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# On the Parametric Excitation of Some Nonlinear Aeroelastic Oscillators 

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#### Abstract

In this chapter the parametric excitation of two one-degree-of-freedom nonlinear aeroelastic oscillators in cross-flow is considered. This is relevant for understanding of aeroelastic oscillations of bridge stay cables induced by bridge deck or bridge tower oscillations, see Da Costa et al. [12]. In particular the parametric excitation for a plunge oscillator and a seesaw oscillator are discussed. The following model equation for the parametric excitation of the aeroelastic oscillators is considered: $$
\begin{equation*} \ddot{z}+2 \beta \dot{z}+(1-\kappa \cos (\omega t)) z=F(z, \dot{z}, U) . \tag{0.1} \end{equation*}
$$

Here, for the plunge oscillator, $z$ denotes the vertical displacement of a mass-springdamper system, and for the seesaw oscillator, $z$ denotes the rotation of a seesaw structure around the hinge axis. $F$ represents the nonlinear aeroelastic force which depends on $z, \dot{z}$ and wind velocity $U$. Assuming $F$, the coefficients of parametric excitation $\kappa$ and structural damping $\beta$ to be small, the averaging method can be applied to study equation (0.1). Note that equation (0.1) is a nonlinear Mathieu equation. Without the parametric excitation one typically finds an aerodynamic instability for a critical wind velocity above which finite amplitude, periodic oscillations result. The parametric excitation complicates this simple picture, especially for the seesaw oscillator. Depending on the ratio of $\kappa$ and $\beta$ a critical wind velocity may still exist. For some cases though, increasing the wind velocity above the critical wind velocity, a restabilization of the trivial solution may occur. Next to the familiar periodic solutions


[^0]

Figure 1: Schematic sketch of the structure of the plunge oscillator.
also solutions with periodically modulated amplitudes and phases are obtained. Criteria for the stability of the trivial solution, the existence and the stability of various nontrivial (periodic) solutions and their bifurcations are given.

Keywords: Aeroelasticity, nonlinear oscillation, parametric excitation, averaging, bifurcation analysis.

## 1 Introduction

This chapter is concerned with the parametric excitation of two aeroelastic oscillators, a plunge oscillator and a seesaw oscillator respectively, exposed to a homogeneous and uniform wind flow. A schematic sketch of the plunge oscillator and the seesaw oscillator is given in Figure 1 and Figure 2, respectively. The first oscillator is a spring supported cylinder with linear damping. It is restricted to oscillate perpendicular to the flow direction. If the cylinder has a non-circular cross section and is exposed to a homogeneous and uniform wind flow, self-excited galloping oscillations may arise, see Blevins, Parkinson, Haaker and van der Burgh $[9,10,3]$. Here we assume that the spring supporting the cylinder has a periodically varying stiffness, causing a parametric excitation, see Figure 1. The second oscillator is a seesaw like structure consisting of a rigid bar hinged around an axis. The bar holds at its right end a cylinder. On the other end a counter weight is fixed balancing the cylinder with respect to the hinge axis. A pendulum weight fixed to the bar provides for a restoring moment. Haaker and van der Burgh [3] modelled and analyzed the equation of motion of the seesaw oscillator for low velocity and constant linear stiffness. The wind forces then act as perturbation on the linear Hamiltonian system modelling the unforced oscillations of the seesaw structure for small amplitudes. Lumbantobing and Haaker [5, 6] considered the aeroelastic oscillations of seesaw type oscillation under strong wind conditions. Van Oudheusden [11] investigated the galloping oscillations with a single rotational


Figure 2: Schematic sketch of the structure of the seesaw oscillator with an electrical rotor.
degree of freedom under the combined effect of both viscous and frictional damping. He studied how the additional effect of even slight amounts of frictional damping affects the galloping curve. He provided results finding from wind tunnel experiments confirm the major findings of analysis. Again self-excited galloping oscillations may arise. Here we assume that an electrical rotor is mounted on the hinge axis and connected to the pendulum weight via a string. The rotor causes the pendulum weight to slide periodically up and down the arm holding the pendulum, see Figure 2. In the model equation this sliding pendulum weight causes a parametric excitation, see for example van der Burgh [1].
The main purpose of this chapter is to study the influence of a parametric excitation on the well-known aeroelastic behaviour of the plunge and the seesaw oscillators. Such a study is relevant for the understanding of the aeroelastic behaviour of coupled structural elements exposed to wind flow. For example, Da Costa, et al. [12] studied the excitation of bridge stay-cable oscillations due to deck and/or towers motions. As a model equation for the first mode for a vertical taut string they found a nonlinear Mathieu equation, with cubic nonlinear stiffness effect. They found that the most dangerous situations arise when the frequency of the parametric excitation is close to one or two times the natural frequency. Here we take into account both the nonlinear aeroelastic forces as well as the parametric excitation due to the prescribed cable anchorage oscillation. Tondl et al. [8] considered parametric excitations for general oscillators with special nonlinear damping. They applied the averaging method to study the the stability of the equilibrium solution. Lumbantobing and Haaker [7] considered the parametric excitation of a nonlinear aeroelastic seesaw oscillator. They applied the averaging method to study the behaviour of the trivial and the nontrivial solutions. Here we consider parametric excitation for a plunge oscillator and a seesaw oscillator, respectively. The model equation we obtain is a nonlinear Mathieu equation. To study the behaviour of the solutions of this equation we apply the averaging method.

This chapter is organized as follows. In section 2, the parametric excitation of the nonlinear aeroelastic plunge oscillator is considered. The analysis is started with the derivation of the model equation in section 2.1 and then followed by the analysis of the model equation in section 2.2. Also some phase portraits for the averaged equations are presented. In section 3, the parametric excitation of the nonlinear aeroelastic seesaw oscillator is considered. The derivation of the model equation for the aeroelastic response of the seesaw oscillator with parametric excitation is given in section 3.1. The analysis of this equation is presented in section 3.2. Again some phase portraits for the averaged equations are presented. In section 4, some conclusions will be given. Finally, the analytical proofs of Hopf bifurcation and Pitchfork bifurcation are shown in appendix 5.

## 2 Parametric excitation of a nonlinear aeroelastic plunge oscillator

In this section, the parametric excitation of a nonlinear aeroelastic plunge oscillator is considered. To model the aeroelastic force a quasi steady approximation is used, see the introduction of this thesis or Blevins [9]. As a model equation a nonlinear Mathieu equation is obtained. The analysis of this equation is presented, which is based on the averaging method.

### 2.1 Derivation of the model equation

In $[9,2,10]$, the equation of motion for the plunge oscillator is derived as follows

$$
\begin{equation*}
m \ddot{y}+b \dot{y}+k y=\frac{1}{2} \rho d l U^{2} C_{N}(\alpha) \tag{2.2}
\end{equation*}
$$

where $\rho, d, l, y, U$ and $C_{N}(\alpha)$ denote air density, a characteristic length of cross section, the length of the cylinder, the distance along the vertical is measured along the y axis which is pointing downward, wind velocity and aerodynamic coefficient curve, respectively. Furthermore, $\alpha$ denotes the so called angle of attack, which in the dynamic situation may be approximated through $\alpha=-\frac{\dot{y}}{U}$.
Here we assume a periodically varying stiffness according to $k=k_{0}-k_{1} \cos (\Omega t)$. We introduce the system parameters $\omega$ (frequency), $\epsilon$ (small parameter) and $\mu$ (reduced velocity) according to $\omega^{2}=\frac{k_{0}}{m}, \epsilon=\frac{\rho d^{2} l}{2 m}$ and $\mu=\frac{U}{\omega d}$. Note that $\epsilon$ may be assumed a small parameter due to air density $\rho$ being of the order $10^{-3}$. Defining a new damping coefficient $\beta$ and parametric excitation coefficient $a$ according to $\epsilon \beta=\frac{b}{2 m \omega}$ and $\epsilon a=\frac{k_{1}}{m \omega^{2}}$, respectively, and introducing new variables $x$ and $\tau$ according to $x=\frac{\omega y}{U}$ and $\tau=\omega t$, respectively then one gets the equation

$$
\begin{equation*}
x^{\prime \prime}+\left(1-\epsilon a \cos \left(\frac{\Omega}{\omega} \tau\right)\right) x=\epsilon\left(-2 \beta x^{\prime}+\mu C_{N}(\alpha)\right), \tag{2.3}
\end{equation*}
$$

with $\alpha=-x^{\text {s }}$ and " " denotes the differentiation with respect to $\tau$.
Finally we take for the aerodynamic coefficient curve $C_{N}$ a cubic polynomial $C_{N}=c_{1} \alpha+$ $c_{2} \alpha^{2}+c_{3} \alpha^{3}$, with $c_{1}<0, c_{3}>0$ (relevant for cylinders with nearly circular cross section, see for example Nigol \& Buchan [13]).
Then the model equation we obtain is

$$
\begin{equation*}
x^{\prime \prime}+\left(1-\epsilon a \cos \left(\frac{\Omega}{\omega} \tau\right)\right) x=\epsilon\left(-\left(2 \beta+c_{1} \mu\right) x^{\iota}+c_{2} \mu x^{〔 2}-c_{3} \mu x^{〔 3}\right) . \tag{2.4}
\end{equation*}
$$

This equation is a nonlinear Mathieu equation.

### 2.2 Analysis of the model equation

In this subsection an amplitude-phase transformation is applied to transform the system (2.4) to a suitable form for applying the averaging method.

