since by Theorem 3.1

$$\sigma_{Y|X} = \sqrt{\frac{1}{2}(1 - \rho_{XY}^2)(\frac{1}{4} - X^2)}, \quad \sigma_{Z|X} = \sqrt{\frac{1}{2}(1 - \rho_{XZ}^2)(\frac{1}{4} - X^2)}$$

then

$$\begin{aligned} \rho_{YZ;X} &= \frac{\rho E\left(\sqrt{\frac{1}{2}(1 - \rho_{XY}^2)(\frac{1}{4} - X^2)}\sqrt{\frac{1}{2}(1 - \rho_{XZ}^2)(\frac{1}{4} - X^2)}\right)}{\sigma_X^2 \sqrt{(1 - \rho_{XY}^2)(1 - \rho_{XZ}^2)}} \\ &= \frac{r\frac{1}{2}E(\frac{1}{4} - X^2)}{\sigma_X^2} = \frac{r\frac{1}{2}(\frac{1}{4} - \frac{1}{12})}{\frac{1}{12}} = \rho. \quad \square \end{aligned}$$

## 5 Conclusions

1. Elliptical copulae are continuous and can realize all correlation values  $\rho \in (-1, 1)$ .
2. This copula has linear regression and for variables joined by this copula we showed that partial and constant conditional correlations are equal.
3. Similar properties characterize normal and Fréchet distribution.
4. Combining elliptical copulae with graphical model called vines, presents attractive way of representing high dimensional distribution and can be used in direct sampling procedures.

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