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SUPPRESSION OF WIND-INDUCED VIBRATIONS OF A SEESAW-TYPE
OSCILLATOR BY MEANS OF A DYNAMIC ABSORBER

H. LUMBANTOBING

Suppression of Wind-Induced Vibrations of a Seesaw-Type Oscillator by means of a Dynamic Absorber

H. Lumbantobing[*][†]

Department of Applied Mathematical Analysis,
Faculty of Information Technology and Systems, Delft University of Technology,
Mekelweg 4, 2628 CD Delft,
The Netherlands

Abstract

In this paper the suppression of wind-induced vibrations of a seesaw-type oscillator by means of a dynamic absorber is considered. With suppression the shift of the critical flow velocity to higher values as well as the reduction of vibration amplitudes is meant. The equations of motion are derived using Lagrange's formalism. From a linear analysis the optimal absorber tuning is obtained. That is, the optimal frequency and damping combination such that the highest critical flow velocity is obtained. A strong increase of the critical flow velocity is obtained when the absorber has a frequency close to the oscillator's frequency. From the nonlinear analysis the suppression in terms of a reduction of amplitude is shown. For a specific case, a comparison between the original and the suppressed vibration behaviour is shown.

1 Introduction

In this paper the suppression of wind-induced vibrations of a one-degree-of-freedom seesaw-type oscillator, by means of a dynamic absorber, is considered. A schematic sketch of the seesaw oscillator with an absorber is given in Figure 1. It consists of a rigid bar, holding a cylinder at the right end. On the other end a counter weight is fixed, balancing the cylinder with respect to a hinge axis. Two springs provide for a restoring moment. It is assumed that the cylinder has a uniform cross-section along its axis. Inside of the cylinder, a dynamic absorber is placed. The absorber consists of a mass, a spring and a damper.

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[*]Supervised by Dr. ir. T. I. Haaker.
[†]On leave from Department of Mathematics - FKIP, University of Cenderawasih, Papua, Indonesia

The motion of the absorber mass is directed perpendicular to the oscillator's rigid bar.
If the cylinder has a non-circular cross-section and is exposed to a steady wind flow, self-excited so called galloping oscillation may arise [1]. Several authors have studied galloping for one-degree-of-freedom structures. Notably, Parkinson & Smith [2] modelled and analyzed plunge galloping for a square prism. Van der Burgh, Haaker, van Oudheusden and Lumbantobing analyzed and modelled rotational galloping of seesaw-type oscillators [3, 4, 5, 7, 8, 9, 13]. These authors applied a quasi-steady theory [1] to model the aeroelastic forces acting on the cylinder. For low to moderate wind velocities, these forces may be assumed small. Then a mathematical analysis of the equations of motion can be based on an asymptotic method [14, 15]. Typically, one finds a critical flow velocity above which the equilibrium position becomes unstable and stable galloping oscillations occur. The amplitude of these oscillations grows if the flow velocity is increased. For higher flow velocities, aeroelastic stiffness forces may arise that are of the same order of magnitude as the structural stiffness force [6, 7, 8]. The resulting aeroelastic behaviour is of a potentially catastrophic nature, as a nonlinear divergence phenomenon may occur [6].
The addition of the dynamic absorber is an important means for suppressing the wind-induced vibrations. With suppression the shift of the critical flow velocity to higher values as well as the reduction of vibration amplitudes is meant. Tondl [16, 17] investigated the effect of a dynamic absorber on vibration quenching of pendulum type systems. He applied the harmonic balance method to analyze the quenching effect. In [10, 11, 12], Matsuhisa, *et al.* investigated the efectiveness of dynamic absorbers for ropeway gondolas, chairlifts, ropeway carrier, ships and cable suspension bridges.
Here the effectiveness of a dynamic absorber for suppression of wind-induced vibrations for a seesaw-type oscillator is considered.
This paper is organized as follows. In section 2, the equation of motion for the seesaw oscillator with dynamic absorber is derived using Lagrange's formalism. In section 3, the analysis of the model equation is presented. In section 3.1, a linear analysis based on the harmonic balance method is given. The optimal tuning for the absorber is derived, such that the critical velocity is shifted to the highest possible value. In section 3.2, a nonlinear analysis based on the two-time-scales method [15, 18] is given. In particular the reduction of vibration amplitudes is considered. For a specific example, a comparison between the original and the suppressed vibration behaviour is given. This paper is ended with some conclusions in section 4.

