

DELFT UNIVERSITY OF TECHNOLOGY

REPORT 03-05

ON A RAYLEIGH WAVE EQUATION WITH BOUNDARY DAMPING

DARMAWIJOYO, W.T. VAN HORSSSEN, AND PH. CLÉMENT

ISSN 1389-6520

Reports of the Department of Applied Mathematical Analysis

Delft 2003

Copyright © 2003 by Department of Applied Mathematical Analysis, Delft, The Netherlands.

No part of the Journal may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise, without the prior written permission from Department of Applied Mathematical Analysis, Delft University of Technology, The Netherlands.

Abstract

In this paper an initial-boundary value problem for a weakly nonlinear string (or wave) equation with non-classical boundary conditions is considered. One end of the string is assumed to be fixed and the other end of the string is attached to a dashpot system, where the damping generated by the dashpot is assumed to be small. This problem can be regarded as a simple model describing oscillations of flexible structures such as overhead transmission lines in a windfield. An asymptotic theory for a class of initial-boundary value problems for nonlinear wave equations is presented. It will be shown that the problems considered are well-posed for all time t . A multiple time-scales perturbation method in combination with the method of characteristics will be used to construct asymptotic approximations of the solution. It will also be shown that all solutions tend to zero for a sufficiently large value of the damping parameter. For smaller values of the damping parameter it will be shown how the string-system eventually will oscillate. Some numerical results are also presented in this paper.

Key words: Wave equation, galloping, boundary damping, asymptotics, two-timescales perturbation method.

AMS subject classifications: 35A05, 35B20, 35B40, 35L05, 74H10, 74H45.

1 Introduction.

There are a number of examples of flexible structures (such as suspension bridges, overhead transmission lines, dynamically loaded helical springs) that are subjected to oscillations due to different causes. Simple models which describe these oscillations can be expressed in initial-boundary value problems for wave equations like in [1, 2, 3, 4, 5, 6, 11, 19] or for beam equations like in [7, 8, 9, 10, 18]. Simple models which describe these oscillations can involve linear or nonlinear second and fourth order partial differential equations with classical or non-classical boundary conditions. Some of these problems have been studied in [1, 2, 3, 4, 5, 6, 7, 8] using a two-timescales perturbation method or a Galerkin-averaging method to construct approximations.

In some flexible structures (such as an overhead transmission line or a cable of a suspension bridge) various types of wind-induced mechanical vibrations can occur. Vortex shedding for instance causes usually high frequency oscillations with small amplitudes, whereas low frequency vibrations with large amplitudes can be caused by flow-induced oscillations (galloping) of cables on which ice or snow has accreted. These vibrations can give rise to for instance material fatigue. To suppress these oscillations various types of dampers have been applied in practice (see [18, 19]).

In most cases simple, classical boundary conditions are applied (such as in [1, 4, 5, 6, 7, 8, 9]) to construct approximations of the oscillations. More complicated, non-classical boundary conditions (see for instance [2, 3, 10, 11, 12, 13, 14]) have been considered only for linear partial differential equations. For nonlinear wave equations with boundary damping the approximations have been obtained only numerically (see for instance [19]). In [19] it has been shown that for large values of the damping parameter the solutions tend to zero. It is, however, not clear how and for what values of the damping parameter the solutions will tend to zero (or not). In this paper we will study an initial-boundary value problem for a weakly nonlinear partial differential equation for which one of the boundary conditions is of non-classical type. It will also be shown in this paper that the use of boundary damping

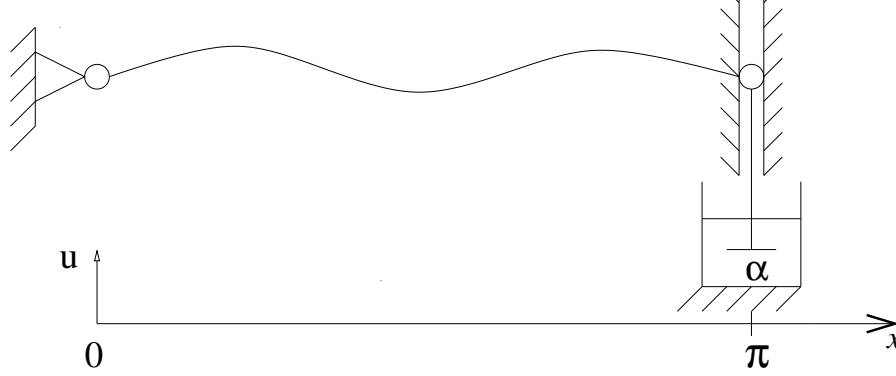


Figure 1. A simple model of an aero-elastic oscillator.

can be used effectively to suppress the oscillation-amplitudes. Asymptotic approximations of the solution will be constructed. In fact we will consider the nonlinear vibrations of a string which is fixed at $x = 0$ and is attached to a dashpot system at $x = \pi$ (see also Figure 1). This problem can be considered as a simple model to describe wind-induced vibrations of an overhead transmission line or a bridge (see [6, 7]). To our knowledge the use of boundary damping and the explicit construction of approximations of oscillations (which are described by a nonlinear PDE) have not yet been investigated deeply. The aim of this paper is also to give a contribution to the foundations of the asymptotic methods for weakly nonlinear hyperbolic partial differential equations with boundary damping. The outline of this paper is as follows. In section 2 of this paper a simple model of the galloping oscillations of overhead transmission lines will be discussed briefly. The following initial - boundary value problem for the function $u(x, t)$ will be obtained:

$$u_{tt} - u_{xx} = \epsilon \left(u_t - \frac{1}{3} u_t^3 \right), \quad 0 < x < \pi, \quad t > 0, \quad (1)$$

$$u(0, t) = 0, \quad t \geq 0, \quad (2)$$

$$u_x(\pi, t) = -\epsilon \alpha u_t(\pi, t), \quad t \geq 0, \quad (3)$$

$$u(x, 0) = \phi(x), \quad 0 < x < \pi, \quad (4)$$

$$u_t(x, 0) = \psi(x), \quad 0 < x < \pi, \quad (5)$$

where ϵ is a small parameter, and where α is a non-negative constant. For Dirichlet boundary conditions a similar problem has been considered in [4, 6], and in a suitable Banach space it has been shown in [4, 6] that a unique solution exists on a time-scale of order $\frac{1}{\epsilon}$. In section 3 of this paper it will be shown for the initial-boundary value problem (1) - (5) (and for more general problems) that global existence of the solution (that is, the solution exists for all time) can be obtained in a suitable Hilbert space. In section 4 of this paper a two-time-scales perturbation method in combination with the method of characteristic coordinates will be used to construct formal approximations of the solution of the initial-boundary value problem (1) - (5). It will also be shown in section 4 that for $\alpha \geq \frac{\pi}{2}$ all solutions tend to zero (up to $O(\epsilon)$), and that for $0 < \alpha < \frac{\pi}{2}$ the solutions (depending on the initial values) tend to bounded functions. In section 5 the asymptotic validity of the constructed approximations is proved. It will turn out in section 4 that only for rather simple initial values (for instance for monochromatic initial values) approximations can be constructed explicitly, and that for more complicated initial values numerical calculation have to be performed. For that reason some numerical results are presented in section 6. Finally in section 7 of this paper some remarks will be made and some conclusions will be drawn.

In this section we will briefly derive a model which describes the galloping oscillations of an overhead conductor line in a windfield. For complete details the reader is referred to [6]. As is well-known galloping is a large amplitude, low frequency, almost purely vertical oscillation of a single conductor line on which ice or snow has accreted. Depending on how the ice or snow has accreted the cable can become aerodynamically unstable. Since the frequency is low a quasi-steady aerodynamic approach can be used. It will be assumed that the conductor line (string) is inextensible and of length l .

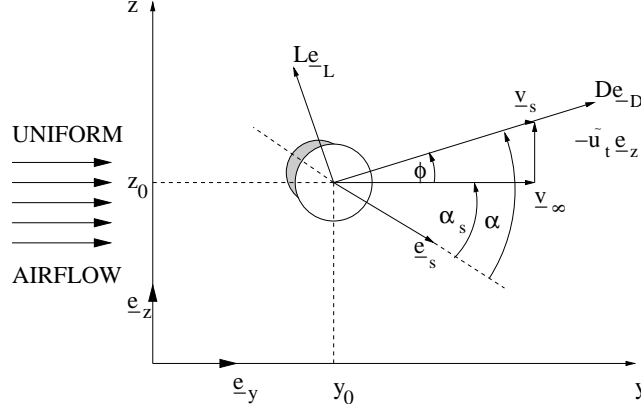


Figure 2. A simple model of an aero-elastic oscillator.

A typical cross-section is depicted in Figure 2. It is assumed that ρ (the mass-density of the string per unit length), T (the tension in the string), $\tilde{\alpha}$ (the damping coefficient of the dashpot) are all positive constants. Moreover, we only consider the vertical displacement $\tilde{u}(x, t)$ of the string in z -direction, where x is the place along the string, and t is time. We neglect internal damping and bending stiffness of the string and consider the weight W of the string per unit length to be constant ($W = \rho g$, g is the gravitational acceleration). We consider a uniform windflow, which causes nonlinear drag and lift forces (F_D , F_L) to act on the structure per unit length. Here the drag force De_D has the direction of the virtual wind velocity $\underline{v}_s = \underline{v}_\infty - \tilde{u}_t \underline{e}_z$, where \underline{v}_∞ is the windflow velocity in y direction, and the lift force Le_L has a direction perpendicular to the virtual wind velocity \underline{v}_s . The angle α is given by

$$\alpha = \alpha_s + \phi = \alpha_s + \arctan\left(-\frac{\tilde{u}_t}{v_\infty}\right), \quad (6)$$

where α_s is the static angle of attack which is assumed to be constant and identical for all cross-sections, and where $\phi \leq \pi$. The influence of the geometrical nonlinearities of the string are assumed to be small (compared to the windforces) and will be neglected in this paper (see [6]). The equation describing the vertical displacement of the string is:

$$\rho \tilde{u}_{tt} - T \tilde{u}_{xx} = -\rho g + F_D + F_L, \quad (7)$$

with boundary conditions

$$\tilde{u}(0, t) = 0 \quad \text{and} \quad T \tilde{u}_x(l, t) + \tilde{\alpha} \tilde{u}_t(l, t) = 0, \quad t \geq 0. \quad (8)$$

In [6] it has been shown that $F_D + F_L$ can be approximated by

$$\frac{\rho_a d v_\infty^2}{2} \left(a_0 + \frac{a_1}{v_\infty} \tilde{u}_t + \frac{a_2}{v_\infty^2} \tilde{u}_t^2 + \frac{a_3}{v_\infty^3} \tilde{u}_t^3 \right), \quad (9)$$

the uniform windflow velocity, and the coefficients a_0, a_1, a_2, a_3 are given explicitly in [6] and depend on certain drag and lift coefficients. To simplify the initial-boundary value problem for $\tilde{u}(x, t)$ we introduce the following transformation

$$\tilde{u}(x, t) = \bar{u}(x, t) + g \frac{\rho}{T} u_s(x), \quad (10)$$

where $u_s(x) = \frac{1}{2}x^2 - lx$ is the stationary (that is, time-independent) solution (due to gravity) of the initial-boundary value problem, and satisfies

$$u_s''(x) - 1 = 0 \quad 0 < x < l, \quad (11)$$

$$u_s(0) = u_s'(l) = 0. \quad (12)$$

Then, we also introduce the following dimensionless variables $\bar{x} = \frac{\pi}{l}x$, $\bar{t} = ct$, $v(\bar{x}, \bar{t}) = \frac{c}{v_\infty} \bar{u}(x, t)$, with $c = (\pi/l) \sqrt{\frac{T}{\rho}}$. In this way equation (7) becomes

$$v_{\bar{t}\bar{t}} - v_{\bar{x}\bar{x}} = \frac{\rho_a d}{2\rho} \frac{v_\infty}{c} (a_0 + a_1 v_{\bar{t}} + a_2 v_{\bar{t}}^2 + a_3 v_{\bar{t}}^3). \quad (13)$$

Now we assume that the windvelocity v_∞ is small with respect to the wave speed c , that is, $\tilde{\epsilon} = \frac{v_\infty}{c}$ is a small parameter. Following the analysis as given in [6] it can be shown that the right-hand side of equation (13) up to order $\tilde{\epsilon}$ is equal to $\frac{\rho_a d}{2\rho} \tilde{\epsilon} (a v_{\bar{t}} - b v_{\bar{t}}^3)$, where a and b are positive constants which depend on the drag and lift coefficients and which are also given explicitly in [6]. Using the transformation $u(\bar{x}, \bar{t}) = \sqrt{\frac{3b}{a}} v(\bar{x}, \bar{t})$, putting $\epsilon = \frac{\rho_a d}{2\rho} a \tilde{\epsilon}$ it follows that (13) becomes the so-called Rayleigh wave equation $u_{\bar{t}\bar{t}} - u_{\bar{x}\bar{x}} = \epsilon (u_{\bar{t}} - \frac{1}{3} u_{\bar{t}}^3)$, where ϵ is a small dimensionless parameter. Finally it is assumed that the damping coefficient $\tilde{\alpha}$ is small, that is, we assume that $\tilde{\alpha} = \epsilon \alpha \sqrt{\rho/T}$. In this paper we will study the following initial-boundary value problem for $u(x, t)$ (for convenience we will drop all the bars):

$$u_{tt} - u_{xx} = \epsilon \left(u_t - \frac{1}{3} u_t^3 \right), \quad 0 < x < \pi, \quad t > 0, \quad (14)$$

$$u(0, t) = 0, \quad t \geq 0, \quad (15)$$

$$u_x(\pi, t) = -\epsilon \alpha u_t(\pi, t), \quad t \geq 0, \quad (16)$$

$$u(x, 0) = \phi(x), \quad 0 < x < \pi, \quad (17)$$

$$u_t(x, 0) = \psi(x), \quad 0 < x < \pi, \quad (18)$$

where ϕ and ψ are the initial displacement and the initial velocity of the string respectively, and where α is a positive constant, and where $0 < \epsilon \ll 1$. In the next section the well-posedness of the initial-boundary value problem (14) - (18) will be investigated.