We define amplitude $r$ and phase $\psi$ or alternatively euclidean coordinates $u$ and $v$ through

$$
\begin{align*}
x & =r(\tau) \cos (\tau+\psi(\tau))  \tag{2.5}\\
x^{‘} & =-r(\tau) \sin (\tau+\psi(\tau)) \tag{2.6}
\end{align*}
$$

or

$$
\begin{align*}
x & =u(\tau) \cos (\tau)+v(\tau) \sin (\tau)  \tag{2.7}\\
x^{\cdot} & =-u(\tau) \sin (\tau)+v(\tau) \cos (\tau) \tag{2.8}
\end{align*}
$$

Note that the following relation between the coordinate pairs holds $u=r \cos (\psi)$ and $v=-r \sin (\psi)$.
To obtain the averaged equations in $(r, \psi)$ or $(u, v)$ coordinates one substitutes expressions (2.5-2.6) or (2.6-2.7) into equation (2.4). To simplify the equations we transform $r \rightarrow \frac{\bar{r}}{\sqrt{c_{3}}}$, or equivalently $(u, v) \rightarrow\left(\frac{\bar{u}}{\sqrt{c_{3}}}, \frac{\bar{v}}{\sqrt{c_{3}}}\right)$. After setting $\frac{\Omega}{\omega}=2+\epsilon \delta$ with $\delta$ a detuning coefficient, and transforming the time $(2+\epsilon \delta) \tau \rightarrow 2 s$, then after neglecting the "bar", one obtains the following averaged equations for $\left(r^{\prime}, \psi^{\prime}\right)$ and for $\left(u^{\prime}, v^{\prime}\right)$, respectively $\left({ }^{\prime}=\frac{d}{d s}\right)$.

$$
\begin{align*}
r^{\prime} & =\epsilon\left(-\left(\frac{a}{4} \sin (2 \psi)+\beta+\frac{1}{2} c_{1} \mu\right) r-\frac{3}{8} \mu r^{3}\right)  \tag{2.9}\\
\psi^{\prime} & =\epsilon\left(-\frac{\delta}{2}-\frac{a}{4} \cos (2 \psi)\right) \tag{2.10}
\end{align*}
$$

and

$$
\begin{align*}
u^{\prime} & =\epsilon\left(-\left(\beta+\frac{1}{2} c_{1} \mu\right) u+\left(\frac{a}{4}-\frac{\delta}{2}\right) v-\frac{3}{8} \mu\left(u v^{2}+u^{3}\right)\right)  \tag{2.11}\\
v^{\prime} & =\epsilon\left(\left(\frac{\delta}{2}+\frac{a}{4}\right) u-\left(\beta+\frac{1}{2} c_{1} \mu\right) v-\frac{3}{8} \mu\left(u^{2} v+v^{3}\right)\right) . \tag{2.12}
\end{align*}
$$

Note that in this chapter we choose $\frac{\Omega}{\omega}=2+\epsilon \delta$, because the most dangerous situations arise when the frequency of the parametric excitation is close to two times the oscillator frequency, see Da Costa et al.[12] and Verhulst [14].
Before starting the analysis we show that both $a$ and $\delta$ may be assumed positive. Suppose a solution of equations $(2.9,2.10)$ is denoted as
$(r(s ; a, \delta), \psi(s ; a, \delta))$. Then, if in equations $(2.9,2.10) a$ is replaced by $-a$, it is readily shown that the solutions of this new equation follows from the solutions of the original equations as follows

$$
(r(s ;-a, \delta), \psi(s ;-a, \delta))=\left(r(s ; a, \delta), \psi(s ; a, \delta)+\frac{\pi}{2}\right)
$$

Similarly, one may show that

$$
(r(s ; a,-\delta), \psi(s ; a,-\delta))=\left(r(s ; a, \delta),-\psi(s ; a, \delta)+\frac{\pi}{2}\right)
$$

### 2.2.1 Stability analysis of the trivial solution

From equations (2.11-2.12), one obtains the eigenvalues evaluated at $(0,0)$ as follows $\lambda_{01}=-\frac{1}{2} c_{1} \mu-\beta-\frac{1}{4} \sqrt{a^{2}-4 \delta^{2}}$ and $\lambda_{02}=-\frac{1}{2} c_{1} \mu-\beta+\frac{1}{4} \sqrt{a^{2}-4 \delta^{2}}$.
Depending on the strength of the parametric excitation $a$ compared to detuning $\delta$ and structural damping $\beta$ one obtains the following cases:

1. In the case $\frac{2 \delta}{a}<1$, both eigenvalues evaluated at $(0,0)$ are real.

Note that $\lambda_{01}<\lambda_{02}$, so the equilibrium position is stable if $\lambda_{02}<0$. Even without wind, i.e. $\mu=0$, the equilibrium position may be unstable, depending on the sign of $-\beta+\frac{1}{4} \sqrt{a^{2}-4 \delta^{2}}$, that is,
(a) if $a^{2}>4 \delta^{2}+16 \beta^{2}$, i.e. if the parametric resonance is strong compared to detuning and damping, then the equilibrium position is a saddle for $\mu=0$. In that case a stable nontrivial critical point $\left(r_{2}, \psi_{2}\right)$ exists. Note that if $\mu$ is increased from zero, one finds that for $\mu=-\frac{2 \beta}{c_{1}}-\frac{1}{2 c_{1}} \sqrt{a^{2}-4 \delta^{2}}$, a pitchfork bifurcation occurs in which an unstable critical point $\left(r_{1}, \psi_{1}\right)$ is born. See Appendix 5 for a proof.
(b) if $a^{2}<4 \delta^{2}+16 \beta^{2}$, then the equilibrium solution is stable for $\mu=0$. On increasing $\mu$ from zero a critical flow velocity $\mu=-\frac{2 \beta}{c_{1}}+\frac{1}{2 c_{1}} \sqrt{a^{2}-4 \delta^{2}}$ is reached for which a pitchfork bifurcation occurs in which a stable critical point $\left(r_{2}, \psi_{2}\right)$ is born. Increasing $\mu$ further such that $\lambda_{01}=0$, one finds the second pitchfork bifurcation in which an unstable critical point $\left(r_{1}, \psi_{1}\right)$ is born.
2. In the case $\frac{2 \delta}{a}=1$, the stability of $(0,0)$ is determined by the sign of $-\frac{1}{2} c_{1} \mu-\beta$, i.e. a critical flow velocity $\mu_{c_{r}}=-\frac{2 \beta}{c_{1}}$ exists such that the equilibrium position is unstable for $\mu>\mu_{c_{r}}$. At $\mu=\mu_{c_{r}}$ a pitchfork bifurcation occurs indicating the loss of stability of the trivial solution and the creation of a stable nontrivial solution.
3. In the case $\frac{2 \delta}{a}>1$, both of the eigenvalues are complex. The equilibrium solution is stable if $\mu<\mu_{c_{r}}$. We see that for $\mu<\mu_{c_{r}},(0,0)$ is a stable focus. At $\mu=\mu_{c_{r}}$, the eigenvalues are purely imaginary which indicates that there exists a Hopf bifurcation leading to the creation of a stable limit cycle. See Appendix 5 for a proof. This limit cycle corresponds to a solution with periodically modulated amplitudes and phases for the original equation. For $\mu>\mu_{c_{r}}$, the equilibrium position becomes an unstable focus.

### 2.2.2 Nonlinear analysis

Note that equations (2.9-2.10) have the following so called rotational symmetry: $(r, \psi) \rightarrow$ $(r, \psi+\pi)$. This indicates that if $(r, \psi)$ is a solution of $(2.9-2.10)$ then $(r, \psi+\pi)$ is also a solution. In the phase portrait the solution $(r, \psi+\pi)$ is obtained by rotating the solution $(r, \psi)$ over $\pi$. In the sequel we only consider $0 \leq \psi \leq \pi$. Setting the right hand side of equation (2.10) to zero, one gets

$$
\begin{equation*}
\cos (2 \psi)=-\frac{2 \delta}{a} \tag{2.13}
\end{equation*}
$$

We consider again the three cases by extending the linear analysis.

1. In the case $\frac{2 \delta}{a}<1$ there are two solutions for equation (2.13), say $\psi_{1}$ and $\psi_{2}$. Assume that $\psi_{1}<\psi_{2}$ then $\sin \left(2 \psi_{1}\right)>0$ and $\sin \left(2 \psi_{2}\right)<0$.
From equation (2.9) one gets that $r=\left\{-\frac{8}{3 \mu}\left(\frac{a}{4} \sin (2 \psi)+\beta+\frac{1}{2} c_{1} \mu\right)\right\}^{\frac{1}{2}}$. From equation (2.13) one finds that $\sin (2 \psi)= \pm \frac{1}{a} \sqrt{a^{2}-4 \delta^{2}}$. For $\psi=\psi_{1}$ we have $r_{1}=$ $\left(\frac{8}{3 \mu} \lambda_{01}\right)^{\frac{1}{2}}$. The solution $\left(r_{1}, \psi_{1}\right)$ exists for $\mu>-\frac{2 \beta}{c_{1}}-\frac{1}{2 c_{1}} \sqrt{a^{2}-4 \delta^{2}}$. Its eigenvalues are $\lambda_{11}=\frac{1}{2} \sqrt{a^{2}-4 \delta^{2}}+2 \beta+c_{1} \mu$ and $\lambda_{12}=\frac{1}{2} \sqrt{a^{2}-4 \delta^{2}}>0$. We conclude that $\left(r_{1}, \psi_{1}\right)$ is always unstable. For $\psi=\psi_{2}$ we have $r_{2}=\left(\frac{8}{3 \mu} \lambda_{02}\right)^{\frac{1}{2}}$. The solution $\left(r_{2}, \psi_{2}\right)$ exists for $\mu>-\frac{2 \beta}{c_{1}}+\frac{1}{2 c_{1}} \sqrt{a^{2}-4 \delta^{2}}$. Its eigenvalues are $\lambda_{21}=-\frac{1}{2} \sqrt{a^{2}-4 \delta^{2}}+2 \beta+c_{1} \mu$ and $\lambda_{22}=$ $-\frac{1}{2} \sqrt{a^{2}-4 \delta^{2}}<0$. The eigenvalue $\lambda_{21}<0$ if and only if $\mu>-\frac{2 \beta}{c_{1}}+\frac{1}{2 c_{1}} \sqrt{a^{2}-4 \delta^{2}}$. From here we conclude that the solution $\left(r_{2}, \psi_{2}\right)$ is always stable.
2. In the case $\frac{2 \delta}{a}=1$ the solution of equation (2.13) is $\psi_{2}=\frac{\pi}{2}$. From equation (2.9) one gets $r=\left\{\frac{8}{3 \mu}\left(-\beta-\frac{1}{2} c_{1} \mu\right)\right\}^{\frac{1}{2}}$. The nontrivial solution is stable and exists for $\mu>\mu_{c_{r}}$.
3. In the case $\frac{2 \delta}{a}>1$ there is no nontrivial solution at all. Instead of this one finds a stable limit cycle born in the Hopf bifurcation for $\mu=\mu_{c_{r}}$.

### 2.2.3 Phase portraits for the averaged equations

Some phase portraits for the averaged equations for the case $\frac{2 \delta}{a}<1$ are given in this subsection. Assume $a=1, \beta=0.5, c_{1}=-1$ and $\delta=0.125$. Then one gets that for a


Figure 3: Phase portraits for $a=1, \beta=0.5, c_{1}=-1$, and $\delta=0.125$.
wind velocity $\mu=0.4$ there is only one solution, i.e. the trivial solution as a stable node, see Figure 3(a). For $\mu=0.8$ one gets the trivial solution as a saddle and two symmetric nontrivial solutions as stable nodes, see Figure 3(b). For $\mu=1.8$ one gets the trivial solution as an unstable node and two pairs of nontrivial solutions, i.e. two stable nodes and two saddles, see Figure 3(c).