2 Derivation of the model equation

To derive the equation of motion the Lagrange's formalism [19] is used. The kinetic and potential energies are given by the following expressions

$$E_k \;=\; \frac{1}{2}a\dot{\psi}^2 + mR\dot{x}\dot{\psi} + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mx^2\dot{\psi}^2, \tag{2.1}$$

$$E_p \;=\; \frac{1}{2}\left(\kappa\psi^2 + k_1 x^2\right) + mgx\cos(\psi), \tag{2.2}$$
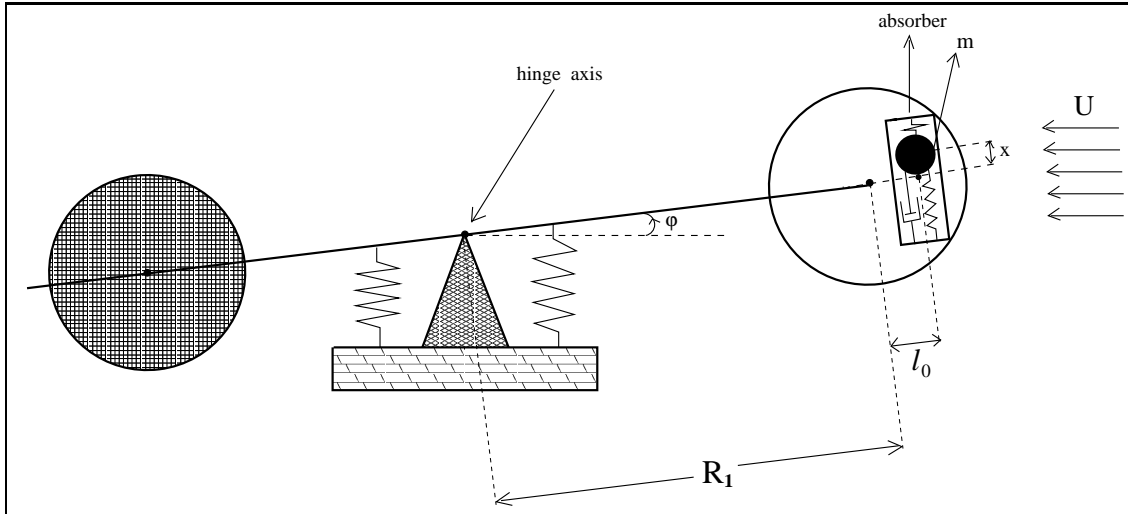
Figure 1: Schematic sketch of the structure of the seesaw oscillator with absorber.

where the "dot" denotes the differentiation with respect to time t, ψ and x denote the angle of rotation of the seesaw structure around the hinge axis and the deflection of the absorber mass from its equilibrium position, respectively. Furthermore, a, m, κ, k_1 and g are the moment of inertia of the seesaw structure including the absorber with respect to the hinge axis, mass of absorber, restoring force coefficient due to the springs attached to the seesaw bar, restoring force coefficient due to the spring of the absorber mass and acceleration of gravity, respectively. Here, $R = R_1 + l_0$, where R_1 is the distance from the cylinder's center to the hinge axis and l_0 is the distance from the equilibrium position of the small mass to the center of the cylinder, respectively.