3 The wellposedness of the problem

In this section we will consider the following initial-boundary value problem

$$u_{tt} - u_{xx} = c u_t - \sigma(u_t), \quad 0 < x < \pi, \quad t > 0, \quad (19)$$

$$u(0, t) = 0, \quad t \geq 0, \quad (20)$$

$$u_x(\pi, t) = -\alpha u_t(\pi, t), \quad t \geq 0, \quad (21)$$

$$u(x, 0) = u_0(x), \quad 0 < x < \pi, \quad (22)$$

$$u_t(x, 0) = u_1(x), \quad 0 < x < \pi, \quad (23)$$

function with $\sigma(0) = 0$, and where u_0 and u_1 have to satisfy certain regularity conditions, which will be given later. It will be shown that the initial-boundary value problem (19) - (23) is well-posed for all times $t > 0$. To show the well-posedness of the problem a semigroup approach will be used. The following theorem will be used.

Theorem 3.1. *Let $(E, <, >)$ be a real Hilbert space, $B : E \longrightarrow E$ be Lipschitz, and $A : D(A) \subseteq E \longrightarrow E$. The abstract Cauchy problem*

$$\frac{dz}{dt}(t) = Az + Bz, \quad z(0) = z_0 \in D(A) \quad (24)$$

has a unique solution $z \in D(A)$ for all $t > 0$ with $z(0) = z_0$, if A is a dissipative operator and if there exists a positive constant λ such that the range $R(\lambda I - A) = E$.

This theorem is a slight modification of the theorem of Kato (see [24]). Let $c > 0$ be a Lipschitz constant. It is easy to see that $B - cI$, where I is an identity operator, is dissipative and Lipschitz and therefore it is m -dissipative. From the m -dissipativity of A and $B - cI$ it follows that $A + B - cI$ is m -dissipative. To complete the proof we now can apply the theorem of Kato which can be found for instance in [4, page 180]. Moreover, if $z : [0, T] \longrightarrow E$ is the solution of (24) then the solution z is Lipschitz continuous and right-differentiable with $z \in D(A)$.

To prove global existence of the problem (19)-(23) we will make use of theorem 3.2. For that reason the initial-boundary value problem (19)-(23) will be put into an abstract Cauchy problem by introducing some new variables and spaces. Let us define $v = u(., t)$, $w = u_t(., t)$, $H_0^1 = \{v \in H^1; v(0) = 0\}$ and $\mathcal{H} := \{z = (v, w) \in H_0^1([0, \pi]) \times L^2([0, \pi])\}$ with the innerproduct

$$\langle z, \tilde{z} \rangle = \int_0^\pi (v_x \tilde{v}_x + w \tilde{w}) dx \quad (25)$$

$$= \langle v, \tilde{v} \rangle_1 + \langle w, \tilde{w} \rangle_2. \quad (26)$$

It can be readily be seen that the vector space \mathcal{H} together with the inner-product (25) forms a real Hilbert space. Next, we define a subspace $D(A)$ of \mathcal{H} , a linear operator B on \mathcal{H} , and a nonlinear operator A on $D(A)$ as follows;

$$D(A) := \{z = (v, w) \in H^2 \cap H_0^1([0, \pi]) \times H_0^1([0, \pi]); v_x(\pi) + \alpha w(\pi) = 0\}, \quad (27)$$

$Bz = (0, cw)^T$, $Az = (w, v_{xx} - \sigma(w))^T$. It should be observed that the boundary conditions in (20) and (21) are included in the space $D(A)$. Now we differentiate z with respect to t according to the following rule $\frac{dz}{dt} = (v_t, w_t)$. It follows from the definition of A and B that

$$\frac{dz}{dt}(t) = Az + Bz, \quad z(0) = z_0 \in D(A), \quad (28)$$

where $z_0 = (u_0, u_1)$. It should be observed that the operator B is linear, and so it satisfies the Lipschitz condition automatically (with a Lipschitz constant c). To show the solvability of the abstract Cauchy problem (28) according to theorem 3.4 we only need the following lemma.

Lemma 3.1. *Let the functional σ be monotonic increasing and continuous with $\sigma(0) = 0$, and let $(u_0, u_1) \in H^2([0, \pi]) \cap H_0^1([0, \pi]) \times H_0^1([0, \pi])$ with $u_0'(\pi) + \alpha u_1(\pi) = 0$. Then the nonlinear operator A is m -dissipative on \mathcal{H} , and $D(A)$ is dense in \mathcal{H} .*

Proof. To prove this lemma we have to show that A is dissipative, and that the range of $\lambda I - A$ is equal to \mathcal{H} for a $\lambda > 0$. Firstly we show that A is a dissipative operator. Let

increasing)

$$\begin{aligned} \langle Az - A\tilde{z}, z - \tilde{z} \rangle &= \int_0^\pi [(w - \tilde{w})_x (v - \tilde{v})_x + ((v - \tilde{v})_{xx} + \sigma(\tilde{w}) - \sigma(w)) (w - \tilde{w})] dx \\ &= -\alpha(w(\pi) - \tilde{w}(\pi))^2 - \int_0^\pi (\sigma(w) - \sigma(\tilde{w}))(w - \tilde{w}) dx \leq 0. \end{aligned} \quad (29)$$

So we have shown that the nonlinear operator A is a dissipative operator. Secondly for any $z_0 \in \mathcal{H}$ with $z_0 = (g, h)$ we will show that there exists a $z \in D(A)$ such that

$$(I - A)z = z_0, \quad (30)$$

or equivalently

$$v = w + g, \quad (31)$$

$$w = v_{xx} - \sigma(w) + h, \quad (32)$$

$$v(0) = 0, \quad v_x(\pi) + \alpha w(\pi) = 0. \quad (33)$$

Let us assume that $g \in H^2 \cap H_0^1$. Then it follows that

$$y = y_{xx} - \sigma(y) + f, \quad (34)$$

$$y(0) = 0, \quad y_x(\pi) + \alpha y(\pi) = -g_x(\pi), \quad (35)$$

where $f = h + g_{xx} \in L^2$ and where $y = v - g$. To show that the boundary-value problem (34) - (35) is solvable we will apply a variational method by introducing the functionals $\langle, \rangle, \varphi$ and J from $H_0^1([0, \pi])$ into \mathbb{R} which are defined by

$$\langle y, y \rangle = \int_0^\pi ((y')^2 + y^2) dx, \quad \varphi(y) = \int_0^\pi f y dx, \quad (36)$$

$$J(y) = \int_0^\pi j(y) dx + \frac{\alpha}{2} \left(y(\pi) + \frac{1}{\alpha} g_x(\pi) \right)^2, \quad (37)$$

where $j(s) = \int_0^s \sigma(\xi) d\xi$ with $s \in \mathbb{R}$. For $y \in H_0^1$ we define the functional $I(\cdot)$ by

$$I(y) = \frac{1}{2} \langle y, y \rangle - \varphi(y) + J(y). \quad (38)$$

It is clear that the functional I is continuous. From the monotonic increasing σ it follows that the functional I is coercive. Next to see that I is convex it enough to show that the functional j is convex. Let $a, b \in \mathbb{R}$ with $a \leq b$. For any $\lambda \in (0, 1)$ it is easy to see that $a < (1 - \lambda)a + \lambda b < b$. By using the mean value theorem and the fact that σ is monotonic increasing it follows that $j((1 - \lambda)a + \lambda b) \leq (1 - \lambda)j(a) + \lambda j(b)$. From convexity, coercivity, and continuity of I it follows that there exists a unique $\bar{y} \in H_0^1$ such that $I(\bar{y}) \leq I(y)$ for all $y \in H_0^1$. Now for arbitrary $y \in H_0^1$ we define $\phi : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\phi(t) := I(\bar{y} + ty). \quad (39)$$

Since ϕ is continuously differentiable and $\phi(0)$ is minimal it follows that $\bar{y} \in H^2$ and satisfies

$$\int_0^\pi (\bar{y} - \bar{y}'' + \sigma(\bar{y}) - f) y dx + (\bar{y}'(\pi) + \alpha \bar{y} + g_x(\pi)) y(\pi) = 0. \quad (40)$$

We have to notice that the equation (40) holds for every $y \in H_0^1$. So \bar{y} is the solution of the boundary value problem (34)-(35) in the sense of distributions.

$g_n \longrightarrow g$ in H^1 . For all $n \in \mathbb{N}^+$ there is a unique $z_n = (v_n, w_n) \in D(A)$ such that

$$v_n = w_n + g_n, \quad (41)$$

$$w_n = v_{n_{xx}} - \sigma(w_n) + h, \quad (42)$$

$$v_n(0) = 0, \quad v_{n_x}(\pi) + \alpha w_n(\pi) = 0. \quad (43)$$

Since A is a dissipative operator on \mathcal{H} an a-priori estimate can be obtained, that is,

$$\|z_n - z_m\|_{\mathcal{H}} \leq \|f_n - f_m\|_{\mathcal{H}}, \quad (44)$$

where $f_n = (g_n, h)^T \in \mathcal{H}$. From (44) it follows that $\{v_n\}$ and $\{w_n\}$ are Cauchy sequences in H^1 and L^2 respectively. Moreover, $(H^1, <, >_1)$ and $(L^2, <, >_2)$ are complete implying that there are $\bar{v} \in H^1$ and $\bar{w} \in L^2$ such that $v_n \longrightarrow \bar{v}$ and $w_n \longrightarrow \bar{w}$. Furthermore, $\{v_n\}$ is also a Cauchy sequence in C^0 with maximum norm. Therefore we obtain $\bar{v} \in H_0^1$. From the continuity of σ it follows from (41) and (42) for $n \longrightarrow \infty$ that

$$w_n = v_n - g_n \longrightarrow \bar{v} - g = \bar{w} \in H_0^1, \quad (45)$$

and

$$v_{n_{xx}} = w_n + \sigma(w_n) - h \longrightarrow \bar{w} + \sigma(\bar{w}) - h \in L^2. \quad (46)$$

Next we will show that $\bar{v} \in H^2 \cap H_0^1$ and that $v_{n_{xx}}$ converges to \bar{v}_{xx} . Since v_n is in H^2 it follows that there are positive constants c_1 and c_2 such that

$$\|(v_n - v_m)'\|_{L^2} \leq c_1 \|v_n - v_m\|_{L^2} + c_2 \|(v_n - v_m)''\|_{L^2}. \quad (47)$$

It can readily be seen from (47) that $v_n' \longrightarrow \bar{v}'$ in L^2 . For $n \longrightarrow \infty$ it follows from $\int_0^\pi v_n' \varphi' dx = -\int_0^\pi v_n'' \varphi dx$ (for arbitrary $\varphi \in H^2$ where φ vanishes for $x = 0$ and π) that $\int_0^\pi \bar{v}' \varphi' dx = -\int_0^\pi (\bar{w} + \sigma(\bar{w}) - h) \varphi dx$. So, $\bar{v} \in H^2$, and $\bar{v}'' = \bar{w} + \sigma(\bar{w}) - h$ in L^2 with $\bar{v}(0) = 0$. It also follows from (46) that $v_{n_{xx}} \longrightarrow \bar{v}''$ in L^2 . Finally we have to show that \bar{v}, \bar{w} satisfy the boundary conditions (33). To show this we integrate (42) once, yielding

$$v_{n_x}(x) = -\int_x^\pi (w_n + \sigma(w_n) - h) dx - \alpha w_n(\pi). \quad (48)$$

It can also be shown that $w_n \longrightarrow \bar{w}$ uniformly in $C^0([0, \pi])$ with the maximum norm. Again by using the continuity of σ and the Lebesgue Monotone Convergence Theorem it follows from (48) that

$$\bar{v}_x(x) = -\int_x^\pi (\bar{w} + \sigma(\bar{w}) - h) dx - \alpha \bar{w}(\pi). \quad (49)$$

We deduce from (45) and (49) that for any $z_0 \in \mathcal{H}$ there is $z \in D(A)$ such that the equations (31)-(33) hold. This completes the proof of the lemma. \square

Remark 3.1. From the theorem of Kato it follows that the unique solution $z : [0, T] \longrightarrow \mathcal{H}$ is absolutely continuous. Moreover, the solution z is Lipschitz continuous and right-differentiable, and when the initial values u_0 and u_1 satisfy the conditions as given in Lemma 3.1 it also follows (observing that $v(t) = u(\cdot, t), w(t) = u_t(\cdot, t)$) that

$$u \in L^\infty([0, T]; H^2([0, \pi])) \cap W^{1, \infty}([0, T]; H^1([0, \pi])) \cap W^{2, \infty}([0, T]; L^2([0, \pi])). \quad (50)$$

From (50) it also follows that

$$u_{tt}, u_{xx}, u_{xt} \in L^\infty([0, T]; L^2([0, \pi])). \quad (51)$$

$$\frac{dz}{dt} = Az + f(t), \quad z(0) = z_0 \in D(A) \quad (52)$$

where $f(t) = (0, cu_t - \sigma(u_t))^T \in W^{1,1}([0, T]; H)$ it can be shown (see for instance [26] and [27] page 400) the solution of (52) is in $C^1([0, T]; H) \cap C([0, T]; D(A))$. It follows that

$$u_{tt}, u_{xx}, u_{xt} \in C([0, T]; L^2([0, \pi])) . \quad (53)$$

Remark 3.2. Alternatively we can also study the problem by using different variables and spaces. By defining the following variables $v = u_x(., t)$, $w = u_t(., t)$, and the space $\mathcal{H} := \{z = (v, w) \in L^2([0, \pi]) \times L^2([0, \pi])\}$ with the innerproduct

$$\langle z, \tilde{z} \rangle = \int_0^\pi (v\tilde{v} + w\tilde{w}) dx \quad (54)$$

the initial-boundary value problem (19)-(23) can be transformed into an abstract Cauchy problem with the operators $Bz = (0, cw)$, $Az = (w_x, v_x - \sigma(w))$, and

$$D(A) = \{z = (v, w) \in H^1([0, \pi]) \times H^1([0, \pi]); w(0) = 0, v(\pi) + \alpha w(\pi) = 0\}. \quad (55)$$

With these operators the following abstract Cauchy problem

$$\frac{dz}{dt}(t) = Az + Bz, \quad z(0) = z_0 \in D(A), \quad (56)$$

will be obtained. Again it can be shown that the solution u of the initial-boundary value problem is unique and satisfies for instance the regularity properties

$$u_{tt}, u_{xx}, u_{xt} \in C([0, T]; L^2([0, \pi])) . \quad (57)$$

By imposing more regularity conditions on the initial values it can be shown that the initial-boundary value problem (19)-(23) has a unique, classical solution. The long proof of this result is beyond of the scope of this paper, but will be published in a forthcoming paper.