## 3 Parametric excitation of a nonlinear aeroelastic seesaw oscillator

In this section the parametric excitation of a nonlinear aeroelastic seesaw oscillator is analyzed. Based on the quasi steady approach the equation of motion is derived and using the averaging method the analysis of this equation is presented.

### 3.1 Derivation of the model equation

In this chapter, $\theta$ denotes the angle of rotation of the seesaw structure around the hinge axis. Following Haaker and van Oudheusden [2, 4], the following equation describes the aeroelastic response of the seesaw oscillator to a homogeneous uniform wind flow

$$
\begin{equation*}
I \ddot{\theta}+c \dot{\theta}+m_{p} g h \sin \theta=\frac{1}{2} \rho d l R U^{2} C_{N}(\alpha) . \tag{3.14}
\end{equation*}
$$

Here $I, c, m_{p}, g$ and $h$ are the structural moment of inertia, the linear viscous damping coefficient, the pendulum mass, the gravity constant and the pendulum length, respectively. The right hand side of equation (3.14) represents the external aerodynamic force where $\rho, d, l$, and $R$ denote air density, a characteristic length of the cross section, the length of the cylinder and the distance from the cylinder's axis to the hinge axis, respectively. Finally, $U$ denotes the wind velocity and $\alpha$ denotes the instantaneous angle of attack. Assuming a periodically varying pendulum-length $h=h_{0}(1-\tilde{h} \cos (\Omega t))$, one gets

$$
\begin{equation*}
I=I_{0}-2 \tilde{h} m_{p} h_{0}^{2} \cos (\Omega t)+O\left(\tilde{h}^{2}\right) \tag{3.15}
\end{equation*}
$$

where $I_{0}$ denotes the structural moment of inertia corresponding with fixed pendulum length $h_{0}$. Furthermore, $\tilde{h}$ denotes the relative amplitude of the periodic variation which is assumed small. So from equations (3.14-3.15) one obtains

$$
\begin{array}{r}
\left(I_{0}-\tilde{h} m_{p} h_{0}^{2} \cos (\Omega t)\right) \ddot{\theta}+c \dot{\theta}+m_{p} g h_{0}(1-\tilde{h} \cos (\Omega t)) \sin \theta= \\
\frac{1}{2} \rho d l R U^{2} C_{N}(\alpha)+O\left(\tilde{h}^{2}\right) \tag{3.16}
\end{array}
$$

Let $\nu=m_{p} h_{0}^{2}$ and approximate $\sin \theta \approx \theta$, one can rewrite equation (3.16) to become

$$
\begin{align*}
\ddot{\theta}+ & \frac{c}{I_{0}}\left(1+\frac{\nu \tilde{h}}{I_{0}} \cos (\Omega t)\right) \dot{\theta}+\frac{m_{p} g h_{0}}{I_{0}}(1-\tilde{h} \cos (\Omega t)) \times \\
& \left(1+\frac{\nu \tilde{h}}{I_{0}} \cos (\Omega t)\right) \theta=\frac{\rho d l R}{2 I_{0}} U^{2} C_{N}(\alpha)\left(1+\frac{\nu \tilde{h}}{I_{0}} \cos (\Omega t)\right) . \tag{3.17}
\end{align*}
$$

Scaling time with $\tau=\omega t$, where $\omega^{2}=\frac{m_{p} g h_{0}}{I_{0}}$ and introducing $\epsilon=\frac{\rho l d R^{3}}{2 I_{0}} ; \mu=\frac{U}{\omega R} ; 2 \beta \epsilon=\frac{c}{I_{0} \omega}$, and assuming $\tilde{h}=\tilde{a} \epsilon$, then one finds ( $"^{\prime} "=\frac{d}{d \tau}$ )

$$
\begin{align*}
\theta^{\prime \prime}+\theta= & \epsilon\left(\tilde{a}\left(1-\frac{\nu}{I_{0}}\right) \cos \left(\frac{\Omega}{\omega} \tau\right) \theta-2 \beta \theta^{\prime}+\mu^{2} C_{N}(\alpha)\right) \\
& +O\left(\epsilon^{2}\right) \tag{3.18}
\end{align*}
$$

Letting $a=\tilde{a}\left(1-\frac{\nu}{I_{0}}\right)>0$, one obtains

$$
\begin{equation*}
\theta^{\prime \prime}+\left(1-\epsilon a \cos \left(\frac{\Omega}{\omega} \tau\right)\right) \theta=\epsilon\left(-2 \beta \theta^{\prime}+\mu^{2} C_{N}(\alpha)\right)+O\left(\epsilon^{2}\right) \tag{3.19}
\end{equation*}
$$

with $\alpha=\theta-\frac{\theta^{\prime}}{\mu}$, see Haaker [2] and $C_{N}(\alpha)=c_{1} \alpha+c_{2} \alpha^{2}+c_{3} \alpha^{3}, c_{1}<0$ and $c_{3}>0$. So the equation becomes

$$
\begin{align*}
\theta^{\prime \prime}+\theta= & \epsilon\left(\left(a \cos \left(\frac{\Omega}{\omega} \tau\right)+c_{1} \mu^{2}\right) \theta-\left(c_{1} \mu+2 \beta\right) \theta^{\prime}+c_{2} \mu^{2} \theta^{2}\right. \\
& \left.+c_{2} \theta^{\prime 2}-2 c_{2} \mu \theta \theta^{\prime}+c_{3} \mu^{2} \theta^{3}-\frac{c_{3}}{\mu} \theta^{\prime 3}-3 c_{3} \mu \theta^{2} \theta^{\prime}+3 c_{3} \theta \theta^{\prime 2}\right) \\
& +O\left(\epsilon^{2}\right) \tag{3.20}
\end{align*}
$$

The equation (3.20) is a nonlinear Mathieu equation.

### 3.2 Analysis of the model equation

To reduce complexity in the first order perturbation analysis we set $\frac{\Omega}{\omega}=2+\delta$, with $\delta=O\left(\epsilon^{2}\right)$ rather than $\delta=O(\epsilon)$, and thus remove $\delta$ from our first order averaged equations. We define amplitude $r$ and phase $\psi$ or alternatively coordinates $u$ and $v$ through

$$
\begin{align*}
\theta & =r(\tau) \cos (\tau+\psi(\tau))  \tag{3.21}\\
\theta^{\prime} & =-r(\tau) \sin (\tau+\psi(\tau)) \tag{3.22}
\end{align*}
$$

or

$$
\begin{align*}
\theta & =u(\tau) \cos (\tau)+v(\tau) \sin (\tau)  \tag{3.23}\\
\theta^{\prime} & =-u(\tau) \sin (\tau)+v(\tau) \cos (\tau) \tag{3.24}
\end{align*}
$$

Substituting equations (3.21-3.24) into (3.20) one obtains the averaged equations for $r$ and $\psi$ or for $u$ and $v$, respectively, as follows

$$
\begin{align*}
r^{\prime} & =\epsilon\left(\frac{1}{4}\left(-4 \beta-2 c_{1} \mu+a \sin (2 \psi)\right) r-\frac{3 c_{3}}{8 \mu}\left(1+\mu^{2}\right) r^{3}\right)  \tag{3.25}\\
\psi^{\prime} & =\epsilon\left(\frac{a}{4} \cos (2 \psi)+\frac{c_{1} \mu^{2}}{2}+\frac{3 c_{3}}{8}\left(1+\mu^{2}\right) r^{2}\right) \tag{3.26}
\end{align*}
$$

or

$$
\begin{align*}
u^{\prime}= & \epsilon\left(-\left(\frac{c_{1}}{2} \mu+\beta\right) u+\frac{1}{4}\left(a-2 c_{1} \mu^{2}\right) v-\frac{3 c_{3}}{8}\left(\frac{1}{\mu}+\mu\right) u v^{2}\right. \\
& \left.-\frac{3 c_{3}}{8}\left(1+\mu^{2}\right) u^{2} v-\frac{3 c_{3}}{8}\left(\frac{1}{\mu}+\mu\right) u^{3}-\frac{3 c_{3}}{8}\left(1+\mu^{2}\right) v^{3}\right),  \tag{3.27}\\
v^{\prime}= & \epsilon\left(+\frac{1}{4}\left(a+2 c_{1} \mu^{2}\right) u-\left(\frac{c_{1}}{2} \mu+\beta\right) v+\frac{3 c_{3}}{8}\left(1+\mu^{2}\right) u v^{2}\right. \\
& \left.-\frac{3 c_{3}}{8}\left(\frac{1}{\mu}+\mu\right) u^{2} v+\frac{3 c_{3}}{8}\left(1+\mu^{2}\right) u^{3}-\frac{3 c_{3}}{8}\left(\frac{1}{\mu}+\mu\right) v^{3}\right) . \tag{3.28}
\end{align*}
$$

### 3.2.1 Stability of the trivial solution

For the linear analysis of the trivial solution we use the $(u, v)$-coordinates, i.e. equations (3.27-3.28). One can find that $(0,0)$ is a critical point of (3.27-3.28).