From (2.1, 2.2) one gets the Lagrangian as follows

$$
\begin{aligned}
L &= E_k - E_p, & (2.3)\\
&= \frac{1}{2}a\dot{\psi}^2 + mR\dot{x}\dot{\psi} + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mx^2\dot{\psi}^2 \\
&\quad - \frac{1}{2}\left(\kappa\psi^2 + k_1 x^2\right) - mgx\cos(\psi). & (2.4)
\end{aligned}
$$

Then one finds that

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\psi}} - \frac{\partial L}{\partial \psi} &= a\ddot{\psi} + mR\ddot{x} + \kappa\psi + mx^2\ddot{\psi} + 2mx\dot{x}\dot{\psi} \\
&\quad - mgx\sin(\psi), & (2.5)\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= m\ddot{x} + mR\ddot{\psi} + k_1 x - mx\dot{\psi}^2 \\
&\quad + mg\cos(\psi). & (2.6)
\end{aligned}
$$

So one gets an unperturbed and undamped system as follows

$$
\begin{aligned}
a\ddot{\psi} + mR\ddot{x} + \kappa\psi + mx^2\ddot{\psi} + 2mx\dot{x}\dot{\psi} - mgx\sin(\psi) &= 0, & (2.7)\\
m\ddot{x} + mR\ddot{\psi} + k_1 x - mx\dot{\psi}^2 + mg\cos(\psi) &= 0. & (2.8)
\end{aligned}
$$

Assuming that b_1 and b_2 are damping coefficients of the main system and the dynamic absorber, respectively, and assuming that a quasi-steady aeroelastic force $F = \frac{1}{2}\rho dl U^2 C_N(\alpha)$ is exerted on the cylinder, then one gets

$$a\ddot{\psi} + mR\ddot{x} + \kappa\psi + b_1\dot{\psi} + mx^2\ddot{\psi} + 2mx\dot{x}\dot{\psi}$$

$$-mgx\sin(\psi) = \frac{1}{2}\rho dl R_1 U^2 C_N(\alpha), \tag{2.9}$$

$$m\ddot{x} + mR\ddot{\psi} + k_1 x + b_2\dot{x} - mx\dot{\psi}^2 + mg\cos(\psi) = 0, \tag{2.10}$$

where ρ, d, l, U and C_N are air density, characteristic diameter of the cylinder cross section, length of the cylinder, wind velocity and aerodynamic coefficient curve, respectively. Furthermore, the dynamic angle of attack α is approximated by $\psi - \frac{R_1\dot{\psi}}{U}$.

Using the approximation for $sin(\psi)$ and $cos(\psi)$ up to quadratic terms and transforming $x \to w - \frac{mg}{k_1}$ then one gets

$$a_0\ddot{\psi} + mR\ddot{w} + a_1\psi + b_1\dot{\psi} + mw^2\ddot{\psi} - \frac{2m^2 g}{k_1}w\ddot{\psi}$$

$$+2mw\dot{w}\dot{\psi} - \frac{2m^2 g}{k_1}\dot{w}\dot{\psi} - mgw\psi = \frac{1}{2}\rho dl R_1 U^2 C_N(\alpha), \tag{2.11}$$

$$m\ddot{w} + mR\ddot{\psi} + k_1 w + b_2\dot{w} - mw\dot{\psi}^2 + \frac{m^2 g}{k_1}\dot{\psi}^2 - \frac{1}{2}mg\psi^2 = 0, \tag{2.12}$$

where $a_0 = a + \frac{m^3 g^2}{k_1^2}$ and $a_1 = \kappa + \frac{m^2 g^2}{k_1}$.