Now we consider the following abstract Cauchy problem

$$\frac{dz}{dt} = Az + Bz + f(t), \quad z(0) = z_0 \in D(A), \quad (58)$$

where A and B satisfy the conditions as stated in theorem 3.2 and where f is continuous. By making use of the dissipativity of $A + B - cI$ (with c a Lipschitz constant) and by putting $\tilde{A} = A + B - cI$, and then by using (58) it follows that

$$\frac{dz}{dt} = \tilde{A}z + cz + f(t), \quad z(0) = z_0 \in D(\tilde{A}). \quad (59)$$

As property of the pseudo-scalar product (see for instance [25], page 185) it is standard to show that

$$\frac{d}{dt}(\|u\|^2) = 2 \langle \frac{dz}{dt}, z \rangle . \quad (60)$$

Suppose that z_1 and z_2 are solutions of (28) and (58) respectively with $f \in L^1([0, T]; \mathcal{H})$ and with the initial values $z_{01}, z_{02} \in \overline{D(A)}$. By making use of (28), (59), and (60) it then follows that

$$\frac{d}{dt}(e^{-2ct} \|z_2 - z_1\|^2) = 2e^{-2ct} \langle f, z_2 - z_1 \rangle . \quad (61)$$

be obtained

$$\| z_1(t) - z_2(t) \| \leq e^{ct} \| z_{0_1} - z_{0_2} \| + \int_0^t e^{c(t-\tau)} \| f \| d\tau, \quad 0 \leq t \leq T. \quad (62)$$

We refer to [24, page 182-183] for the details of the proof. It follows from (62) that the solution of the abstract Cauchy problem (24) depends continuously on the initial values. Moreover, the estimate as given by (62) will be used in section 5 of this paper to prove the asymptotic validity of the approximation as constructed in section 4.

4 The construction of approximations.

In this section formal approximations of the solution of the nonlinear initial-boundary value problem (14) - (18) will be constructed for different values of the damping parameter α . To obtain insight we will first study in section 4.1 the linearized problem, that is, in (14) we will first neglect the nonlinear term $-\frac{\epsilon}{3}u_t^3$. It will turn out for the linearized problem that for $\alpha > \frac{\pi}{2}$ all solutions will tend to zero, whereas for $0 < \alpha < \frac{\pi}{2}$ the solutions of the linearized problem will become unbounded. The nonlinear problem (14) - (18) will be studied in detail in section 4.2. It will turn out in section 4.2 that for $\alpha \geq \frac{\pi}{2}$ all solutions tend to zero, and that for $0 \leq \alpha < \frac{\pi}{2}$ the solutions tend to bounded solutions.

4.1 The linearized problem

In this section we will study the linearized problem (14) - (18), that is,

$$u_{tt} - u_{xx} - \epsilon u_t = 0, \quad 0 < x < \pi, \quad t > 0, \quad (63)$$

$$u(0, t) = 0, \quad t \geq 0, \quad (64)$$

$$u_x(\pi, t) + \epsilon \alpha u_t(\pi, t) = 0, \quad t \geq 0, \quad (65)$$

$$u(x, 0) = \phi(x), \quad 0 < x < \pi, \quad (66)$$

$$u_t(x, 0) = \psi(x), \quad 0 < x < \pi. \quad (67)$$

To solve the problem (63) - (67) the Laplace transform method will be used. By introducing

$$U(x, s) = \int_0^\infty e^{-st} u(x, t) dt, \quad (68)$$

it follows that the initial-boundary value problem (63)-(67) becomes

$$\frac{d^2 U}{dx^2} - (s^2 - \epsilon s)U = -\psi(x) - (s - \epsilon)\phi(x), \quad 0 < x < \pi, \quad s > 0, \quad (69)$$

$$U(0, s) = 0, \quad s > 0, \quad (70)$$

$$\frac{dU}{dx}(\pi, s) + \epsilon \alpha s U(\pi, s) = \epsilon \alpha \phi(\pi), \quad s > 0. \quad (71)$$

In the further analysis it is convenient to put $\delta^2 = s^2 - \epsilon s$ in (69). The following three cases have to be considered: $\delta^2 > 0$, $\delta^2 < 0$, and $\delta^2 = 0$. However, the case $\delta^2 = 0$ only leads to the trivial solution. For that reason only the case $\delta^2 > 0$ and the case $\delta^2 < 0$ have to be studied.

4.1.1 The case $\delta^2 > 0$.

The solution of the boundary value problem (69)-(71) with $\delta^2 > 0$ is given by

$$U(x, s) = C(s) \sinh(\delta x) - \frac{1}{\delta} \int_0^x h(z, s) \sinh(\delta(x - z)) dz, \quad (72)$$

$$C(s) = \frac{\int_0^\pi h(z, s) \left(\frac{\epsilon \alpha s}{\delta} \sinh(\delta(\pi - z)) + \cosh(\delta(\pi - z)) \right) dz}{\epsilon \alpha s \sinh(\delta\pi) + \delta \cosh(\delta\pi)} + \frac{\epsilon \alpha \phi(\pi)}{\epsilon \alpha s \sinh(\delta\pi) + \delta \cosh(\delta\pi)}. \quad (73)$$

To obtain the solution of the initial-boundary value problem (63)-(67) we have to take the inverse Laplace transform of U , that is,

$$u(x, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt} U(x, z) dz, \quad (74)$$

where γ is positive. It should be observed that $z = 0$ or $z = \epsilon$ (or $\delta = 0$) are not poles. The poles of (73) are given by

$$\epsilon \alpha z \sinh(\delta\pi) + \delta \cosh(\delta\pi) = 0 \quad \text{with} \quad \delta^2 = z^2 - \epsilon z, \quad (75)$$

or equivalently by

$$\exp(2\pi\delta) = -\frac{(\delta - \epsilon \alpha z)}{(\delta + \epsilon \alpha z)} \quad \text{with} \quad \delta^2 = z^2 - \epsilon z, \quad (76)$$

and with $\delta \neq 0$. It can be shown (for instance by using Hadamard's factorization theorem) that (76) has infinitely many isolated roots which all have a geometric multiplicity equal to one. It is also obvious that (76) is hard to solve exactly for $\epsilon \neq 0$. However, it can be solved exactly if $\epsilon = 0$. For that reason the solutions of (76) will be approximated by expanding z and δ in power series in ϵ . It will be convenient to put $\delta = \delta_0 + i\delta_1$ and $z = \sigma_0 + i\sigma_1$ with $\delta_0, \delta_1, \sigma_0$ and $\sigma_1 \in \mathbb{R}$. Then by separating the real part and the imaginary part of (76) it follows that

$$e^{2\pi\delta_0} \cos(2\pi\delta_1) = -\frac{(\delta_0 - \epsilon \alpha \sigma_0)(\delta_0 + \epsilon \alpha \sigma_0) + (\delta_1 - \epsilon \alpha \sigma_1)(\delta_1 + \epsilon \alpha \sigma_1)}{(\delta_0 + \epsilon \alpha \sigma_0)^2 + (\delta_1 + \epsilon \alpha \sigma_1)^2}, \quad (77)$$

$$e^{2\pi\delta_0} \sin(2\pi\delta_1) = -\frac{(\delta_0 - \epsilon \alpha \sigma_0)(\delta_1 + \epsilon \alpha \sigma_1) + (\delta_1 - \epsilon \alpha \sigma_1)(\delta_0 + \epsilon \alpha \sigma_0)}{(\delta_0 + \epsilon \alpha \sigma_0)^2 + (\delta_1 + \epsilon \alpha \sigma_1)^2}. \quad (78)$$

It is assumed that z and δ can be expanded in power series in ϵ , that is,

$$\delta_0 = \delta_{00} + \epsilon \delta_{01} + \epsilon^2 \delta_{02} + \dots, \quad \delta_1 = \delta_{10} + \epsilon \delta_{11} + \epsilon^2 \delta_{12} + \dots, \quad (79)$$

$$\sigma_0 = \sigma_{00} + \epsilon \sigma_{01} + \epsilon^2 \sigma_{02} + \dots, \quad \sigma_1 = \sigma_{10} + \epsilon \sigma_{11} + \epsilon^2 \sigma_{12} + \dots \quad (80)$$

It follows from (79), (80), $\delta^2 = z^2 - \epsilon z$, and by taking terms of equal powers in ϵ together that

$$\delta_{00}^2 + \sigma_{10}^2 - \delta_{10}^2 - \sigma_{00}^2 = 0, \quad (81)$$

$$2\delta_{00}\delta_{01} - 2\delta_{10}\delta_{11} - 2\sigma_{00}\sigma_{01} + 2\sigma_{10}\sigma_{11} + \sigma_{00} = 0, \quad (82)$$

$$2\delta_{00}\delta_{02} + \sigma_{01} - \delta_{11}^2 - 2\delta_{10}\delta_{12} + \delta_{01}^2 + \sigma_{11}^2 + 2\sigma_{10}\sigma_{12} - 2\sigma_{00}\sigma_{02} - \sigma_{01}^2 = 0, \dots \quad (83)$$

and

$$-2\sigma_{00}\sigma_{10} + 2\delta_{00}\delta_{10} = 0, \quad (84)$$

$$2\delta_{00}\delta_{11} + 2\delta_{10}\delta_{01} - 2\sigma_{00}\sigma_{11} - 2\sigma_{10}\sigma_{01} + \sigma_{00} = 0, \quad (85)$$

$$-2\sigma_{00}\sigma_{12} + 2\delta_{01}\delta_{11} + 2\delta_{10}\delta_{02} - 2\sigma_{01}\sigma_{11} - 2\sigma_{10}\sigma_{02} + 2\delta_{00}\delta_{12} + \sigma_{01} = 0, \dots \quad (86)$$

terms of equal powers in ϵ together in (77) and (78), and by using (81)-(86) it then can be deduced that

$$\sigma_{00} = 0, \sigma_{01} = \frac{1}{2} - \frac{\alpha}{\pi}, \sigma_{02} = 0, \quad (87)$$

$$\sigma_{10} = \delta_{10}, \sigma_{11} = 0, \sigma_{12} = \frac{1}{2\sigma_{10}} \left(\frac{\alpha}{\pi} - \frac{1}{4} \right), \sigma_{13} = 0, \quad (88)$$

$$\delta_{00} = 0, \delta_{01} = -\frac{\alpha}{\pi}, \delta_{02} = 0, \delta_{03} = \frac{\alpha}{24\pi^2\sigma_{10}^2} (3\pi + 12\alpha - 8\alpha^2\pi\sigma_{10}^2), \quad (89)$$

$$\delta_{10} = n - \frac{1}{2}, \delta_{11} = 0, \delta_{12} = \frac{\alpha}{2\pi\sigma_{10}}, \sigma_{13} = 0, \dots \quad (90)$$

After approximating the roots of (75) we can approximate the solution of (74). For instance, if we approximate the roots (poles) up to order ϵ it now follows from the theorem of residues that an approximation of the solution is given by

$$\begin{aligned} u(x, t) \approx & -\frac{2}{\pi} e^{\sigma_{01}t} \sum_{n=1}^{\infty} \int_0^{\pi} \left(\phi(\tau) [\dots]_1 + \frac{1}{\sigma_{01}^2 + \sigma_{10}^2} (\psi(\tau) - \epsilon\phi(\tau)) (\sigma_{01} [\dots]_1 + \sigma_{10} [\dots]_2) \right) d\tau \\ & + \frac{2\epsilon\alpha\phi(\pi)}{\pi(1 - (\epsilon\alpha)^2)} e^{\sigma_{01}t} \sum_{n=1}^{\infty} \frac{[\dots]_3 [\dots]_5 + [\dots]_4 [\dots]_6}{([\dots]_5)^2 + ([\dots]_6)^2}, \end{aligned} \quad (91)$$

where

$$[\dots]_1 = R(x, t) \cos(\delta_{10}\tau) \sinh(\delta_{01}\tau) - T(x, t) \sin(\delta_{10}\tau) \cosh(\delta_{01}\tau), \quad (92)$$

$$[\dots]_2 = R(x, t) \sin(\delta_{10}\tau) \cosh(\delta_{01}\tau) + T(x, t) \cos(\delta_{10}\tau) \sinh(\delta_{01}\tau), \quad (93)$$

$$[\dots]_3 = \cos(\sigma_{10}t) \cos(\delta_{10}\tau) \sinh(\delta_{01}\tau) - \sin(\sigma_{10}t) \sin(\delta_{10}\tau) \cosh(\delta_{01}\tau), \quad (94)$$

$$[\dots]_4 = \cos(\sigma_{10}t) \cos(\delta_{10}\tau) \sinh(\delta_{01}\tau) + \sin(\sigma_{10}t) \sin(\delta_{10}\tau) \cosh(\delta_{01}\tau), \quad (95)$$

$$[\dots]_5 = \sigma_{01} \cos(\delta_{10}\pi) \sinh(\delta_{01}\pi) - \sigma_{10} \sin(\delta_{10}\pi) \cosh(\delta_{01}\pi), \quad (96)$$

$$[\dots]_6 = \sigma_{10} \cos(\delta_{10}\pi) \sinh(\delta_{01}\pi) + \sigma_{01} \sin(\delta_{10}\pi) \cosh(\delta_{01}\pi), \quad (97)$$

with

$$R(x, t) = \cos(\delta_{10}x) \sinh(\delta_{01}x) \cos(\sigma_{10}t) - \sin(\delta_{10}x) \cosh(\delta_{01}x) \sin(\sigma_{10}t), \quad (98)$$

$$T(x, t) = \sin(\delta_{10}x) \cosh(\delta_{01}x) \cos(\sigma_{10}t) + \cos(\delta_{10}x) \sinh(\delta_{01}x) \sin(\sigma_{10}t). \quad (99)$$

It should be noted from (91) that if $\alpha > \frac{\pi}{2}$ the solution $u(x, t)$ will tend to zero (up to $O(\epsilon)$) as t tends to infinity, and that if $\alpha < \frac{\pi}{2}$ the solution $u(x, t)$ of the linearized problem will tend to infinity as t tends to infinity.