Transforming $u \rightarrow \sqrt{\frac{4\left|c_{1}\right| \mu}{3 c_{3}\left(1+\mu^{2}\right)}} \tilde{u}, v \rightarrow \sqrt{\frac{4\left|c_{1}\right| \mu}{3 c_{3}\left(1+\mu^{2}\right)}} \tilde{v}$, and time $\tau \rightarrow \frac{\left|c_{1}\right|}{2} s$, one finds (after neglecting the "tilde", $\quad, \cdot=\frac{d}{d s}$ )

$$
\begin{align*}
\dot{u}= & \epsilon\left(\left(\mu-\frac{2 \beta}{\left|c_{1}\right|}\right) u+\left(\frac{a}{2\left|c_{1}\right|}+\mu^{2}\right) v-u v^{2}-\mu u^{2} v\right. \\
& \left.-u^{3}-\mu v^{3}\right)  \tag{3.29}\\
\dot{v}= & \epsilon\left(\left(\frac{a}{2\left|c_{1}\right|}-\mu^{2}\right) u+\left(\mu-\frac{2 \beta}{\left|c_{1}\right|}\right) v+\mu u v^{2}-u^{2} v\right. \\
& \left.+\mu u^{3}-v^{3}\right) . \tag{3.30}
\end{align*}
$$

Letting $p=\frac{2 \beta}{\left|c_{1}\right|}$ and $q=\frac{a}{2\left|c_{1}\right|}$ then one obtains

$$
\begin{align*}
\dot{u} & =\epsilon\left((\mu-p) u+\left(\mu^{2}+q\right) v-u v^{2}-\mu u^{2} v-u^{3}-\mu v^{3}\right)  \tag{3.31}\\
\dot{v} & =\epsilon\left(\left(-\mu^{2}+q\right) u+(\mu-p) v+\mu u v^{2}-u^{2} v+\mu u^{3}-v^{3}\right) . \tag{3.32}
\end{align*}
$$

The eigenvalues evaluated at the critical point $(0,0)$ are

$$
\begin{aligned}
& \lambda_{1}=(\mu-p)+\sqrt{q^{2}-\mu^{4}} \\
& \lambda_{2}=(\mu-p)-\sqrt{q^{2}-\mu^{4}}
\end{aligned}
$$

Note that $\operatorname{Re}\left(\lambda_{1}\right)>\operatorname{Re}\left(\lambda_{2}\right)$ always holds, therefore the stability of the trivial solution is determined by $\operatorname{Re}\left(\lambda_{1}\right)<0$. If $q=0$, i.e. in absence of parametric excitation, the stability of the trivial solution is completely determined by the sign of $\mu-p$. If $\mu<p$, the trivial solution is stable; if $\mu>p$ the trivial solution is unstable. In fact $\mu=p$ corresponds with the familiar Den Hartog's criteria for instability as $p=-\frac{2 \beta}{c_{1}}$. If $q \neq 0$ and $\mu \geq \sqrt{q}$ holds then the trivial solution is stable if $\mu<p$ and unstable if $\mu>p$. It remains to check what happens if $q \neq 0$ and $\mu<\sqrt{q}$. In that case both eigenvalues are real. Because of that the trivial solution is stable if eigenvalue $\lambda_{1}$ is negative. One gets for $\mu<\sqrt{q}$

$$
\begin{gathered}
\frac{d \lambda_{1}}{d \mu}=1-\frac{2 \mu^{3}}{\sqrt{q^{2}-\mu^{4}}}, \frac{d^{2} \lambda_{1}}{d \mu^{2}}=-\left(\frac{6 q^{2} \mu^{2}-2 \mu^{6}}{\left(q^{2}-\mu^{4}\right) \sqrt{q^{2}-\mu^{4}}}\right)<0 \\
\frac{d \lambda_{1}}{d \mu}(0)=1, \lim _{\mu \rightarrow \sqrt{q}} \frac{d \lambda_{1}}{d \mu}(\mu)=-\infty, \lambda_{1}(0)=q-p
\end{gathered}
$$

From here one knows that $\lambda_{1}$ first increases with increasing $\mu$ until a maximum value, say $\lambda_{1_{\max }}$, is reached and then strictly decreases. Because of that $\lambda_{1}$ has at most two roots. One can distinguish between two main cases as follows
A. Case $p<\sqrt{q}$.
B. Case $p>\sqrt{q}$.

In the $(q, p)$ parameter plane, the cases A and B are separated by the curve $p=\sqrt{q}$ as shown in Figure 4.

1. Case A: $p<\sqrt{q}$.

In this case one finds that $\lambda_{1}(\sqrt{q})=\sqrt{q}-p>0$. The region may be divided into two parts. The region I is the region below of the curve $p=\sqrt{q}$ and the line $p=q$. The region II is the closed region between the curve $p=\sqrt{q}$ and the line $p=q$, see Figure 4. In region I, one finds $\lambda_{1}(0)=q-p>0$. Combined with $\lambda_{1}(\sqrt{q})>0$ and $\frac{d^{2} \lambda_{1}}{d \mu^{2}}<0$, it follows that $\lambda_{1}(\mu)$ is positive for $0<\mu<\sqrt{q}$. Therefore $(0,0)$ is unstable.
From the properties of $\lambda_{2}(\mu)$ it follows that a single root $\mu_{2}$ for $\lambda_{2}$ exists with $\mu_{2}<$ $\sqrt{q}$. Therefore the trivial solution is a saddle for $\mu<\mu_{2}$ and an unstable node for $\mu_{2}<\mu<\sqrt{q}$. Finally, for $\mu>\sqrt{q}$ it follows from the assumption $p<\sqrt{q}$ that the trivial solution is an unstable focus. In region II, one finds $\lambda_{1}(0)<0$. Combined with $\frac{d \lambda_{1}}{d \mu}(0)>0, \frac{d^{2} \lambda_{1}}{d \mu}<0$, and $\lambda_{1}\left(\sqrt{q}>0\right.$ one finds that a single root $\mu_{1}$ exists for $\lambda_{1}$ in the interval $0<\mu<\sqrt{q}$. Therefore the trivial solution is stable for $\mu<\mu_{1}$ and unstable for $\mu_{1}<\mu<\sqrt{q}$. For $\mu=\mu_{1}$, a pitchfork bifurcation occurs in which a stable nontrivial solution is born. The type of the instability also depends on the sign of $\lambda_{2}$. From the properties of $\lambda_{2}$ it readily follows that a single root $\mu_{2}$ for $\lambda_{2}$


Figure 4: Separation of the (q, p)-plane into five regions.
exists with $\mu_{1}<\mu_{2}<\sqrt{q}$. The trivial solution is a saddle for $\mu_{1}<\mu<\mu_{2}$ and an unstable node for $\mu_{2}<\mu<\sqrt{q}$. Again for $\mu>\sqrt{q}$, the trivial solution is an unstable focus. One gets a stability diagram for the trivial solution in the regions I and II as in Figure 5a and Figure 5b, respectively.
2. Case B: $p>\sqrt{q}$.

In this case one finds that $\lambda_{1}(\sqrt{q})=\sqrt{q}-p<0$. Note that $\frac{d \lambda_{2}}{d \mu}=1+\frac{2 \mu^{3}}{\sqrt{q^{2}-\mu^{4}}}>0$, $\lambda_{2}(0)<0, \lambda_{2}(\sqrt{q})<0$. It shows that $\lambda_{2}(\mu)<0$ for $0<\mu \leq \sqrt{q}$.
Above the line $p=q$ one finds $\lambda_{1}(0)=-p+q<0$. One obtains two roots if $\lambda_{1_{\max }}>0$, no root if $\lambda_{1_{\max }}<0$. The cases are separated by the case $\lambda_{1_{\max }}=0$, i.e. the case for which $\lambda_{1}(\mu)$ has a double root. Solving $\lambda_{1}=0$ and $\frac{d \lambda_{1}}{d \mu}=0$ for $\mu$ one obtains a curve $g(q, p)=0$ in ( $q, p$ )-plane on which a double root occurs, with

$$
g(q, p)=-p+\frac{1}{6} k-\frac{1}{k}+\left(q^{2}-\left(\frac{1}{6} k-\frac{1}{k}\right)^{4}\right)^{\frac{1}{2}}
$$

where $k=\left(54 p+6 \sqrt{6+81 p^{2}}\right)^{\frac{1}{3}}$.
Above $g=0$, in the region III, we find $\lambda_{1_{\max }}<0$, and no root exist. The trivial solution is a stable node for $0<\mu<\sqrt{q}$, a stable focus for $\sqrt{q}<\mu<p$ and an unstable focus for $\mu>p$. The stability diagram is shown in Figure 5c. Below $g=0$, in region IV, we find $\lambda_{1_{\max }}>0$, and two roots $\mu_{11}$ and $\mu_{12}$ for $\lambda_{1}$ are found. The trivial solution is a stable node for $0<\mu<\mu_{11}$. It is a saddle for $\mu_{11}<\mu<\mu_{12}$


Figure 5: Stability diagrams for the trivial solution in the five regions.
and again a stable node for $\mu_{12}<\mu<\sqrt{q}$. Furthermore, it is a stable focus for $\sqrt{q}<\mu<p$ and an unstable focus for $\mu>p$. The stability diagram is shown in Figure 5d.
Finally, in region V, between the curves $p=q$ and $p=\sqrt{q}$, one has $\lambda_{1}(0)=-p+q>$ 0. In that case a single root $\mu_{1}$ for $\lambda_{1}$ exists. Then the trivial solution is a saddle for $0<\mu \leq \mu_{1}$, a stable node for $\mu_{1}<\mu<\sqrt{q}$, stable focus for $\sqrt{q}<\mu<p$, and an unstable focus for $\mu>p$. The stability diagram is shown in Figure 5e.
Furthermore one can see that for the case $p>\sqrt{q}$ at $\mu=p$, the system has a purely imaginary pair of eigenvalues. It indicates the occurrence of a Hopf bifurcation. After a straightforward calculation, one gets the Lyapunov number

$$
\sigma=-\frac{1}{8}\left(1-\frac{p^{2}+q}{-p^{2}+q}\right)<0 .
$$

See Appendix 5 for a proof. This Hopf bifurcation therefore indicates the creation of a stable limit cycle in the averaging equations. This corresponds to oscillations in the original system with periodically modulated amplitudes and phases.
We finally remark that the zeroes of the eigenvalues $\lambda_{1}$ and $\lambda_{2}$ are related to pitchfork bifurcations which involve the creation of branches of nontrivial critical points. See Appendix 5 for a proof.

