Computing key parameters of continued fraction dynamical systems

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The classical continued fraction expansion of a real number x returns a string of integers $a_0, a_1, \ldots$, with $a_j > 0$ for $j > 0$;

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}.$$

Natural questions include the following:

1. What is the typical distribution of 1's, 2's, and so on among the a_k?
2. Conditional information, such as that $a_k = N$, influences the likely distribution of a_{k+1}, and, to a lesser extent, the likely distribution of a_{k+j}. How fast does this influence decay as j tends to infinity?

Many such questions boil down to questions about the spectrum of the transfer operator L acting on 'the' space V of 'reasonable' functions on $[0,1]$ for the dynamical system given by the state space $E = [0,1)\backslash\mathbb{Q}$ and the step function $T : E \to E$, with

$$Tx = \frac{1}{x} - \lfloor\frac{1}{x}\rfloor.$$

The naive optimist would think that linear operators are given by matrices, that the polynomials $(t-1)^n$ form a basis for V, and that all we need to do is just get the coefficients of the matrix for L and extract eigenvalues and eigenvectors.

And he'd be right. One can get away with this, and justify it, and compute several key parameters to high precision in this fashion. For instance, the 'influence' described above decays like λ^j, where

$$\lambda = 0.30366\ 30028\ 98732\ 65859\ 74481\ 21902.$$

The idea carries over nicely to variants of the classical continued fraction algorithm, including variants for which no closed form answers to these questions are known. This last is the fruit of joint work with Dajani, Kraaikamp, and Masarotto.