4.1.2 The case $\delta^2 < 0$.

By putting $\delta = i\rho$, it follows that the solution of the boundary value problem (69)- (71) is given by

$$U(x, s) = C(s) \sin(\rho x) - \frac{1}{\rho} \int_0^x h(z, s) \sinh(\rho(x - z)) dz, \quad (100)$$

where $h(z, s) = \psi(z) + (\epsilon - s)\phi(z)$ and

$$C(s) = \frac{\int_0^{\pi} h(\tau, s) \left(\frac{\epsilon\alpha s}{\rho} \sin(\rho(\pi - \tau)) + \cos(\rho(\pi - \tau)) \right) d\tau + \epsilon\alpha\phi(\pi)}{\epsilon\alpha s \sin(\rho\pi) + \rho \cos(\rho\pi)}. \quad (101)$$

inverse Laplace transform of U , that is,

$$u(x, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} U(x, z) dz, \quad (102)$$

where γ is positive. It should be observed that $z = 0$ or $z = \epsilon$ (or $\rho = 0$) are not poles. The poles of (102) are given by

$$\epsilon \alpha z \sin(\rho \pi) + \rho \cos(\rho \pi) = 0, \text{ with } -\rho^2 = z^2 - \epsilon z. \quad (103)$$

By putting $\rho = \rho_0 + i\rho_1$ and $z = \sigma_0 + i\sigma_1$ with ρ_0, ρ_1, σ_0 , and $\sigma_1 \in \mathbb{R}$, and by using the following relations

$$\sin(z) = \sin(\sigma_0) \cosh(\sigma_1) + i \cos(\sigma_0) \sinh(\sigma_1), \quad (104)$$

$$\cos(z) = \cos(\sigma_0) \cosh(\sigma_1) - i \sin(\sigma_0) \sinh(\sigma_1) \quad (105)$$

it can be shown (by separating the real part and the imaginary part) that (103) is equivalent to the following equations

$$e^{2\pi\rho_1} \cos(2\pi\rho_0) = -\frac{(\rho_0 - \epsilon\alpha\sigma_1)(\rho_0 + \epsilon\alpha\sigma_1) + (\rho_1 - \epsilon\alpha\sigma_0)(\rho_1 + \epsilon\alpha\sigma_0)}{(\rho_0 - \epsilon\alpha\sigma_1)^2 + (\rho_1 + \epsilon\alpha\sigma_0)^2}, \quad (106)$$

$$e^{2\pi\sigma} \sin(2\pi\rho_0) = -\frac{(\rho_1 + \epsilon\alpha\sigma_0)(\rho_0 + \epsilon\alpha\sigma_1) - (\rho_0 - \epsilon\alpha\sigma_1)(\rho_1 - \epsilon\alpha\sigma_0)}{(\rho_0 - \epsilon\alpha\sigma_1)^2 + (\rho_1 + \epsilon\alpha\sigma_0)^2}, \quad (107)$$

where ρ_0, ρ_1, σ_0 and σ_1 are related by

$$\sigma_1^2 - \sigma_0(\sigma_0 - \epsilon) = \rho_0^2 - \rho_1^2 \quad \text{and} \quad \sigma_1(\epsilon - 2\sigma_0) = 2\rho_0\rho_1. \quad (108)$$

We should observe (noticing that $\delta = i\rho$) that the equations (103) -(107) and (75)-(78) are equivalent. In fact it can be shown that the approximate solutions for $\delta^2 > 0$ and for $\delta^2 < 0$ are coinciding.

4.2 The nonlinear problem

Now we consider the nonlinear initial-boundary value problem (14) - (18). A straightforward perturbation expansion $u_0(x, t) + \epsilon u_1(x, t) + \dots$ will cause secular terms. For that reason a two-timescales perturbation method (see [15, 16, 17]) will be used in this section to construct formal asymptotic approximations for the solution of the initial-boundary value problem (14) - (18). We will encounter some computational difficulties whenever we assume an infinite series representation for the solution of the nonlinear initial-boundary value problem (14) - (18). To avoid these difficulties we will use additionally the method of characteristic coordinates to approximate the solution. By using a two-timescales perturbation method the function $u(x, t)$ is supposed to be a function of x, t , and τ , where $\tau = \epsilon t$. We put

$$u(x, t) = v(x, t, \tau; \epsilon). \quad (109)$$

By substituting (109) into the initial-boundary value problem (14) - (18) we obtain

$$v_{tt} - v_{xx} = -2\epsilon v_{t\tau} + \epsilon \left(v_t + \epsilon v_\tau - \frac{1}{3}(v_t + \epsilon v_\tau)^3 \right) - \epsilon^2 v_{\tau\tau}, \quad 0 < x < \pi, \quad t > 0, \quad (110)$$

$$v(0, t, \tau; \epsilon) = 0, \quad t \geq 0, \quad (111)$$

$$v_x(\pi, t, \tau; \epsilon) = -\epsilon \alpha (v_t(\pi, t, \tau) + \epsilon v_\tau(\pi, t)), \quad t \geq 0, \quad (112)$$

$$v(x, 0, 0; \epsilon) = \phi(x), \quad 0 < x < \pi, \quad (113)$$

$$v_t(x, 0, 0; \epsilon) = -\epsilon v_\tau(x, 0, 0; \epsilon) + \psi(x), \quad 0 < x < \pi. \quad (114)$$

$$v(x, t, \tau; \epsilon) = v_0(x, t, \tau) + \epsilon v_1(x, t, \tau) + \dots \quad (115)$$

As usual it is assumed that v, v_0, v_1, \dots are of order one on a time-scale of order ϵ^{-1} . By substituting (115) into (110)-(114), and by equating the coefficients of like powers in ϵ , it follows from the power 0 and 1 of ϵ respectively, that v_0 should satisfy

$$v_{0tt} - v_{0xx} = 0, \quad 0 < x < \pi, \quad t > 0, \quad (116)$$

$$v_0(0, t, \tau) = 0, \quad v_{0x}(\pi, t, \tau) = 0, \quad t \geq 0, \quad (117)$$

$$v_0(x, 0, 0) = \phi(x), \quad 0 < x < \pi, \quad (118)$$

$$v_{0t}(x, 0, 0) = \psi(x), \quad 0 < x < \pi, \quad (119)$$

and that v_1 should satisfy

$$v_{1tt} - v_{1xx} = v_{0t} - 2v_{0t\tau} - \frac{1}{3}v_{0t}^3, \quad 0 < x < \pi, \quad t > 0, \quad (120)$$

$$v_1(0, t, \tau) = 0, \quad v_{1x}(\pi, t, \tau) = -\alpha v_{0t}(\pi, t, \tau) \quad t \geq 0, \quad (121)$$

$$v_1(x, 0, 0) = 0, \quad 0 < x < \pi, \quad (122)$$

$$v_{1t}(x, 0, 0) = -v_{0\tau}(x, 0, 0), \quad 0 < x < \pi. \quad (123)$$

We will solve the initial-boundary value problem (116) - (119) by using the characteristic coordinates $\delta = x - t$ and $\xi = x + t$. The initial-boundary value problem (116) - (119) then has to be replaced by an initial value problem. This can be accomplished by extending all functions in 4π -periodic function in x which are odd and even around at $x = 0$ and at $x = \pi$ respectively. It then follows that the solution of the initial-boundary value problem (116) - (119) is given by $v_0(x, t, \tau) = f_0(\sigma, \tau) + g_0(\xi, \tau)$, where $\sigma = x - t$, and $\xi = x + t$. Applying the initial conditions (118) - (119) it also follows that f_0 and g_0 have to satisfy $f_0(\sigma, 0) + g_0(\sigma, 0) = \phi(\sigma)$ and $-f_0'(\sigma, 0) + g_0'(\sigma, 0) = \psi(\sigma)$, where the prime denotes differentiation with respect to the first argument. It also follows from the odd/even, and 4π -periodic extension in x of the dependent variable v_0 that f_0 and g_0 have to satisfy $f_0(\sigma, \tau) = -g_0(-\sigma, \tau)$ and $f_0(\sigma, \tau) = f_0(\sigma + 4\pi, \tau)$ for $-\infty < \sigma < \infty$ and $\tau \geq 0$. The behaviour of f_0 and g_0 with respect to τ will be determined completely by demanding that v_1 does not contain secular terms. To solve the initial-boundary value problem (120) - (123) for v_1 it is convenient to make the boundary condition (121) at $x = \pi$ homogeneous by introducing the following transformation

$$v_1(x, t, \tau) = w(x, t, \tau) - \alpha H(x)v_{0t}(\pi, t, \tau), \quad (124)$$

where $H(x) = x$ for $0 \leq x \leq \pi$, odd around $x = 0$, even around $x = \pi$, and 4π -periodic. The initial-boundary value problem (120) - (123) then becomes

$$w_{tt} - w_{xx} = v_{0t} - 2v_{0t\tau} - \frac{1}{3}v_{0t}^3 + \alpha H(x)v_{0ttt}(\pi, t, \tau), \quad 0 < x < \pi, \quad t > 0, \quad (125)$$

$$w(0, t, \tau) = 0, \quad t \geq 0, \quad (126)$$

$$w_x(\pi, t, \tau) = 0, \quad t \geq 0, \quad (127)$$

$$w(x, 0, 0) = \alpha H(x)v_{0t}(\pi, 0, 0), \quad 0 < x < \pi, \quad (128)$$

$$w_t(x, 0, 0) = -v_{0\tau}(x, 0, 0) + \alpha H(x)v_{0tt}(\pi, 0, 0), \quad 0 < x < \pi. \quad (129)$$

To solve the initial-boundary value problem (125)-(129) it will turn out (in order to avoid computational difficulties with infinite sums (Fourier-series)) that it is convenient to use again characteristic coordinates. So, all functions should again be extended in 4π -periodic functions in x , which are odd around $x = 0$ and even around $x = \pi$. However, to recognize the terms in

series representation for f_0 , v_0 , and $H(x)$ is helpful. For that reason we write f_0 , v_0 , and $H(x)$ as

$$f_0(\sigma, \tau) = \frac{A_0(\tau)}{2} + \sum_{n=1}^{\infty} \left(\frac{A_n(\tau)}{2} \cos((n-1/2)\sigma) + \frac{B_n(\tau)}{2} \sin((n-1/2)\sigma) \right). \quad (130)$$

$$v_0(x, t, \tau) = \sum_{n=1}^{\infty} (A_n(\tau) \sin((n-1/2)t) + B_n(\tau) \cos((n-1/2)t)) \sin((n-1/2)x). \quad (131)$$

and

$$H(x) = \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{(-1)^{n+1}}{(n-\frac{1}{2})^2} \sin((n-1/2)x), \quad (132)$$

respectively. Now it should be observed that the terms $H(x)v_{0ttt}(\pi, t, \tau)$ in the right-hand side of (125) can be written as

$$\begin{aligned} H(x)v_{0ttt}(\pi, t, \tau) &= -\frac{2}{\pi} \sum_{n=1}^{\infty} \left(n - \frac{1}{2} \right) \left[A_n(\tau) \cos\left(\left(n - \frac{1}{2} \right) t \right) - B_n(\tau) \sin\left(\left(n - \frac{1}{2} \right) t \right) \right] \\ &\quad \times \sin\left(\left(n - \frac{1}{2} \right) x \right) + G(x, t, \tau), \\ &= -\frac{2}{\pi} v_{0t}(x, t, \tau) + G(x, t, \tau), \end{aligned} \quad (133)$$

where

$$\begin{aligned} G(x, t, \tau) &= -\frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{(k-\frac{1}{2})^3}{(n-\frac{1}{2})^2} (-1)^{k+n+2} \left(A_k(\tau) \cos\left(\left(k - \frac{1}{2} \right) t \right) - B_k(\tau) \sin\left(\left(k - \frac{1}{2} \right) t \right) \right) \\ &\quad \times \sin\left(\left(n - \frac{1}{2} \right) x \right). \end{aligned} \quad (134)$$