### 3.2.2 Nonlinear analysis

In the analysis of the nontrivial solutions, the averaged equations in polar coordinates are used. Transforming equations (3.31-3.32) into polar coordinates by using $u=r \cos (\psi)$ and $v=-r \sin (\psi)$ one gets

$$
\begin{align*}
\dot{r} & =\epsilon\left((\mu-p-q \sin (2 \psi)) r-r^{3}\right),  \tag{3.33}\\
\dot{\psi} & =\epsilon\left(\mu^{2}-q \cos (2 \psi)-\mu r^{2}\right) . \tag{3.34}
\end{align*}
$$

Multiplying equation (3.33) by $r$ and setting $R=r^{2}$ yields

$$
\begin{align*}
\dot{R} & =\epsilon\left(2(\mu-p-q \sin (2 \psi)) R-2 R^{2}\right)  \tag{3.35}\\
\dot{\psi} & =\epsilon\left(\mu^{2}-q \cos (2 \psi)-\mu R\right) . \tag{3.36}
\end{align*}
$$

Setting the right hand sides of equations (3.35) and (3.36) to zero one obtains

$$
\begin{equation*}
A R^{2}+B R+C=0, \quad R>0 \tag{3.37}
\end{equation*}
$$

with

$$
\begin{aligned}
& A=1+\mu^{2} \\
& B=-2\left(\mu-p+\mu^{3}\right) \\
& C=\mu^{4}-q^{2}+(\mu-p)^{2}
\end{aligned}
$$

The zeroes of equation (3.37) are

$$
R_{1,2}=\frac{-B \pm \sqrt{B^{2}-4 A C}}{2 A}
$$

Note that every solution of equation (3.37) corresponds to two solutions, $\left(R_{0}, \psi_{0}\right)$ and $\left(R_{0}, \psi_{0}+\pi\right)$, of equations (3.35-3.36). Let $D$ denotes the discriminant of (3.37), then $D=4\left(\left(1+\mu^{2}\right) q^{2}-\mu^{2} p^{2}\right)$. One gets two positive zeroes of (3.37) if $C>0, B<0$, and $D>0$, one positive zero if $C<0$, a double positive zero if $D=0$ and $B<0$, no positive zero in other cases.
Note that a pitchfork bifurcation occurs for $C=0$, a saddle-node bifurcation occurs for $D=$ 0 . Figures 6 and 7 show the location of saddle-node bifurcation (SN), pitchfork bifurcation (P), Hopf bifurcation (H) and saddle-connection bifurcation (SC) in the ( $\mu, q$ )-plane for $p>1$ and $p<1$, respectively. The saddle-node and pitchfork bifurcation curves have an intersection point Q , in which one gets $B=C=D=0$. The coordinates of Q are $\left(\mu_{Q}, q_{Q}\right)$, with $\mu_{Q}=\frac{1}{6} k-\frac{2}{k}, q_{Q}=\frac{\sqrt{6}}{6}\left(\frac{p}{k}\left(-k^{2}+6 p k+12\right)\right)^{\frac{1}{2}}$, and $k=\left(108 p+12 \sqrt{12+81 p^{2}}\right)^{\frac{1}{3}}$.
The equations for the pitchfork bifurcation curve $(\mathrm{P})$ and the saddle-node bifurcation curve (SN) are

$$
q_{p}=\left(\mu^{4}+(\mu-p)^{2}\right)^{\frac{1}{2}} \text { and } q_{s n}=\frac{p \mu}{\sqrt{1+\mu^{2}}}, \text { respectively. }
$$



Figure 6: Number of nontrivial solutions, Hopf (H), saddle connection (SC), saddle-node (SN), and pitchfork (P) bifurcations for case $p>1$.


Figure 7: Number of nontrivial solutions, Hopf (H), saddle connection (SC), saddle-node (SN), and pitchfork ( P ) bifurcations for case $p<1$.

The coordinate of the minimum point of the pitchfork bifurcation curve is $\left(\mu_{t}, q_{t}\right)$ with $\mu_{t}=\frac{1}{6} t-\frac{1}{t}, q_{t}=\left(\mu_{t}^{4}+\left(\mu_{t}-p\right)^{2}\right)^{\frac{1}{2}}$, and
$t=\left(54 p+6 \sqrt{6+81 p^{2}}\right)^{\frac{1}{3}}$.
The Hopf bifurcation is found for $\mu=p$ and $q<p^{2}$. In the Figures 6 and 7 , the numbers denote the number of nontrivial critical point of equations (3.35-3.36) found in each region. The roman numerals to the right correspond with the stability regions of the linear analysis. The pitchfork and the saddle-node bifurcation curves divide the $(\mu, q)$-parameter plane into three big regions, i.e.

- The region where one finds no nontrivial solution, below the pitchfork and saddlenode curves.
- The region where one finds two nontrivial solutions, above the pitchfork curve.
- The region where one finds four nontrivial solutions, between the pitchfork and saddle-node curves.

The number of nontrivial solution in each region is determined from the number of positive roots of the equation (3.37) in that region.
We now consider Figure 6, i.e. the case $p>1$. For a fixed value of $q=q_{0}$ such that $q_{0}>p^{2}$, starting with a small value of $\mu$, one finds readily two stable nontrivial critical points. Increasing the value of $\mu$ such that the pitchfork curve is reached then one gets a pitchfork bifurcation which leads to the creation of two new nontrivial critical points of saddle type. At the point $\left(p, p^{2}\right)$, on the pitchfork curve, one finds that both of the eigenvalues of the trivial solution are zero. From this point, the Hopf (H) and the saddle connection (SC) bifurcation curves start.
For $q=q_{0}$ such that $p<q_{0}<p^{2}$, starting with a small value of $\mu$, one finds two stable nontrivial critical points. Increasing the value of $\mu$ such that the pitchfork curve is reached one obtains another two nontrivial critical points (unstable). At $\mu=p$, a Hopf bifurcation occurs and a stable limit cycle is born. Increasing the value of $\mu$ further such that the saddle connection (SC) curve is reached the limit cycle disappears in the saddle connection bifurcation (SC). The four nontrivial solutions survive though.
For $q=q_{0}$ such that $q_{Q}<q_{0}<p$ starting with a small value of $\mu$, one gets only a stable trivial solution. Increasing the value of $\mu$ such that the pitchfork bifurcation curve is passed, one finds two stable nontrivial critical points and the trivial solution becomes unstable. Increasing again the value of $\mu$ such that the pitchfork bifurcation curve is passed once more, another two unstable critical points are born. However the trivial solution is now stable again. Increasing the value of $\mu$ such that the Hopf bifurcation curve is passed one gets a stable limit cycle and four nontrivial solutions. The critical solution is now unstable focus. After increasing the value of $\mu$ passed the saddle-node bifurcation curve (SN), both pairs of nontrivial solutions disappear in a saddle-node bifurcation and only the stable limit cycle survives. But if one reaches the saddle-node bifurcation curve (SN) first, then after passing the saddle-node bifurcation curve (SN) both pairs of nontrivial solutions disappear in the saddle-node bifurcation. Finally, increasing the value of $\mu$ one gets a stable limit cycle after passing the Hopf bifurcation curve and the trivial solution is an unstable focus.
For $q=q_{0}$ such that $q_{t}<q_{0}<q_{Q}$, starting with a small value for $\mu$, then one gets only a stable trivial solution. Increasing the value of $\mu$ such that the first pitchfork point is reached then one finds one pair of stable critical points bearing from the pitchfork bifurcation and the trivial solution becomes unstable. Increasing again the value of $\mu$, such that the second pitchfork point is reached then one obtains that the nontrivial critical points disappear and the trivial solution becomes stable. Furthermore, increasing the value of $\mu$, then one obtains a Hopf bifurcation from which the stable limit cycle is born and the trivial solution becomes an unstable focus.
For $q=q_{0}$ such that $q_{0}<q_{t}$, starting with a small value for $\mu$, then one gets a stable trivial solution. Increasing the value of $\mu$, one gets a Hopf bifurcation from which the stable limit cycle is born and the trivial solution becomes an unstable focus.


Figure 8: Phase portraits in the region III for $p=2$ and $q=1$.

The story for Figure 7 is nearly the same as for Figure 6. The difference is the saddle connection (SC) bifurcation in Figure 7, that now corresponds with the creation of a stable limit cycle.
To consider the stability of the nontrivial critical points, one can determine the eigenvalues of the Jacobian from the equation (3.35-3.36) evaluated at $\left(r_{0}, \psi_{0}\right)$, where $r_{0}=\sqrt{R}$ and $\psi_{0}$ are determined from the zeroes of equations (3.35-3.36).

### 3.2.3 Phase portraits for the averaged equations

In this section some phase portraits for the averaged equations (3.31-3.32) are given. Some special values for the parameters are considered to depict the behaviour of the trivial and the nontrivial solutions in the $(u, v)$-plane. Depending on the wind velocity $\mu$, one finds how the stability of the trivial solution changes and how the nontrivial critical points and a stable limit cycle appear.
Assuming $p=2$ and choosing $q=1$ then $q<q_{t}<p$ and one is in the region III, see Figures 4 and 6. The stability diagram of the trivial solution is shown in Figure 5c. Varying the wind velocity $\mu$ one gets the following: At $\mu=0.5$, one gets only the trivial solution as a stable node. The phase portrait in the $(u, v)$-plane is shown in Figure 8(a). Increasing the value of $\mu$ to $\mu=1.5$ one obtains the trivial solution is a stable focus. The phase portrait is shown in Figure 8(b). Furthermore, increasing the value of $\mu$ to $\mu=2.3$ one finds that the trivial solution is an unstable focus. There is also a stable limit cycle bearing from the Hopf bifurcation at $\mu=p=2$. The limit cycle corresponds to oscillations in the original system with periodically modulated amplitudes and phases. The phase portrait is shown in Figure 8(c).
Assuming $p=2$ and choosing $q=1.38$ then $q_{Q}<q<p$, one is in the region IV, see Figures 4 and 6. The stability diagram of the trivial solution is shown in the Figure 5d with $\mu_{11}=0.72, \mu_{12}=0.94$ and $\sqrt{q}=1.17$. Varying the wind velocity $\mu$ one gets the following: At $\mu=0.1$ one gets only the trivial solution as a stable node. The phase portrait in the ( $u, v$ )-plane is shown in Figure 9(a). Increasing the value of $\mu$ to $\mu=0.8$ one finds a symmetric pair of nontrivial critical points as stable nodes and the trivial


Figure 9: Phase portraits in the region IV for $p=2$ and $q=1.38$.
solution is a saddle. The phase portrait in the $(u, v)$-plane is shown in Figure 9(b). In this case, the amplitudes of the symmetrical pair of nontrivial solutions are increasing with increasing $\mu$ up to certain value of $\mu$ after which they are shrinking to zero in the second pitchfork bifurcation at $\mu=\mu_{12}$ and the trivial solution becomes stable. At $\mu=1.2$ one gets the trivial solution is a stable focus and there is no nontrivial solution at all. The phase portrait in $(u, v)$-plane is given in Figure 9(c). Increasing again the value of $\mu$ to $\mu=3.2$ one gets the trivial solution as an unstable focus. There is also a stable limit cycle bearing from the Hopf bifurcation at $\mu=p=2$. The phase portrait is shown in Figure 9(d).
Assuming $p=2$ and $q=1.9$ then $q_{Q}<q<p$, one is in the region IV, see Figures 4 and 6. The stability diagram of the trivial solution is shown in the Figure 5 d , with $\mu_{11}=0.10$, $\mu_{12}=1.33$ and $\sqrt{q}=1.378$. Compared to the case $p=2$ and $q=1.38$, here the second pitchfork bifurcation, i.e. at $\mu=\mu_{12}$ corresponds to the creation of another pair of nontrivial critical points. Varying the wind velocity $\mu$ one gets the following: For $\mu=0.06$ one finds the trivial solution as a stable node without nontrivial critical points. The phase