It should be noted that the function G as given by (134) does not contain terms giving rise to secular terms in w . By introducing the characteristic coordinates $\sigma = x - t$ and $\xi = x + t$ and by putting $w(x, t, \tau) = \tilde{w}(\sigma, \xi, \tau)$ equation (125) then becomes

$$\begin{aligned} -4\tilde{w}_{\sigma\xi}(\sigma, \xi, \tau) &= \left(2f_{0\sigma\tau}(\sigma, \tau) + \left(\left(\frac{2\alpha}{\pi} - 1 \right) + f_{0\xi}^2(-\xi, \tau) \right) f_{0\sigma}(\sigma, \tau) + \frac{1}{3}f_{0\sigma}^3(\sigma, \tau) \right) \\ &\quad - \left(2f_{0\xi\tau}(-\xi, \tau) + \left(\left(\frac{2\alpha}{\pi} - 1 \right) + f_{0\sigma}^2(\sigma, \tau) \right) f_{0\xi}(-\xi, \tau) + \frac{1}{3}f_{0\xi}^3(-\xi, \tau) \right) \\ &\quad + \alpha\tilde{G}(\sigma, \xi, \tau), \end{aligned} \quad (135)$$

where

$$\tilde{G}(\sigma, \xi, \tau) = \frac{1}{\pi} \sum_{n=1}^{\infty} \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{(k-\frac{1}{2})^3}{(n-\frac{1}{2})^2} (-1)^{k+n+2} T_{nk}(\sigma, \xi, \tau) \quad (136)$$

with

$$\begin{aligned} T_{nk}(\sigma, \xi, \tau) &= A_k(\tau) \left(\cos \left[\frac{k-n}{2} \xi + \frac{1-k-n}{2} \sigma \right] - \cos \left[\frac{k-n}{2} \sigma + \frac{1-k-n}{2} \xi \right] \right) + \\ &\quad B_k(\tau) \left(\sin \left[\frac{k-n}{2} \xi + \frac{1-k-n}{2} \sigma \right] + \sin \left[\frac{k-n}{2} \sigma + \frac{1-k-n}{2} \xi \right] \right). \end{aligned} \quad (137)$$

it follows that

$$\begin{aligned}
4\tilde{w}_\sigma(\sigma, \xi, \tau) &= 4\tilde{w}_\sigma(\sigma, \sigma, \tau) - (\xi - \sigma) \left(2f_{0_{\sigma\tau}}(\sigma, \tau) + \left(\frac{2\alpha}{\pi} - 1 \right) f_{0_\sigma}(\sigma, \tau) + \frac{1}{3} f_{0_\sigma}^3(\sigma, \tau) \right) \\
&+ \int_\sigma^\xi \left(2f_{0_{\xi\tau}}(-\xi, \tau) + \left(\left(\frac{2\alpha}{\pi} - 1 \right) + f_{0_\sigma}^2(\sigma, \tau) \right) f_{0_\xi}(-\xi, \tau) + \frac{1}{3} f_{0_\xi}^3(-\xi, \tau) \right) d\xi \\
&- f_{0_\sigma}(\sigma, \tau) \int_\sigma^\xi f_{0_\xi}^2(-\xi, \tau) d\xi + \alpha \int_\sigma^\xi \tilde{G}(\sigma, \xi, \tau) d\xi + h(\sigma, \tau),
\end{aligned} \tag{138}$$

where $h(\sigma, \tau)$ will be determined later on. Now it should be observed in (138) that the integrals which involve the (4π) -periodic functions $f_{0_\xi}^2(-\xi, \tau)$ and $f_{0_\xi}^3(-\xi, \tau)$ can become unbounded if the integrals over a period of 4π are nonzero. It turns out these integrals can be rewritten in a part which is $O(1)$ for all values of σ and ξ and in a part which is linear in $2t = \xi - \sigma$, that is, for $n = 2, 3$ we have

$$\int_\sigma^\xi f_{0_\xi}^n(-\xi, \tau) d\xi = \int_\sigma^\xi \left(f_{0_\xi}^n(-\xi, \tau) - \frac{1}{4\pi} \int_0^{4\pi} f_{0_\theta}^n(-\theta, \tau) d\theta \right) d\xi + \frac{\xi - \sigma}{4\pi} \int_0^{4\pi} f_{0_\theta}^n(-\theta, \tau) d\theta. \tag{139}$$

Noticing that $\xi - \sigma = 2t$ it follows that $\xi - \sigma$ is of order ϵ^{-1} on a time-scale of order ϵ^{-1} . So \tilde{w}_σ will be of order ϵ^{-1} . In a similar way it can also be shown that \tilde{w}_ξ will contain secular terms. To avoid these secular terms it follows that f_0 and g_0 have to satisfy

$$2f_{0_{\sigma\tau}} + \left(\frac{2\alpha}{\pi} - 1 \right) f_{0_\sigma} + \frac{1}{3} f_{0_\sigma}^3 + f_{0_\sigma} I_2(\tau) - \frac{1}{3} I_3(\tau) = 0, \tag{140}$$

$$2g_{0_{\xi\tau}} + \left(\frac{2\alpha}{\pi} - 1 \right) g_{0_\xi} + \frac{1}{3} g_{0_\xi}^3 + g_{0_\xi} Y_2(\tau) - \frac{1}{3} Y_3(\tau) = 0, \tag{141}$$

where

$$I_n(\tau) = \frac{1}{4\pi} \int_0^{4\pi} f_{0_\theta}^n(-\theta, \tau) d\theta \quad \text{for } n=1, 2, 3, \dots, \tag{142}$$

$$Y_n(\tau) = \frac{1}{4\pi} \int_0^{4\pi} g_{0_\theta}^n(\theta, \tau) d\theta \quad \text{for } n=1, 2, 3, \dots. \tag{143}$$

From the relation $g_0(\theta, \tau) = -f_0(-\theta, \tau)$ it follows that the equations (140) and (141) are equivalent. It then follows that \tilde{w}_σ and \tilde{w}_ξ will be of order one on a time-scale of order ϵ^{-1} if f_0 satisfies

$$2f_{0_{\sigma\tau}} + \left(\frac{2\alpha}{\pi} - 1 \right) f_{0_\sigma} + \frac{1}{3} f_{0_\sigma}^3 + f_{0_\sigma} I_2(\tau) - \frac{1}{3} I_3(\tau) = 0. \tag{144}$$

The equation (144) for f_0 is usually hard to solve. For a Rayleigh wave equation with Dirichlet boundary conditions (see [6, 21, 22, 23]) only limited results are known when $I_3(\tau) \equiv 0$, and it is still an open problem when $I_3(\tau) \neq 0$. For monochromatic initial values (that is, for $\phi(x) = a_n \sin(nx)$ and $\psi(x) = b_n \sin(nx)$) explicit approximations of the solution are found (see [6, 21]). The behaviour of the solutions have been determined for $t \rightarrow \infty$ if the initial values are symmetric around the midpoint $x = \frac{\pi}{2}$ (see [22]), or if there is no initial displacement (see [23]). It turns out that $I_3(\tau)$ is identically equal to zero for these three cases (monochromatic initial values, symmetry around the midpoint, and no initial displacement). For the Rayleigh wave equation with boundary damping we will now show for what type of initial values the integral $I_3(\tau)$ is equal to zero. For that reason we study I_n (given by (142)) by making use of (144). The analysis as presented in [20] will now be followed partly. We

$$\begin{aligned}
\frac{dI_n}{d\tau} &= \frac{d}{d\tau} \left(\frac{1}{4\pi} \int_0^{4\pi} f_{0_\theta}^n(-\theta, \tau) d\theta \right) = \frac{1}{4\pi} \int_0^{4\pi} n f_{0_\theta}^{n-1}(-\theta, \tau) f_{0_{\theta\tau}}^n(-\theta, \tau) d\theta \\
&= \frac{n}{4\pi} \int_0^{4\pi} f_{0_\theta}^{n-1}(-\theta, \tau) f_{0_{\theta\tau}}(-\theta, \tau) d\theta \\
&= \frac{n}{4\pi} \int_0^{4\pi} f_{0_\theta}^{n-1}(-\theta, \tau) \cdot \frac{1}{2} \cdot \left(f_{0_\theta} \left[1 - \frac{2\alpha}{\pi} - I_2 \right] - \frac{1}{3} f_{0_\theta}^3 + \frac{1}{3} I_3 \right) d\theta \\
&= \frac{n}{2} \left(I_n \left[1 - \frac{2\alpha}{\pi} - I_2 \right] - \frac{1}{3} I_{n+2} + \frac{1}{3} I_{n-1} I_3 \right). \tag{145}
\end{aligned}$$

From this system of infinitely many ordinary differential equations for I_n it can readily be deduced that the integral $I_3(\tau)$ is identically equal to zero if and only if $I_{2n+1}(0) = 0$ for all $n \in \mathbb{N}^+$, or equivalently (by noticing that $f_{0_\sigma}(\sigma, 0) = \phi'(\sigma) - \psi(\sigma)$)

$$\int_0^{4\pi} (\phi'(\sigma) - \psi(\sigma))^{2n+1} d\sigma = 0 \quad \text{for all } n \in \mathbb{N}^+. \tag{146}$$

So, if (146) is satisfied then f_0 has to satisfy $2f_{0_{\sigma\tau}} + (\frac{2\alpha}{\pi} - 1 + I_2)f_{0_\sigma} + \frac{1}{3}f_{0_\sigma}^3 = 0$ else f_0 has to satisfy (144). It can be verified that (146) is satisfied for instance for monochromatic initial values, that is, for $\phi(x) = a_n \sin((n - \frac{1}{2})x)$ and $\psi(x) = b_n \sin((n - \frac{1}{2})x)$. In this paper we will study (140) for monochromatic initial values in detail. Now to solve (140) we use a procedure originally developed in [21]. First it is assumed in [21] that $I_2(\tau)$ is constant, and then (144) is solved. In the so-obtained solution all constants of integration are replaced by arbitrary functions in τ . In this way the solution of (144) take the form

$$f_{0_\sigma}(\sigma, \tau) = \frac{r(\tau)(\phi'(\sigma) - \psi(\sigma))}{\sqrt{1 + s(\tau)(\phi'(\sigma) - \psi(\sigma))^2}}, \tag{147}$$

where r and s will be determined later on. By substituting (147) into (140) we obtain

$$2r'(\tau) + \left(\frac{2}{\pi} \alpha - 1 + C(\tau) \right) r(\tau) = 0, \quad \tau > 0, \tag{148}$$

$$s'(\tau) - \frac{1}{3} r^2(\tau) = 0, \quad \tau > 0, \tag{149}$$

where $r(0) = \frac{1}{2}, s(0) = 0$, and where

$$C(\tau) = \frac{r^2(\tau)}{s(\tau)} \left(1 - \frac{1}{4\pi} \int_0^{4\pi} \frac{1}{1 + s(\tau)h^2(-\xi)} d\xi \right) \tag{150}$$

with $h(\sigma) = \phi'(\sigma) - \psi(\sigma)$. From (149) and (150) it follows for $\tau > 0$ that $s(\tau) > 0$ and $C(\tau) > 0$. It then follows from (148) for $\alpha \geq \frac{\pi}{2}$ that $r(\tau)$ decays exponentially. And so, for $\alpha \geq \frac{\pi}{2}$ f_0 (see (147)) and v_0 decay to zero for $t \rightarrow \infty$. In the further analysis it will now be assumed that $\phi(x) = a_n \sin((n - \frac{1}{2})x)$ and $\psi(x) = b_n \sin((n - \frac{1}{2})x)$, and so, $h(\sigma) = a_n(n - \frac{1}{2}) \cos((n - \frac{1}{2})\sigma) - b_n \sin((n - \frac{1}{2})\sigma)$. For these initial values the integral in (150) can be calculated explicitly, and so also $C(\tau)$. The equation (148) then becomes

$$-2 \frac{r'(\tau)}{r(\tau)} + 1 - \frac{2}{\pi} \alpha = 3 \frac{s'(\tau)}{s(\tau)} \left(\frac{\sqrt{1 + 2s(\tau)A_n^2} - 1}{\sqrt{1 + 2s(\tau)A_n^2}} \right), \tag{151}$$

where $A_n = \left(\frac{(a_n(n - \frac{1}{2}))^2 + b_n^2}{2} \right)^{1/2}$. The function f_0 can now be calculated from (147)-(151) yielding

$$f_0(\sigma, \tau) = \frac{\sqrt{3}}{n - \frac{1}{2}} \sqrt{\frac{s'(\tau)}{s(\tau)}} \arcsin \left(\sqrt{\frac{2s(\tau)A_n^2}{1 + 2s(\tau)A_n^2}} \sin \left(\beta + \left(n - \frac{1}{2} \right) \sigma \right) \right) + k(\tau), \tag{152}$$

$$v_0(x, t, \tau) = \frac{\sqrt{3}}{n - \frac{1}{2}} \sqrt{\frac{s'(\tau)}{s(\tau)}} \left[\arcsin \left(\sqrt{\frac{2s(\tau)A_n^2}{1 + 2s(\tau)A_n^2}} \sin \left(\beta + \left(n - \frac{1}{2} \right) (x - t) \right) \right) - \arcsin \left(\sqrt{\frac{2s(\tau)A_n^2}{1 + 2s(\tau)A_n^2}} \sin \left(\beta - \left(n - \frac{1}{2} \right) (x + t) \right) \right) \right], \quad (153)$$

where k is an arbitrary function in τ and satisfies $k(0) = 0$, where β is defined by

$$\cos(\beta) = \frac{a_n(n - \frac{1}{2})}{\sqrt{(a_n(n - \frac{1}{2}))^2 + b_n^2}}, \quad \sin(\beta) = \frac{b_n}{\sqrt{(a_n(n - \frac{1}{2}))^2 + b_n^2}}, \quad (154)$$

and where $s(\tau)$ will be determined in the next subsections for the cases: (i) $0 < \alpha < \frac{\pi}{2}$, and (ii) $\alpha \geq \frac{\pi}{2}$.