Figure 10: Phase portraits in the region IV for $p=2$ and $q=1.9$.
portrait is shown in Figure 10(a). Increasing the value of $\mu$ to $\mu=0.4$ one obtains the trivial solution as a saddle and a pair of nontrivial solutions as stable nodes. The phase portrait is shown in Figure $10(\mathrm{~b})$. Increasing the value of $\mu$ such that $\mu=\mu_{12}$ then one gets the second pitchfork bifurcation leading to the exchange of the stability of the trivial solution and the creation of the second pair of nontrivial critical points. At $\mu=1.35$ one gets two pairs of nontrivial critical points, one pair as saddles and the other pair as stable foci. The trivial solution becomes a stable node. The phase portrait is shown in Figure 10(c).
Increasing again the value of $\mu$ to $\mu=1.4$, one finds that the trivial solution is a stable focus and two pairs of nontrivial critical points. One pair of them as saddles and the other pair as stable foci. The phase portrait is shown in Figure 10(d). Furthermore, increasing $\mu$ to $\mu=2.5$, one obtains two pairs of nontrivial critical points, one pair as saddles and the other pair as stable foci. The trivial solution is an unstable focus. There is also a stable limit cycle bearing from the Hopf bifurcation at $\mu=p=2$. The phase portrait is shown in Figure 10(e). Increasing again the value of $\mu$ to $\mu=3.1$, one gets only a stable limit cycle and the trivial solution as an unstable focus. Both pairs of nontrivial critical points are disappeared in the saddle node bifurcation. The phase portrait is shown in Figure 10(f). Assuming $p=2$ and choosing $q=3.333$ then $p<q<p^{2}$, one is in the region V , see Figures 4 and 6. The stability diagram of the trivial solution is shown in Figure 5e, with $\mu_{1}=1.82$ and $\sqrt{q}=1.825$. Varying the wind velocity $\mu$ one gets the following: At $\mu=1.7$,


Figure 11: Phase portraits in the region V for $p=2$ and $q=3.333$.
one finds two nontrivial critical points as stable foci and the trivial critical point as a saddle. The phase portrait in the $(u, v)$-plane is shown in Figure 11(a). Increasing the value of $\mu$, to $\mu=1.9$, one obtains two pairs of nontrivial critical points, one pair as saddles and the other pair as stable foci. The trivial solution is a stable focus. The phase portrait is shown in Figure 11(b). Increasing the value of $\mu$, to $\mu=2.1$, one gets two pairs of nontrivial critical points, one pair as saddles and the other pair as stable foci. The trivial solution is an unstable focus. There is also a stable limit cycle bearing from the Hopf bifurcation at $\mu=p=2$. The phase portrait is shown in Figure 11(c). For $\mu=4$ one finds that the limit cycle has disappeared in a heteroclinic loop connecting the two symmetric saddle points, when passing the saddle connection curve. The phase portrait is shown in Figure 11(d). Finally, for $\mu=4.1$ one obtains that the saddle connection is broken. All solutions tend to one of the stable foci. The phase portrait is shown in Figure 11(e).
Assuming $p=2$ and $q=5$ then $q>p^{2}$, one is in the region I, see Figures 4 and 6 . The stability diagram of the trivial solution is shown in Figure 5a, with $\mu_{2}=2.23$ and $\sqrt{q}=2.24$. Varying the wind velocity $\mu$ one gets the following: For value of $\mu<\mu_{c}$ one finds a pair of symmetric nontrivial critical points as stable foci and the trivial solution is a saddle. The phase portrait is given in Figure 12(a) for $\mu=2$. Increasing the value of $\mu$ to $\mu=2.5$, one obtains two pairs of symmetric nontrivial critical points, one pair as saddles and the other pair as stable foci. The trivial solution is an unstable focus. The phase portrait is shown in Figure 12(b).


Figure 12: Phase portraits in the region I for $p=2$ and $q=5$.


Figure 13: Phase portraits in the region III for $p=0.5$ and $q=0.1$.

Finally we consider phase portraits for the case $p<1$.
Assuming $p=0.5$ and $q=0.1$ then $0<q<q_{t}$, one is in the region III, see Figures 4 and 7. The diagram of the stability of the trivial solution is given in Figure 5c with $\sqrt{q}=0.32$. Varying the wind velocity $\mu$ one gets the following: At $\mu=0.1$ one finds only the trivial solution as a stable node. The phase portrait is shown in Figure 13(a). Increasing the value of $\mu$ such that $\mu>\sqrt{q}$ then both of the eigenvalues are complex and the trivial solution changes from a stable node to a stable focus. At $\mu=0.4$ one finds the trivial solution is a stable focus. The phase portrait is shown in Figure 13(b). Furthermore, increasing the value of $\mu$ to $\mu=0.6$, one gets that the trivial solution is an unstable focus. There is also a stable limit cycle bearing from the Hopf bifurcation at $\mu=0.5$. Its phase portrait is shown in Figure 13(c).
Assuming $p=0.5$ and $q=0.24$ then $q_{t}<q<q_{Q}$ one is in the region IV, see Figures 4 and 7. The diagram of the stability of the trivial solution is the same as in the Figure 5d, with $\mu_{11}=0.27, \mu_{12}=0.4897$, and $\sqrt{q}=0.4898$.
Varying the wind velocity $\mu$ one gets the following: At $\mu=0.1$, one finds only the trivial solution as a stable node. Its phase portrait is shown in Figure 14(a). Increasing the value of $\mu$ to $\mu=0.3$, one obtains the trivial solution as a saddle and a pair of symmetric


Figure 14: Phase portraits in the region IV for $p=0.5$ and $q=0.24$.
nontrivial critical points as stable nodes. The phase portrait is shown in Figure 14(b). Increasing the value of $\mu$ to $\mu=0.4897$, one gets two pairs of nontrivial critical points, one pair as saddles and the other pair as stable foci. The trivial solution becomes a stable focus. The phase portrait is shown in Figure 14(c). For larger value of $\mu$ such that $\mu=0.53$ one gets the trivial solution is an unstable focus. There is a stable limit cycle bearing from the saddle connection bifurcation. The two pairs of nontrivial points still survive, see the phase portrait in Figure $14(\mathrm{~d})$. Increasing the value of $\mu$ to $\mu=0.6$, one gets only the stable limit cycle and the trivial solution as an unstable focus. There is no nontrivial solution at all. The nontrivial points are destroyed in the saddle node bifurcation. Its phase portrait is given in Figure 14(e).
Assuming $p=0.5$ and choosing $q=0.26$ then $p^{2}<q<p$ and one is in region II, see Figures 4 and 7. The stability diagram of the trivial solution is shown in Figure 5b, with $\mu_{1}=0.25, \mu_{2}=0.5097$, and $\sqrt{q}=0.5099$. Varying the wind velocity $\mu$ one gets the behaviour of the solutions as follows: Figure 15 (a) depicts the phase portrait for $\mu=0.1$. It shows the trivial solution as a stable node and no other critical points exist.
Increasing the value of $\mu$ to $\mu=0.4$, one gets a pair of nontrivial solutions as stable nodes and the zero solution as a saddle. The phase portrait in $(u, v)$-plane as in the Figure 15(b). Increasing the value of $\mu$ further such that $\mu=0.57$ then one gets the zero solution is an unstable focus. The phase portrait in the $(u, v)$-plane is given in Figure 15(c). Increasing again the value of $\mu$ to $\mu=0.6084$, one gets a stable limit cycle bearing from the saddle


Figure 15: Phase portraits in the region II for $p=0.5$ and $q=0.26$.
connection bifurcation. Domains of attraction are separated by the stable manifolds of the two saddles. Inside those manifolds the flow tends to the limit cycle, outside it tends to one of the stable node. The phase portrait is shown in Figure 15(d). Increasing the value of $\mu$ further to $\mu=0.8$, one finds saddle-node bifurcations in which the stable and unstable nontrivial solutions disappear. Only the stable limit cycle survives and all flow tends to this limit cycle, see Figure 15(e).
Assuming $p=0.5$ and $q=1$ then $p<q$, one is in the region I, see Figures 4 and 7 . The stability of the trivial solution is given in Figure 5a with $\mu_{2}=0.946$. Varying the wind velocity $\mu$ one gets the following behaviour: At $\mu=0.8$ one gets two symmetric nontrivial critical points as stable foci and the trivial solution is a saddle. The phase portrait is shown in Figure 16(a). Increasing the value of $\mu$ to $\mu=0.96$, one finds that the trivial solution is an unstable node and two pairs of symmetric nontrivial critical points, one pair as stable foci and the other pair as saddles. The phase portrait is given in Figure 16(b). Furthermore, increasing the value of $\mu$ to $\mu=1.2$, one obtains that the trivial solution is an unstable focus and two pairs of symmetric nontrivial critical points exist, one pair as stable foci and the other pair as saddles. The phase portrait is given in Figure 16(c).


Figure 16: Phase portraits in the region I for $p=0.5$ and $q=1$.

## 4 Conclusions

In this chapter the parametric excitation of two nonlinear aeroelastic oscillators was studied. In practice, parametric excitation of structural elements, like bridge stay cables, may be caused by the periodic motion of the bridge deck or the bridge tower. The induced vibrations are most prominent if the excitation frequency is nearly twice the natural frequency.
For the plunge oscillator the following results were obtained;

1. For strong parametric excitation, $\frac{\delta}{2 a}<1$, i.e. when the ratio of detuning $\delta$ and parametric force coefficient $a$ is small, the critical flow velocity is shifted to a lower value. For flow velocities above this value stable periodic oscillations were found. If there is a small structural damping, the critical flow velocity shifts to zero, indicating that even in absence of wind the equilibrium position is unstable.
2. For weak parametric excitation, $\frac{\delta}{2 a}>1$, the well-known critical wind velocity according to Den Hartog's criterion was found. Rather than a stable periodic oscillation, one obtains a stable solution with periodically modulated amplitude and phase.

For the seesaw oscillator the following results were obtained:

1. The observed behaviour for strong and weak parametric excitation is identical to the behaviour observed for the plunge oscillator.
2. Interesting new dynamics is found when the parametric excitation, the structural damping and the linear aeroelastic force are in close range. In that case a critical flow velocity may still exist, above which the equilibrium position becomes unstable. However, in some cases the trivial solution re-stabilizes when the flow velocity is increased above a certain value. This may be understood from the fact that the flow velocity causes an increased detuning between the excitation frequency and the natural frequency, which effectively reduces the parametric excitation. Increasing the flow velocity further though, an aeroelastic instability is found for a flow velocity
corresponding to Den Hartog's criterion. Next to periodic solutions with constant amplitudes also stable solutions, with periodically modulated amplitudes and phases, are found bearing from limit cycles in the averaged equations. Both Hopf and saddle connection bifurcations were found responsible for the appearance of these limit cycles. In some cases constant amplitude periodic solutions and periodically modulated solutions were found to co-exist.