4.2.1 The case $0 < \alpha < \frac{\pi}{2}$.

By integrating both sides of (151) with respect to τ and by making use of (149) it follows that

$$s'(\tau) = \frac{2^4}{3} \left(\sqrt{1 + 2s(\tau)A_n^2} + 1 \right)^{-6} e^{(1-2\alpha/\pi)\tau}. \quad (155)$$

By putting $m(\tau) = \sqrt{1 + 2s(\tau)A_n^2} + 1$ the solution of (155) is given by

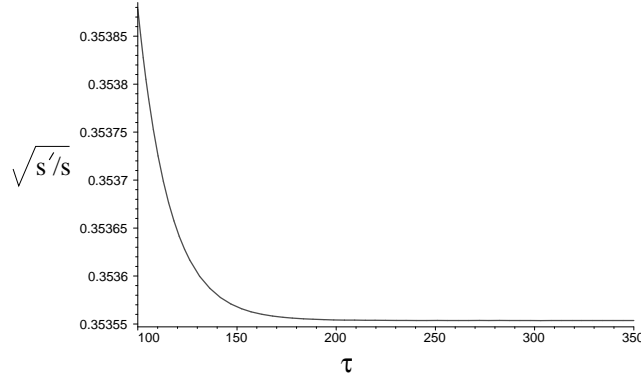


Figure 3. $\sqrt{s'/s}$ vs time (τ) for $a_n = 0.2, b_n = 0.02, n = 1, \alpha = \frac{\pi}{4}$.

$$s(\tau) = \frac{1}{2A_n^2} \left(m^2(\tau) - 2m(\tau) \right), \quad (156)$$

where m satisfies

$$m^8(\tau) - \frac{8}{7}m^7(\tau) = \frac{2^7}{3} \frac{A_n^2}{1 - \frac{2}{\pi}\alpha} \left(e^{(1-\frac{2}{\pi}\alpha)\tau} - 1 \right) + \frac{3}{7}2^8, \quad (157)$$

with $m(0) = 2$. Next, by defining

$$\mu(s) = \frac{1}{4\pi} \int_0^{4\pi} \frac{h^2(-\xi, 0)}{1 + sh^2(-\xi, 0)} d\xi \quad (158)$$

it also follows from (148)-(150) that

$$s'' + \left(\frac{2}{\pi}\alpha - 1 + 3s'\mu(s) \right) s' = 0. \quad (159)$$

$$s'(\tau) = \frac{2^{-2}}{3} e^{(1-\frac{2}{\pi}\alpha)\tau} e^{-3 \int_0^{s(\tau)} \mu(y) dy} \geq 0. \quad (160)$$

From $s(0) = 0$ it can be deduced that s is strictly positive for $\tau > 0$. From (158) we should note that

$$\mu(s) \leq \frac{1}{4\pi} \int_0^{4\pi} f_{0\xi}^2(-\xi, 0) d\xi = R. \quad (161)$$

It follows from (161) and by solving the inequality (160) that for $R > 0$

$$s(\tau) \geq \frac{1}{3R} \ln \left(\frac{3R}{12(1-\frac{2}{\pi}\alpha)} \left(e^{(1-\frac{2}{\pi}\alpha)\tau} - 1 \right) + 1 \right), \quad (162)$$

and that for $R = 0$

$$s(\tau) = \frac{1}{12(1-\frac{2}{\pi}\alpha)} \left(e^{(1-\frac{2}{\pi}\alpha)\tau} - 1 \right). \quad (163)$$

Hence it follows from (162) and (163) that $s(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$, and so $m(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$. It then follows from (156), (157), and (160) that (see also Figure 3)

$$\lim_{\tau \rightarrow \infty} \frac{s'(\tau)}{s(\tau)} = \frac{1 - \frac{2}{\pi}\alpha}{4}. \quad (164)$$

So, it can be seen from (153) and (164) that v_0 will tend to a standing triangular wave with amplitude $\pi\sqrt{3(1-2\alpha/\pi)}/(2(n-1/2))$ and period $2\pi/(n-1/2)$ as $\tau \rightarrow \infty$.

4.2.2 The case $\alpha \geq \frac{\pi}{2}$.

For $\alpha = \frac{\pi}{2}$ it follows from (155) that

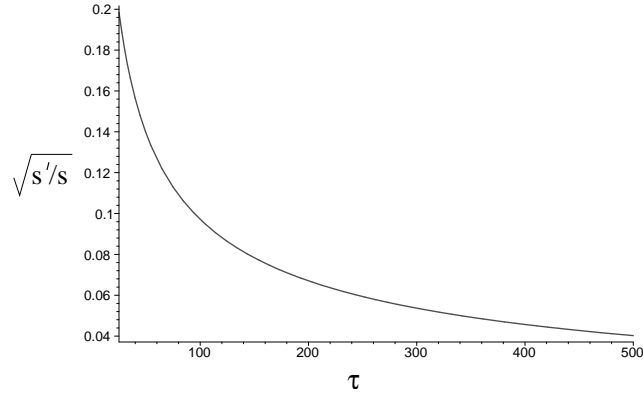


Figure 4. $\sqrt{s'/s}$ vs time (τ) for $a_n = 0.2, b_n = 0.02, n = 1, \alpha = \frac{\pi}{2}$.

$$m(\tau)^8 - \frac{8}{7}m(\tau)^7 = A_n^2 \frac{2^7}{3} \tau + \frac{3}{7}2^8, \quad (165)$$

and again using (161) and by solving inequality (160) it follows that

$$s(\tau) \geq \frac{1}{3R} \ln \left(1 + \frac{R}{4} \tau \right). \quad (166)$$

$$\lim_{\tau \rightarrow \infty} \frac{s'(\tau)}{s(\tau)} = \frac{2^5}{3} \frac{A_n^2}{m(\tau)^8 - 2m(\tau)^7} = 0. \quad (167)$$

It follows from (153) and (167) that the string vibrations will finally come to rest up to $O(\epsilon)$ as time t tends to infinity.

For $\alpha > \frac{\pi}{2}$ it follows that

$$\frac{s'(\tau)}{s(\tau)} = \frac{2^5}{3} \frac{A_n^2}{m(\tau)^7(m(\tau) - 2)} e^{(1 - \frac{2}{\pi}\alpha)\tau}. \quad (168)$$

Since s is a monotonic increasing and strictly positive function for $\tau > 0$ it follows from (155) that $m(\tau) > m(0) = 2$ for all $\tau > 0$. It then follows from (168) that

$$\lim_{\tau \rightarrow \infty} \frac{s'(\tau)}{s(\tau)} = 0. \quad (169)$$

We can conclude from (167) and (169) that the amplitude of oscillation of the string for $\alpha > \frac{\pi}{2}$ tends to zero up to order ϵ on a timescale of order ϵ^{-1} as time t tends to infinity. We depict this phenomenon for $\alpha = 4\pi/5$ in Figure 5. So far it has been shown that the solution of

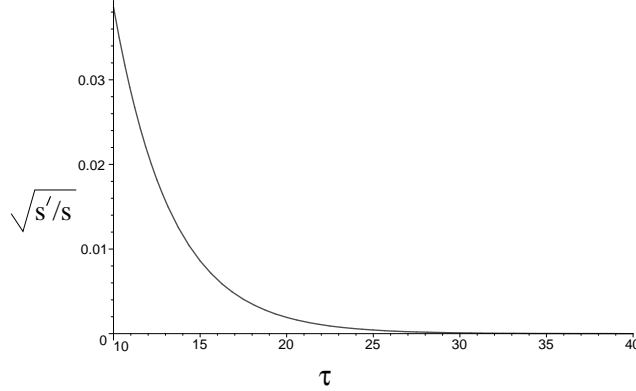


Figure 5. $\sqrt{s'/s}$ vs time (τ) for $a_n = 0.2, b_n = 0.02, n = 1, \alpha = 4\pi/5$.

the initial - boundary value problem will be damped to zero (up to $O(\epsilon)$) if the damping parameter satisfies $\alpha \geq \pi/2$. The analysis as carried out so far in this section is restricted to monochromatic initial conditions. However, from section 3 it can be deduced for arbitrary (but sufficiently smooth) initial values that the solutions will be bounded for $0 \leq \alpha < \frac{\pi}{2}$ as time t tends to infinity, and from section 4.1 and this section it can be concluded that for $\alpha \geq \frac{\pi}{2}$ all solutions will tend to zero (up to $O(\epsilon)$) as time t tends to infinity. As has been shown in this section it is difficult (or impossible) to solve (144) for arbitrary initial values. For that reason we will present some numerical results for arbitrary initial values in section 6.

After eliminating the terms in (135) that give rise to secular terms it follows from (135) that \tilde{w} is given by

$$\begin{aligned} \tilde{w}(\sigma, \xi, \tau) = & -\frac{1}{4} \int_{\sigma}^{\xi} f_{0\bar{\sigma}}(\bar{\sigma}, \tau) \int_{\bar{\sigma}}^{\xi} \left(g_{0\theta}^2(\theta, \tau) - \frac{1}{4\pi} \int_0^{4\pi} g_{0\delta}(\delta, \tau) d\delta \right) d\theta d\bar{\sigma} \\ & + \frac{1}{4} \int_{\sigma}^{\xi} \left(f_{0\theta}^2(\theta, \tau) - \frac{1}{4\pi} \int_0^{4\pi} f_{0\delta}(\delta, \tau) d\delta \right) (g_0(\xi, \tau) - g_0(\theta, \tau)) d\theta \\ & - \frac{1}{4} \alpha \int_{\sigma}^{\xi} \int_{\delta}^{\xi} \tilde{G}(\theta, \delta, \tau) d\theta d\delta + f_1(\sigma, \tau) + g_1(\xi, \tau), \end{aligned} \quad (170)$$

f_1, g_1 are still arbitrary. The undetermined behaviour of f_1, g_1 with respect to τ can be used to avoid secular terms in v_2 . It is, however, our goal to construct a function \bar{u} that satisfies the partial differential equation, the boundary conditions, and the initial values up to order ϵ^2 . For that reason the functions f_1 and g_1 are taken to be equal to their initial values $f_1(\sigma, 0)$ and $g_1(\xi, 0)$ respectively which can be determined from (128) and (129). So far we have constructed a formal approximation $\bar{u} = v_o + \epsilon v_1$ for u that satisfies the partial differential equation, the boundary conditions, and the initial values up to order ϵ^2 . The asymptotic validity of this formal approximation on a time-scale of $O(\frac{1}{\epsilon})$ will be proved in the next section.

5 The asymptotic validity of a formal approximation.

In [4, 5, 6] asymptotic theories have been presented for wave equations having Dirichlet boundary conditions and similar nonlinearities. In those papers the constructed formal approximations have been shown to be asymptotically valid, that is, the differences between the approximations and the exact solutions are of order ϵ on timescales of order ϵ^{-1} as $\epsilon \rightarrow 0$. It will be shown in this section that the formal approximation as constructed in section 4.2 is asymptotically valid on a timescale of order ϵ^{-1} . The approximation \bar{u} satisfies

$$\bar{u}_{tt} - \bar{u}_{xx} = \epsilon \left(\bar{u}_t - \frac{1}{3} \bar{u}_t^3 \right) + \epsilon^2 f(x, t, \epsilon), \quad 0 < x < \pi, \quad t > 0, \quad (171)$$

$$\bar{u}(0, t) = 0, \quad t \geq 0, \quad (172)$$

$$\bar{u}_x(\pi, t) + \epsilon \alpha \bar{u}_t(\pi, t) = \epsilon^2 g(t, \epsilon), \quad t \geq 0, \quad (173)$$

$$\bar{u}(x, 0) = \phi(x), \quad 0 < x < \pi, \quad (174)$$

$$\bar{u}_t(x, 0) = \psi(x) + \epsilon^2 v_{1\tau}(x, 0, 0), \quad 0 < x < \pi, \quad (175)$$

where f and g are given by

$$\begin{aligned} f(x, t, \epsilon) = & 2v_{1t\tau} + v_{0\tau\tau} + \epsilon v_{1\tau\tau} + (v_{1t} + v_{0\tau} \epsilon v_{1\tau})(v_{0t}^2 - \epsilon^2) + \epsilon v_{0t}(v_{1t} + v_{0\tau} + \epsilon v_{1\tau})^2 \\ & + \frac{1}{3} \epsilon^2 (v_{1t} + v_{0\tau} + \epsilon v_{1\tau})^3 \end{aligned} \quad (176)$$

and

$$g(t, \epsilon) = \alpha(v_{1t}(\pi, t) + v_{0\tau}(\pi, t) + \epsilon v_{1\tau}(\pi, t)) \quad (177)$$

respectively. To estimate the difference between the exact solution u and the approximation \bar{u} the theory as presented in section 3 will be used. It is convenient to move the inhomogeneous term g in the boundary condition at $x = \pi$ to the PDE so that the boundary condition at $x = \pi$ becomes homogeneous. The boundary condition at $x = 0$ remains homogeneous. For that reason the following transformation will be introduced

$$\bar{v}(x, t) = \bar{u}(x, t) + \epsilon^2 \sin(x) g(t; \epsilon). \quad (178)$$