## 5 Appendix

In this appendix the existence of the Hopf and pitchfork bifurcations are considered.

### 5.1 Hopf bifurcation

Consider the system below

$$
\begin{align*}
& \dot{u}=\epsilon\left(-(\gamma \mu+\lambda) u+\left(1-\gamma \mu^{2}\right) v-u v^{2}-\mu u^{2} v-u^{3}-\mu v^{3}\right),  \tag{5.38}\\
& \dot{v}=\epsilon\left(\left(1+\gamma \mu^{2}\right) u-(\gamma \mu+\lambda) v+\mu u v^{2}-u^{2} v+\mu u^{3}-v^{3}\right) . \tag{5.39}
\end{align*}
$$

Letting

$$
\mathbf{X}=\binom{u}{v}
$$

then we have

$$
\begin{align*}
\dot{\mathbf{X}}= & \epsilon\left(\left(\begin{array}{cc}
-(\gamma \mu+\lambda) & 1-\gamma \mu^{2} \\
1+\gamma \mu^{2} & -(\gamma \mu+\lambda)
\end{array}\right) \mathbf{X}\right. \\
& \left.+\binom{f^{1}(\mu, u, v)}{f^{2}(\mu, u, v)}\right), \tag{5.40}
\end{align*}
$$

with

$$
\begin{aligned}
& f^{1}(\mu, u, v)=-u v^{2}-\mu u^{2} v-u^{3}-\mu v^{3} \\
& f^{2}(\mu, u, v)=\mu u v^{2}-u^{2} v+\mu u^{3}-v^{3}
\end{aligned}
$$

Letting $a=-(\gamma \mu+\lambda), b=1-\gamma \mu$, and $c=1+\gamma \mu^{2}$ then we have

$$
\begin{equation*}
\dot{\mathbf{X}}=\epsilon(\mathbf{A X}+\mathbf{F}) \tag{5.41}
\end{equation*}
$$

with

$$
\mathbf{A}=\left(\begin{array}{ll}
a & b \\
c & a
\end{array}\right)
$$

and

$$
\mathbf{F}=\binom{f^{1}(\mu, u, v)}{f^{2}(\mu, u, v)}
$$

We now translate the equation (5.41) to normal form with transformation

$$
\begin{equation*}
\mathbf{X}=\mathbf{T Z} \tag{5.42}
\end{equation*}
$$

where

$$
\mathbf{T}=\left(\begin{array}{cc}
0 & \frac{\sqrt{-b c}}{c} \\
1 & 0
\end{array}\right)
$$

and

$$
\mathbf{Z}=\binom{z_{1}}{z_{2}}
$$

We get

$$
\mathbf{T}^{-\mathbf{1}}=\left(\begin{array}{cc}
0 & 1 \\
\frac{c}{\sqrt{-b c}} & 0
\end{array}\right)
$$

From equation (5.42) we get $u=\frac{\sqrt{-b c}}{c} z_{2}$ and $v=z_{1}$.
So from (5.41) and (5.42) we get

$$
\begin{equation*}
\dot{\mathbf{Z}}=\epsilon\left(\left(\mathbf{T}^{-1} \mathbf{A T}\right) \mathbf{Z}+\mathbf{G}\right) \tag{5.43}
\end{equation*}
$$

with

$$
\begin{gather*}
\mathbf{G} \equiv \mathbf{T}^{-\mathbf{1}} \mathbf{F}=\binom{g^{1}\left(\mu, z_{1}, z_{2}\right)}{g^{2}\left(\mu, z_{1}, z_{2}\right)}, \\
\mathbf{T}^{-\mathbf{1}} \mathbf{A T}=\left(\begin{array}{cc}
a \\
\sqrt{-b c} & \sqrt{-b c}
\end{array}\right), \\
g^{1}\left(\mu, z_{1}, z_{2}\right)=  \tag{5.44}\\
g^{2}\left(\mu, z_{1}, z_{2}\right)=  \tag{5.45}\\
=-z_{1}^{2} z_{2}-\mu \frac{\sqrt{-b c}}{c} z_{1}^{2} z_{2}+\frac{b}{c} z_{1} z_{2}^{2}-\mu \frac{b}{c^{2}} \sqrt{-b c} z_{2}^{3}-z_{1}^{3}, \\
z_{1} z_{2}^{2}+\frac{b}{c} z_{2}^{3}-\mu \frac{c}{\sqrt{-b c}} z_{1}^{3} .
\end{gather*}
$$

From equations $(5.44-5.45)$ we get $g_{z_{1} z_{1} z_{1}}^{1}=-1, g_{z_{1} z_{2} z_{2}}^{1}=\frac{b}{c}, g_{z_{2} z_{2} z_{2}}^{2}=\frac{b}{c}$, and $g_{z_{1} z_{1} z_{2}}^{2}=-1$. The Lyapunov number evaluated at $(0,0)$ is

$$
\begin{align*}
\sigma & =\frac{1}{16}\left(g_{z_{1} z_{1} z_{1}}^{1}+g_{z_{1} z_{2} z_{2}}^{1}+g_{z_{2} z_{2} z_{2}}^{2}+g_{z_{1} z_{1} z_{2}}^{2}\right) \\
& =\frac{1}{16}\left(-1+\frac{b}{c}+\frac{b}{c}-1\right) \\
& =-\frac{1}{8}\left(1-\frac{b}{c}\right) \\
& <0 \tag{5.46}
\end{align*}
$$

Eigenvalues of $(5.41)$ evaluated at $(0,0)$ are
$\lambda_{1}=a+\sqrt{b c}$ and $\lambda_{2}=a-\sqrt{b c}$
To have a purely imaginary pair of eigenvalues, we have to set $a=-(\gamma \mu+\lambda)=0$, i.e. $\mu \equiv \mu_{c}=-\frac{\lambda}{\gamma}$ and $b c=1-\gamma^{2} \mu^{4}<0$, i.e. $\lambda^{2}>|\gamma|$.
Then the system becomes

$$
\dot{\mathbf{Z}}=\epsilon\left(\left(\begin{array}{cc}
0 & -\omega  \tag{5.47}\\
\omega & 0
\end{array}\right) \mathbf{Z}+\binom{g^{1}\left(\mu_{c}, z_{1}, z_{2}\right)}{g^{2}\left(\mu_{c}, z_{1}, z_{2}\right)}\right) .
$$

The equation (5.47) is a normal form from (5.38-5.39) evaluated at $\mu=\mu_{c}$. The Lyapunov number at $\mu=\mu_{c}$ is

$$
\begin{aligned}
\sigma & =-\frac{1}{8}\left(1+\frac{-\left(1-\frac{\lambda^{4}}{\gamma^{2}}\right)}{\left(1+\frac{\lambda^{2}}{\gamma}\right)^{2}}\right) \\
& <0
\end{aligned}
$$

Using the method in the book of Wiggins [15] on the page 277, we conclude that there is a stable limit cycle bifurcates at $\mu=\mu_{c}$.

## 5.2 pitchfork bifurcation

We consider the system

$$
\begin{equation*}
\dot{\mathbf{X}}=\epsilon(\mathbf{A X}+\mathbf{F}) \tag{5.48}
\end{equation*}
$$

with

$$
\begin{gathered}
\mathbf{X}=\binom{u}{v}, \quad \mathbf{A}=\left(\begin{array}{ll}
a & b \\
c & a
\end{array}\right) \\
\mathbf{F}=\binom{f^{1}(\mu, u, v)}{f^{2}(\mu, u, v)}
\end{gathered}
$$

with

$$
\begin{aligned}
& f^{1}(\mu, u, v)=-u v^{2}-\mu u^{2} v-u^{3}-\mu v^{3} \\
& f^{2}(\mu, u, v)=\mu u v^{2}-u^{2} v+\mu u^{3}-v^{3}
\end{aligned}
$$

$a=-(\gamma \mu+\lambda), \quad b=1-\gamma \mu^{2}>0$, and $c=1+\gamma \mu^{2}$.
In this case $b c=1-\gamma^{2} \mu^{4}>0$. Therefore $c>0$. The eigenvalues of A are

$$
\lambda_{1}=a-\sqrt{b c} \quad \text { and } \quad \lambda_{2}=a+\sqrt{b c} .
$$

Let

$$
\begin{gathered}
\mathbf{T}=\left(\begin{array}{cc}
\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} \frac{c}{\sqrt{b c}} & \frac{1}{2} \frac{c}{\sqrt{b c}}
\end{array}\right) \text { then } \mathbf{T}^{-\mathbf{1}}=\left(\begin{array}{cc}
1 & -\frac{\sqrt{b c}}{\frac{c}{c}} \\
1 & \frac{\sqrt{b c}}{c}
\end{array}\right) \\
\text { and } \quad \mathbf{T}^{-\mathbf{1}} \mathbf{A T}=\left(\begin{array}{cc}
a-\sqrt{b c} & 0 \\
0 & a+\sqrt{b c}
\end{array}\right) .
\end{gathered}
$$

We transform equation (5.48) with transformation

$$
\begin{equation*}
\mathbf{X}=\mathbf{T} \mathbf{Z} \tag{5.49}
\end{equation*}
$$

where

$$
\mathbf{Z}=\binom{z_{1}}{z_{2}}
$$

Then we get $u=\frac{1}{2}\left(z_{1}+z_{2}\right), v=r_{0}\left(-z_{1}+z_{2}\right)$, with $r_{0}=\frac{1}{2} \sqrt{\frac{c}{b}}$,

$$
\begin{aligned}
& f^{1}\left(z_{1}, z_{2}\right)=a_{30} z_{1}^{3}+a_{21} z_{1}^{2} z_{2}+a_{12} z_{1} z_{2}^{2}+a_{03} z_{2}^{2} \\
& f^{2}\left(z_{1}, z_{2}\right)=b_{30} z_{1}^{3}+b_{21} z_{1}^{2} z_{2}+b_{12} z_{1} z_{2}^{2}+b_{03} z_{2}^{2}
\end{aligned}
$$

and

- $a_{30}=\mu r_{0}^{3}-\frac{1}{2} r_{0}^{2}+\frac{1}{4} \mu r_{0}-\frac{1}{8}$,
- $a_{21}=-3 \mu r_{0}^{3}+\frac{1}{2} r_{0}^{2}+\frac{1}{4} \mu r_{0}-\frac{3}{8}$,
- $a_{12}=3 \mu r_{0}^{3}+\frac{1}{2} r_{0}^{2}-\frac{1}{4} \mu r_{0}-\frac{3}{8}$,
- $a_{03}=-\mu r_{0}^{3}-\frac{1}{2} r_{0}^{2}-\frac{1}{4} \mu r_{0}-\frac{1}{8}$,
- $b_{30}=r_{0}^{3}+\frac{1}{2} \mu r_{0}^{2}+\frac{1}{4} r_{0}+\frac{1}{8} \mu$,
- $b_{21}=-3 r_{0}^{3}-\frac{1}{2} \mu r_{0}^{2}+\frac{1}{4} r_{0}+\frac{3}{8} \mu$,
- $b_{12}=3 r_{0}^{3}-\frac{1}{2} \mu r_{0}^{2}-\frac{1}{4} r_{0}+\frac{3}{8} \mu$,
- $b_{03}=-r_{0}^{3}+\frac{1}{2} \mu r_{0}^{2}-\frac{1}{4} r_{0}+\frac{1}{8} \mu$.

Consider that

$$
\begin{align*}
\dot{\mathbf{Z}}= & \epsilon\left(\left(\mathbf{T}^{-\mathbf{1}} \mathbf{A} \mathbf{T}\right) \mathbf{Z}+\mathbf{T}^{-\mathbf{1}} \mathbf{F}\right) \\
= & \epsilon\left(\left(\begin{array}{cc}
a-\sqrt{b c} & 0 \\
0 & a+\sqrt{b c}
\end{array}\right)\binom{z_{1}}{z_{2}}\right. \\
& \left.+\binom{g^{1}\left(z_{1}, z_{2}\right)}{g^{2}\left(z_{1}, z_{2}\right)}\right), \tag{5.50}
\end{align*}
$$

where

$$
\begin{align*}
g^{1}\left(z_{1}, z_{2}\right) & =\alpha_{30} z_{1}^{3}+\alpha_{21} z_{1}^{2} z_{2}+\alpha_{12} z_{1} z_{2}^{2}+\alpha_{03} z_{2}^{3}  \tag{5.51}\\
g^{2}\left(z_{1}, z_{2}\right) & =\beta_{30} z_{1}^{3}+\beta_{21} z_{1}^{2} z_{2}+\beta_{12} z_{1} z_{2}^{2}+\beta_{03} z_{2}^{3} \tag{5.52}
\end{align*}
$$

where

- $\alpha_{30}=-\sqrt{\frac{b}{c}}\left(r_{0}^{3}+\frac{1}{2} \mu r_{0}^{2}+\frac{1}{4} r_{0}+\frac{1}{8} \mu\right)+\frac{\mu}{4} r_{0}^{3}-\frac{1}{2} r_{0}^{2}+\frac{1}{4} \mu r_{0}-\frac{1}{8}$
- $\alpha_{21}=-\sqrt{\frac{b}{c}}\left(-3 r_{0}^{3}-\frac{1}{2} \mu r_{0}^{2}+\frac{1}{4} r_{0}+\frac{3}{8} \mu\right)-3 \mu r_{0}^{3}+\frac{1}{2} r_{0}^{2}+\frac{1}{4} \mu r_{0}-\frac{3}{8}$
- $\alpha_{12}=-\sqrt{\frac{b}{c}}\left(3 r_{0}^{3}-\frac{1}{2} \mu r_{0}^{2}-\frac{1}{4} r_{0}+\frac{3}{8} \mu\right)+3 \mu r_{0}^{3}+\frac{1}{2} r_{0}^{2}-\frac{1}{4} \mu r_{0}-\frac{3}{8}$
- $\alpha_{03}=-\sqrt{\frac{b}{c}}\left(-r_{0}^{3}+\frac{1}{2} \mu r_{0}^{2}-\frac{1}{4} r_{0}+\frac{1}{8} \mu\right)-\mu r_{0}^{3}-\frac{1}{2} r_{0}^{2}-\frac{1}{4} \mu r_{0}-\frac{1}{8}$
- $\beta_{30}=\sqrt{\frac{b}{c}}\left(r_{0}^{3}+\frac{1}{2} \mu r_{0}^{2}+\frac{1}{4} r_{0}+\frac{1}{8} \mu\right)+\mu r_{0}^{3}-\frac{1}{2} r_{0}^{2}+\frac{1}{4} \mu r_{0}-\frac{1}{8}$
- $\beta_{21}=\sqrt{\frac{b}{c}}\left(-3 r_{0}^{3}-\frac{1}{2} \mu r_{0}^{2}+\frac{1}{4} r_{0}+\frac{3}{8} \mu\right)-3 \mu r_{0}^{3}+\frac{1}{2} r_{0}^{2}+\frac{1}{4} \mu r_{0}-\frac{3}{8}$
- $\beta_{12}=-\sqrt{\frac{b}{c}}\left(3 r_{0}^{3}-\frac{1}{2} \mu r_{0}^{2}-\frac{1}{4} r_{0}+\frac{3}{8} \mu\right)+3 \mu r_{0}^{3}+\frac{1}{2} r_{0}^{2}-\frac{1}{4} \mu r_{0}-\frac{3}{8}$
- $\beta_{03}=\sqrt{\frac{b}{c}}\left(-r_{0}^{3}+\frac{1}{2} \mu r_{0}^{2}-\frac{1}{4} r_{0}+\frac{1}{8} \mu\right)-\mu r_{0}^{3}-\frac{1}{2} r_{0}^{2}-\frac{1}{4} \mu r_{0}-\frac{1}{8}$

From equation (5.50), $\lambda_{1}=0$ if $a=\sqrt{b c}$ and $\lambda_{2}>0$. Let $\mu=\mu_{\lambda_{20}}$ is a zero of $\lambda_{2}$ such that $\lambda_{2}\left(\mu_{\lambda_{20}}\right)=0$. Of course $(0,0)$ is unstable at $\mu=\mu_{\lambda_{20}}$, because $\lambda_{1}\left(\mu_{\lambda_{20}}\right)>0$.
We now consider the pitchfork bifurcation at $\mu=\mu_{\lambda_{10}}$.
We apply the center manifold theory to our system, i.e. to equation (5.50). Letting

$$
\begin{align*}
z_{2} \equiv h\left(z_{1}\right) & =\alpha z_{1}^{2}+\beta z_{1}^{3}+O(4)  \tag{5.53}\\
\frac{d h}{d z_{1}} & =2 \alpha z_{1}+3 \beta z_{1}^{2}+O(3) \tag{5.54}
\end{align*}
$$

At $\mu=\mu_{\lambda_{10}}$ we get

$$
\begin{equation*}
a_{4} z_{1}^{4}+O(4) \equiv \eta_{2} z_{1}^{2}+\eta_{3} z_{1}^{3}+O(4) \tag{5.55}
\end{equation*}
$$

with

$$
\begin{aligned}
& a_{4}=2 \alpha\left(\mu r_{0}^{3}-\frac{1}{2} r_{0}^{2}+\frac{1}{4} \mu r_{0}-\frac{1}{8}-\sqrt{\frac{b}{c}}\left(r_{0}^{3}+\frac{1}{2} \mu r_{0}^{2}+\frac{1}{4} r_{0}+\frac{1}{8} \mu\right)\right) \\
& \eta_{2}=(a+\sqrt{b c}) \alpha \\
& \eta_{3}=\mu r_{0}^{3}-\frac{1}{2} r_{0}^{2}+\frac{1}{2} \mu r_{0}+(a+\sqrt{b c}) \beta-\frac{1}{8}+\sqrt{\frac{b}{c}}\left(r_{0}^{3}+\frac{1}{2} \mu r_{0}^{2}+\frac{1}{4} r_{0}+\frac{1}{8} \mu\right)
\end{aligned}
$$

After equaling the coefficients of (5.55), then we get $\alpha=0$ and
$\beta=-\frac{1}{8 c(a+\sqrt{b c})}\left(8(\sqrt{b c}+\mu c) r_{0}^{3}+4(\mu \sqrt{b c}-c) r_{0}^{2}+2(c \mu+\sqrt{b c}) r_{0}+\sqrt{b c} \mu-c\right)$.
From here we get that

$$
h\left(z_{1}\right)=\beta z_{1}^{3}+O(4)
$$

The flow in the center manifold is determined by the 1-dimensional equation

$$
\begin{equation*}
\dot{\omega}=p_{1} \omega+p_{3} \omega^{3}, \tag{5.56}
\end{equation*}
$$

with $p_{1}=a-\sqrt{b c}$ and $p_{3}=-\frac{1}{8 c}\left(8(\sqrt{b c}-c \mu) r_{0}^{3}\right.$
$\left.+4(c+\sqrt{b c} \mu) r_{0}^{2}+2(\sqrt{b c}-c \mu) r_{0}+c+\sqrt{b c} \mu\right)$.
We rewrite the equation (5.56) to becomes

$$
\begin{equation*}
\dot{\omega}=q(\mu) \omega-\omega^{3}, \tag{5.57}
\end{equation*}
$$

where
$q(\mu)=\frac{-8(a-\sqrt{b c})}{\left(-3+\left(\frac{c}{b} \mu-3 \mu\right) \sqrt{\frac{b}{c}}-\frac{c}{b}-\mu \sqrt{\frac{b}{c}}\right)}$ with $\frac{c}{b}<1$.
We see that $\frac{-8}{\left(-3+\left(\frac{c}{b} \mu-3 \mu\right) \sqrt{\frac{b}{c}}-\frac{c}{b}-\mu \sqrt{\frac{b}{c}}\right)}>0$.
We can see that

1. If $\mu<\mu_{\lambda_{10}}$ then $\lambda_{1}<0$.

In this case we can also see that $q(\mu)<0$
2. If $\mu=\mu_{\lambda_{10}}$ then $\lambda_{1}=0$.

In this case we can also see that $q(\mu)=0$
3. If $\mu>\mu_{\lambda_{10}}$ then $\lambda_{1}>0$.

In this case we can also see that $q(\mu)>0$
So from (5.57) we conclude that

1. For $\mu \leq \mu_{\lambda_{10}}$ there is only one trivial solution exists.
2. For $\mu>\mu_{\lambda_{10}}$ there are three critical points we get, i.e. a zero solution and two nontrivial solutions.

## Note

In this paper, we only consider the positive nontrivial solution.
We conclude that from (5.57) the pitchfork bifurcation is born at $\mu=\mu_{\lambda_{10}}$, i.e. the zero of the eigenvalue $\lambda_{1}$.

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