Substituting (178) into the initial-boundary value problem (171)-(175) it follows that

$$\bar{v}_{tt} - \bar{v}_{xx} = \epsilon \left(\bar{v}_t - \frac{1}{3} \bar{v}_t^3 \right) + \epsilon^2 F(\bar{v}_t, x, t, \epsilon), \quad 0 < x < \pi, \quad t > 0, \quad (179)$$

$$\bar{v}(0, t) = 0, \quad t \geq 0, \quad (180)$$

$$\bar{v}_x(\pi, t) + \epsilon \alpha \bar{v}_t(\pi, t) = 0, \quad t \geq 0, \quad (181)$$

$$\bar{v}(x, 0) = \phi(x) + \epsilon^2 G(x), \quad 0 < x < \pi, \quad (182)$$

$$\bar{v}_t(x, 0) = \psi(x) + \epsilon^2 H(x), \quad 0 < x < \pi, \quad (183)$$

$F(\bar{v}_t, x, t, \epsilon) = f(x, t, \epsilon) + \sin(x)(g'' + g) - \epsilon(\sin(x)g')(1 - [3\bar{v}_t^2 - 3\bar{v}_t \sin(x)g' + (\sin(x)g')^2])$, $G(x) = \sin(x)g(0)$, and $H(x) = v_{1\tau}(x, 0, 0) + \sin(x)g'(0, \epsilon)$. It should be observed that F, G , and H are bounded functions in $W^{1,\infty}([0, T]; C^2[0, \pi])$. By putting the initial-boundary value problem (179) - (183) into an abstract Cauchy problem (see also section 3) and by making use of (58) and (62) it then easily follows that

$$\|z - \bar{z}\| \leq \epsilon^2 e^{\epsilon t} \|\Theta\| + \epsilon^2 \int_0^t e^{\epsilon(t-\tau)} \|\Omega\| d\tau, \quad (184)$$

where $z = (u, u_t)^T$, $\bar{z} = (\bar{v}, \bar{v}_t)^T$, and where $\Theta = (G, H)^T$, $\Omega = (0, F)^T$. Since F, G , and H are smooth functions it follows that there are positive constants M_0 and M_1 such that

$$\|z - \bar{z}\| \leq \epsilon^2 e^{\epsilon t} M_0 + \epsilon^2 \int_0^t e^{\epsilon(t-\tau)} M_1 d\tau. \quad (185)$$

For $0 \leq t \leq \frac{L}{\epsilon}$ (in which L is a positive constant independent of ϵ) it follows from (185) that

$$\|z - \bar{z}\| \leq \epsilon M_2 \quad (186)$$

for some positive constant M_2 . It should be observed that all calculations have been done in the function space \mathcal{H} . It also follows from (186) that

$$\max_{0 \leq x \leq \pi, 0 \leq t \leq \frac{L}{\epsilon}} |u(x, t) - \bar{v}(x, t)| < \epsilon \sqrt{\pi} M_2, \quad (187)$$

implying that $u - \bar{u} = O(\epsilon)$ in $C^0([0, \frac{L}{\epsilon}], [0, \pi])$. So far we have obtained the asymptotic validity of the formal approximations. Hence, it is easy to see that the first order approximations v_0 are also order ϵ approximations of the exact solutions on timescales of order ϵ^{-1} .

It will now be interesting to see the behaviour of the asymptotic approximation v_0 (as given by (143)) of the solution u of the initial-boundary value problem. For some values of the damping parameter α (that is, for $\alpha = \frac{\pi}{4}$, $\alpha = \frac{\pi}{2}$, and $\alpha = \frac{3\pi}{2}$) and $\epsilon = 0.1$ we have plotted v_0 in Figure 6-8. These figures can be compared with the figures which will be presented in the next section and which will be obtained by directly applying a numerical method to the initial-boundary value problem.

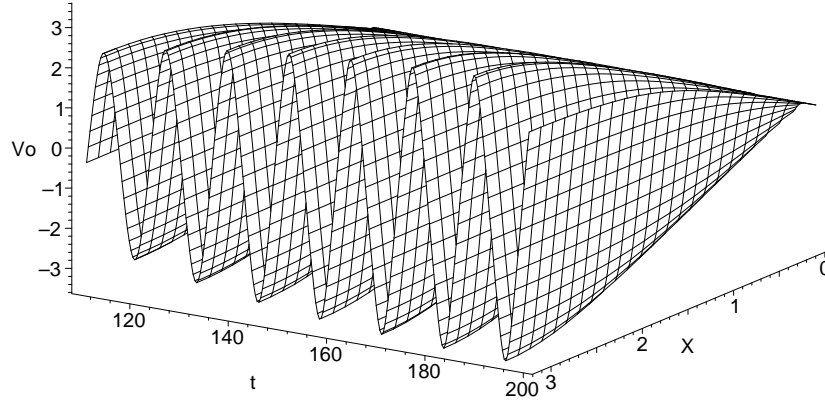


Figure 6. Displacement v_0 vs. the space variable x and the time variable t for $a_n = 0.2, b_n = 0.02, n = 1, \alpha = \pi/4, \epsilon = 0.1$.

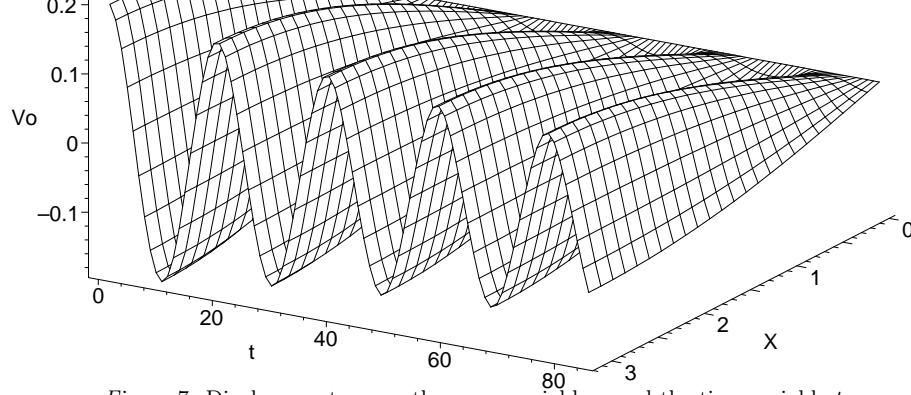


Figure 7. Displacement v_0 vs. the space variable x and the time variable t for $a_n = 0.2, b_n = 0.02, n = 1, \alpha = \pi/2, \epsilon = 0.1$.

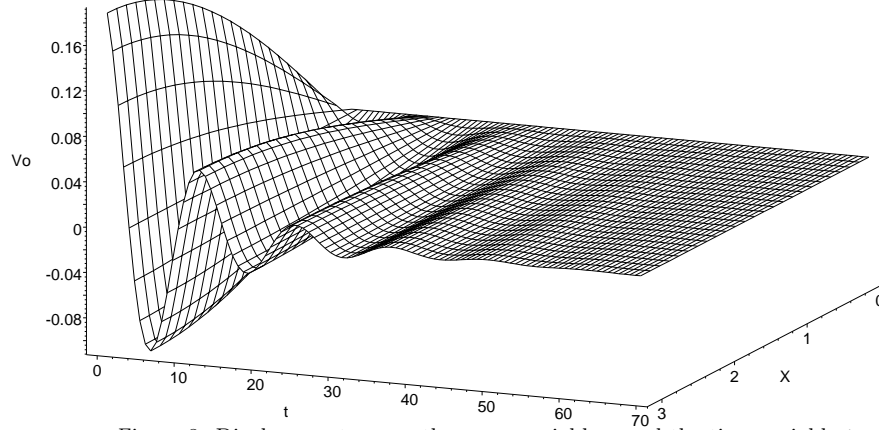


Figure 8. Displacement v_0 vs. the space variable x and the time variable t for $a_n = 0.2, b_n = 0.02, n = 1, \alpha = 3\pi/2, \epsilon = 0.1$.

6 Numerical Results.

For various values of the damping parameter α we will present in this section numerical approximations of the solution of the initial-boundary value problem (14)-(18). The numerical results will confirm the analytical results as obtained in the previous sections. Moreover, numerical approximations of the solution(s) are obtained for initial-boundary value problems with initial values which could not be treated in section 4.2 because of the complicated equation which had to be solved (see (144) with $I_3(\tau) \neq 0$). To solve the initial-boundary value problem (14)-(18) numerically and in a convenient way the second order PDE (14) is rewritten in a hyperbolic system of two first order PDEs by introducing

$$v(x, t) = \frac{1}{2}(u_t(x, t) - u_x(x, t)), \quad 0 < x < \pi, \quad t > 0, \quad (188)$$

$$w(x, t) = \frac{1}{2}(u_t(x, t) + u_x(x, t)), \quad 0 < x < \pi, \quad t > 0. \quad (189)$$

The initial-boundary value problem (14)-(18) then becomes

$$v_t(x, t) + v_x(x, t) = \frac{\epsilon}{2}f(v(x, t) + w(x, t)), \quad 0 < x < \pi, \quad t > 0, \quad (190)$$

$$w_t(x, t) - w_x(x, t) = \frac{\epsilon}{2}f(v(x, t) + w(x, t)), \quad 0 < x < \pi, \quad t > 0, \quad (191)$$

$$v(\pi, t) = \frac{1 - \epsilon\alpha}{1 + \epsilon\alpha} w(\pi, t), \quad t \geq 0, \quad (193)$$

$$w(x, 0) = \frac{1}{2}(\psi(x) + \phi'(x)), \quad 0 < x < \pi, \quad (194)$$

$$v(x, 0) = \frac{1}{2}(\psi(x) - \phi'(x)), \quad 0 < x < \pi, \quad (195)$$

where $f(v(x, t) + w(x, t)) = v(x, t) + w(x, t) - \frac{1}{3}(v(x, t) + w(x, t))^3$. From (188) and (189), and from the boundary condition $u(0, t) = 0$ it follows that

$$u(x, t) = \int_0^x (w(y, t) - v(y, t)) dy. \quad (196)$$

Using an "upwind scheme" the initial-boundary value problem (14)-(18) is solved numerically. As long as the space discretization is larger than the time discretization this numerical method will be stable. As usual the space discretizations for $w_x(j\Delta x, n\Delta t)$ and $v_x(j\Delta x, n\Delta t)$, and the time discretizations for $w_t(j\Delta x, n\Delta t)$ and $v_t(j\Delta x, n\Delta t)$ are given by $\frac{w_{j+1}^n - w_j^n}{\Delta x}$, $\frac{v_j^n - v_{j-1}^n}{\Delta x}$, $\frac{w_j^{n+1} - w_j^n}{\Delta t}$, and $\frac{v_j^{n+1} - v_j^n}{\Delta t}$ respectively. In the Figures 9-12 the energy $E(t) = \int_0^\pi (u_t^2 + u_x^2) dx$ is approximated, and in the Figure 13 -16 the solution is approximated for various values of α and for various initial values. The numerical results as given in these figures confirm the analytically obtained results as presented in the previous sections.

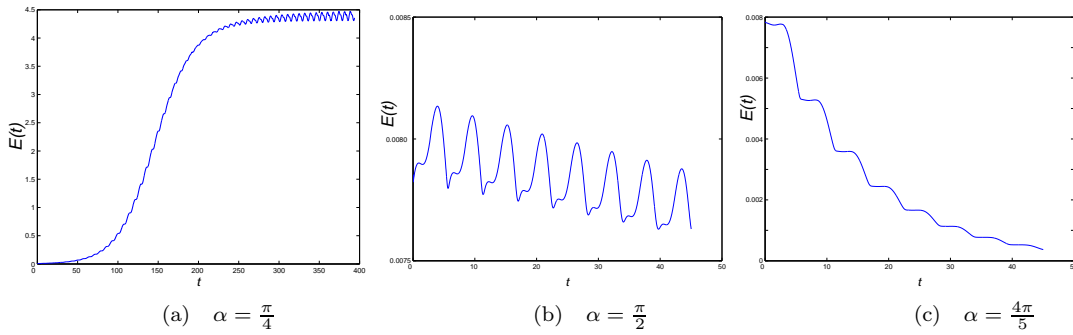


Figure 9. Energy $E(t)$ (vertical) vs. time t (horizontal) for $\phi(x) = 0.1 \sin(0.5x)$, $\psi(x) = 0.05 \sin(0.5x)$, and $\epsilon = 0.1$.

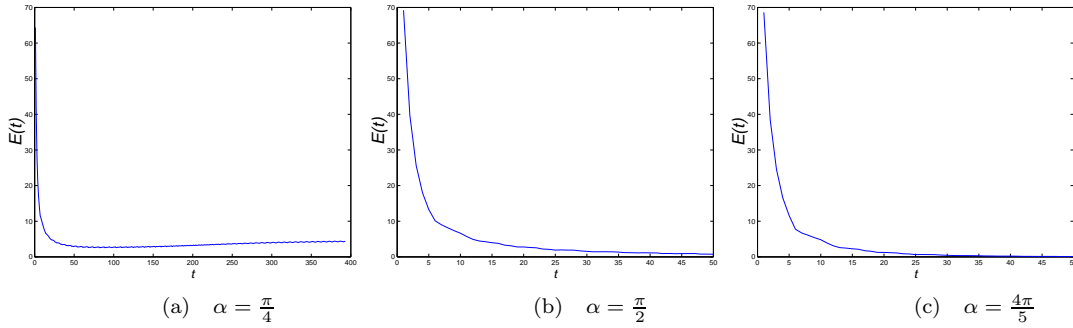


Figure 10. Energy $E(t)$ (vertical) vs. time t (horizontal) for $\phi(x) = 2.5 \sin(3.5x)$, $\psi(x) = 1.5 \sin(3.5x)$, and $\epsilon = 0.1$.

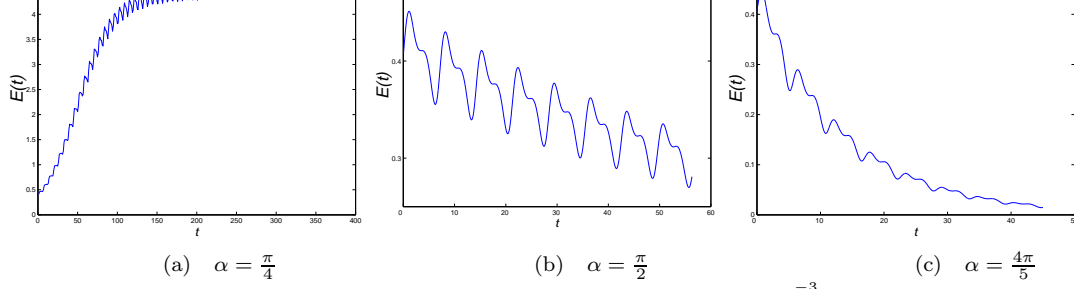


Figure 11. Energy $E(t)$ (vertical) vs. time t (horizontal) for $\phi(x) = 0.01x^3e^{\frac{-3}{\pi}x}$, $\psi(x) = 0.2x(\pi - x)$, and $\epsilon = 0.1$.

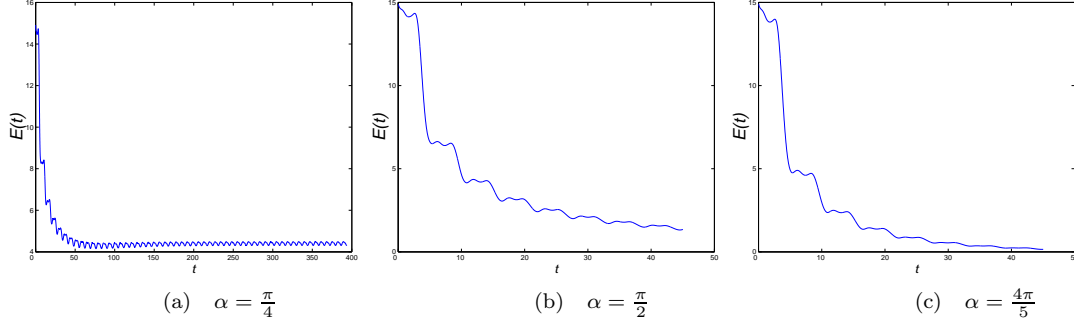


Figure 12. Energy $E(t)$ (vertical) vs. time t (horizontal) for $\phi(x) = \sin(x) + x$, $\psi(x) = x(\pi - x)$, and $\epsilon = 0.1$.

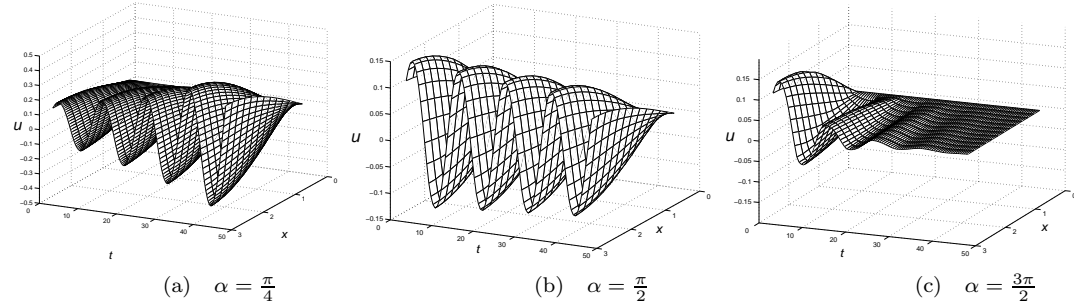


Figure 13. Displacement u vs. x and t for $\phi(x) = 0.1 \sin(0.5x)$, $\psi(x) = 0.05 \sin(0.5x)$, and $\epsilon = 0.1$.

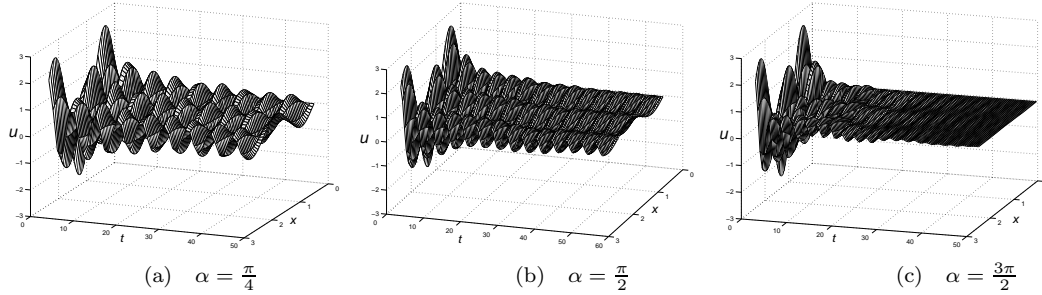


Figure 14. Displacement u vs. x and t for $\phi(x) = 2.5 \sin(3.5x)$, $\psi(x) = 0.05 \sin(3.5x)$, and $\epsilon = 0.1$.

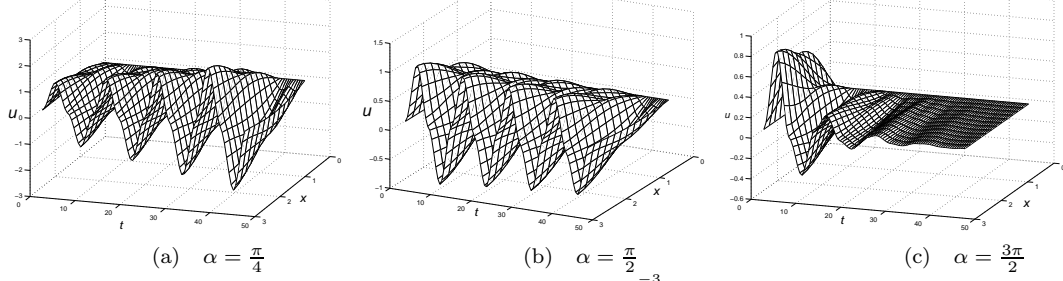


Figure 15. Displacement u vs. x and t for $\phi(x) = 0.01x^3e^{\frac{-3}{\pi}x}$, $\psi(x) = 0.2x(\pi - x)$, and $\epsilon = 0.1$.

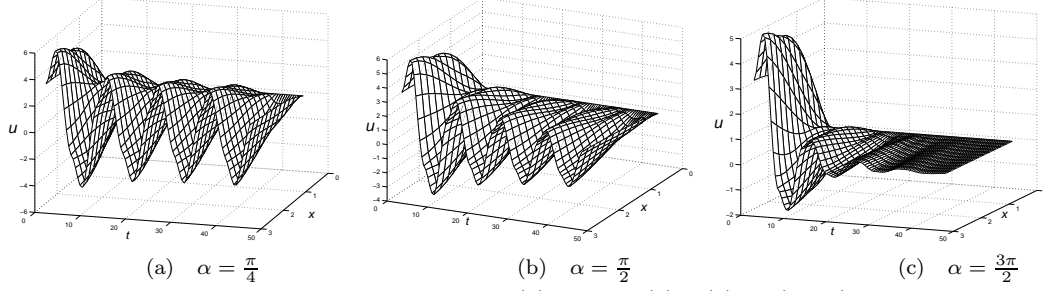


Figure 16. Displacement u vs. x and t for $\phi(x) = x + \sin(x)$, $\psi(x) = x(\pi - x)$, and $\epsilon = 0.1$.

7 Conclusions

In this paper an initial-boundary value problem for a weakly nonlinear wave equation (a Rayleigh equation) has been studied. The problem can be considered to be a simple model to describe the galloping oscillations of overhead power transmission lines in a windfield. One end of the transmission line is assumed to be fixed, whereas the other end of the line is assumed to be attached to a dashpot system. Using a semi-group approach it has been shown that the initial-boundary value problem is well-posed for all time t . Moreover, it has been shown that the solution is bounded for all time t . Formal asymptotic approximations of the exact solution have been constructed by using a two-timescales perturbation method. Also it has been shown that the formal approximations are indeed order ϵ asymptotic approximations (as $\epsilon \rightarrow 0$) of the solution(s) for $0 \leq x \leq \pi$ and $0 \leq t \leq L\epsilon^{-1}$, in which L is an ϵ -independent positive constant. For the damping parameter α larger than or equal to $\frac{\pi}{2}$ it has been shown that all solutions will tend to zero as time t tends to infinity. For $0 < \alpha < \frac{\pi}{2}$ all solution will be bounded. In particular it has been shown for monochromatic initial values and for $0 < \alpha < \frac{\pi}{2}$ that the solution will tend to a standing triangular wave with amplitude $\pi\sqrt{3(1 - 2\alpha/\pi)/(2(n - 1/2))}$ and period $2\pi/(n - 1/2)$ as time t tends to infinity. For more complicated initial values numerical approximations have been constructed by using a numerical method (an upwind scheme). The numerical results confirm the aforementioned analytical results.

Acknowledgements

The authors wish to thank Y. A. Erlangga for making the numerical scheme available in Matlab.

- [1] Keller, J. B. and Kogelman, S., 'Asymptotic Solutions of Initial Value Problems for Nonlinear Partial Differential Equations', *SIAM Journal on Applied Mathematics* **18**, 1970, 748 - 758.
- [2] Darmawijoyo, and van Horssen W. T., 'On the weakly Damped Vibrations of A String Attached to A Spring-Mass-Dashpot System', 2002 to appear in Journal of Vibration and Control.
- [3] Morgül, Ö., Rao, B. P., and Conrad, F., 'On the stabilization of a cable with a Tip Mass', *IEEE Transactions on automatic control* **39**, no.10, 1994, 2140-2145.
- [4] Krol, M.S., 'On a Galerkin - Averaging Method for Weakly nonlinear Wave Equations', *Mathematical Methods in the Applied Sciences* **11**, 1989, 649 - 664.
- [5] van Horssen, W. T. and van der Burgh, A. H. P., 'On initial-boundary value problems for weakly semi-linear telegraph equations. Asymptotic theory and application', *SIAM Journal on Applied Mathematics* **48**, 1988, 719-736.
- [6] van Horssen, W. T., 'An asymptotic Theory for a class of Initial - Boundary Value Problems for Weakly Nonlinear Wave Equations with an application to a model of the Galloping Oscillations of Overhead Transmission Lines', *SIAM Journal on Applied Mathematics* **48**, 1988, 1227 - 1243.
- [7] Boertjens, G. J. and van Horssen, W. T., 'On Mode Interactions for a Weakly Nonlinear Beam Equation', *Nonlinear Dynamics* **17**, 1998, 23 - 40.
- [8] Boertjens, G. J. and van Horssen, W. T., 'An Asymptotic Theory for A weakly Nonlinear Beam Equation With A Quadratic Perturbation', *SIAM Journal on Applied Mathematics* **60**, 2000, 602-632.
- [9] Conrad, F. and Morgül, Ö., 'On the Stabilization of a Flexible Beam with a Tip Mass', *SIAM Journal on Control and Optimization* **36**, 1998, 1962 - 1986.
- [10] Castro, C. and Zuazua, E., 'Boundary Controllability of a Hybrid System Consisting in two Flexible Beams Connected by a Point Mass', *SIAM Journal on Control Optimization* **36**, 1998, 1576 - 1595.
- [11] Durant, N.J., 'Stress in a dynamically loaded helical spring', *The Quarterly Journal of Mechanics and Applied Mathematics* **xiii**, Pt.2, 1960, 251-256.
- [12] Cox, S. and Zuazua, E., 'The Rate at which Energy Decays in a String Damped at one End', *Indiana University Mathematics Journal* **44**, 1995, 545 - 573.
- [13] Rao, B. P., 'Decay Estimates of Solutions for a Hybrid System of Flexible Structures', *European Journal of Applied Mathematics* **4**, 1993, 303 - 319.
- [14] Rao, B. P., 'Uniform Stabilization of a Hybrid System of Elasticity', *SIAM Journal on Control Optimization* **33**, 1995, 440 - 454.
- [15] Nayfeh, A. H., *Perturbation Methods*, Wiley, New York, 1973.
- [16] Kevorkian, J. and Cole, J. D., *Multiple Scale and Singular Perturbation Methods*, Springer-Verlag, New York, 1996.
- [17] Holmes, M. H., *Introduction to Perturbation Methods*, Springer-Verlag, New York, 1995.

- [19] Wang, H., Elcrat, A. R., and Egbert, R. I., 'Modelling and Boundary Control of Conductor Galloping', *Journal of Sound and Vibration* **161**, 1993, 301 - 315.
- [20] van Horssen, W. T., 'Asymptotics for a class of weakly nonlinear wave equations with applications to some problems', proc. of the first world congress of nonlinear analysts, edited by V. Lakshmikantham, Tampa, Florida 19-26 August 1992, pp.945-952.
- [21] Chikwendu, S.C., Kevorkian, J., 'A perturbation method for hyperbolic equations with small nonlinearities', *SIAM Journal on Applied Mathematics* **22**, 1972, 235 - 258.
- [22] Hall, W. S., 'The Rayleigh wave equation, an analysis', *Nonlinear Analysis* **2**, no.2, 1978, 129 - 156.
- [23] Lardner, R. W., 'Asymptotic solutions of nonlinear wave equations using the methods of averaging and two-timing', *Quarterly of Applied Mathematics* **35**, 1977, 225 - 238.
- [24] Showalter R.E., *Monotone operators in Banach spaces and nonlinear partial differential equations*, Mathematical surveys and monographs Vol.**49**, American Mathematical Society, 1997.
- [25] Belleni-Morante A., McBride A. C., *Applied Nonlinear Semigroups*, Mathematical Methods in Practice, John Wiley&Sons, New York, 1998.
- [26] Da Prato G., Sinestrari E., 'Differential operators with non-dense domain', *Ann. Scuola Norm Sup. Pisa Cl. Sci.* **2**, no.14, 1987, 285 - 344.
- [27] Renardy M., Rogers R. C., *An introduction to partial differential equations*, Text in applied Math.no.**13**, Springer-Verlag, New York, 1993.