Introducing $\Omega_1^2 = \frac{a_1}{a_0}$, $\Omega_2^2 = \frac{k_1}{m}$ and $Q = \frac{\Omega_2}{\Omega_1}$ then after transforming time $\tau \to \Omega_1 t$ one gets

$$\ddot{\psi} + \frac{mR}{a_0}\ddot{w} + \psi + \frac{b_1}{a_0\Omega_1}\dot{\psi} + \frac{m}{a_0}w^2\ddot{\psi} - \frac{2m^2 g}{a_0 k_1}w\ddot{\psi}$$

$$+\frac{2m}{a_0}w\dot{w}\dot{\psi} - \frac{2m^2 g}{a_0 k_1}\dot{w}\dot{\psi} - \frac{mg}{a_0\Omega_1^2}w\psi = \frac{1}{2a_0\Omega_1^2}\rho dl R_1 U^2 C_N(\alpha), \tag{2.13}$$

$$\ddot{w} + R\ddot{\psi} + Q^2 w + \frac{b_2}{m\Omega_1}\dot{w} - w\dot{\psi}^2 + \frac{mg}{k_1}\dot{\psi}^2 - \frac{g}{2\Omega_1^2}\psi^2 = 0, \tag{2.14}$$

where $\alpha = \psi - \frac{R_1\Omega_1}{U}\dot{\psi}$ and the "dot" now denotes differentiation with respect to τ. One can assume that the wind force is small as the air density, ρ, is $O(10^{-3})$. Introducing a small parameter $\epsilon = \frac{1}{2a_0}\rho dl R_1^3$ and a reduced velocity $\mu = \frac{U}{\Omega_1 R_1}$ and assuming $\frac{m}{a_0} = O(\epsilon)$ and $\frac{b_1}{a_0} = O(\epsilon)$, then one gets

$$\ddot{\psi} + \psi = \epsilon\left(-\lambda_1 w^2\ddot{\psi} + \lambda_2 w\ddot{\psi} - \lambda_3\ddot{w} - \beta_1\dot{\psi} - 2\lambda_1 w\dot{w}\dot{\psi}\right.$$

$$\left.+\lambda_2\dot{w}\dot{\psi} + \lambda_4 w\psi + \mu^2 C_N(\alpha)\right), \tag{2.15}$$

$$\ddot{w} + R\ddot{\psi} + Q^2 w + \beta_2\dot{w} = w\dot{\psi}^2 - \lambda_5\dot{\psi}^2 + \lambda_6\psi^2, \tag{2.16}$$

where $\alpha = \psi - \frac{\dot{\psi}}{\mu}$, $\epsilon\beta_1 = \frac{b_1}{a_0\Omega_1}$, $\beta_2 = \frac{b_2}{m\Omega_1}$, $\epsilon\lambda_1 = \frac{m}{a_0}$, $\epsilon\lambda_2 = \frac{2m^2 g}{a_0 k_1}$, $\epsilon\lambda_3 = \frac{mR}{a_0}$, $\epsilon\lambda_4 = \frac{mg}{a_0\Omega_1^2}$, $\lambda_5 = \frac{mg}{k_1}$, $\lambda_6 = \frac{g}{2\Omega_1^2}$.

Assuming the cylinder cross section to be symmetric, then one can approximate the aerodynamic coefficient curve up to the cubic term as follows $C_N(\alpha) = c_1\alpha + c_3\alpha^3$, see [20]. After substituting the aerodynamic coefficient curve into equation (2.15) and defining $\mathcal{U} = -\frac{c_1\mu}{\beta_1}$ then one gets

$$
\begin{aligned}
\ddot{\psi} + \psi \;=\; & \epsilon\left(-\lambda_3\ddot{w} - \beta_1(1-\mathcal{U})\dot{\psi} + \frac{\beta_1^2}{c_1}\mathcal{U}^2\psi - \lambda_1 w^2\ddot{\psi} \right. \\
& + \lambda_2 w\ddot{\psi} - 2\lambda_1 w\dot{w}\dot{\psi} + \lambda_2\dot{w}\dot{\psi} + \lambda_4 w\psi \\
& + c_3\left(\frac{\beta_1^2\mathcal{U}^2}{c_1^2}\psi^3 + \frac{3\beta_1\mathcal{U}}{c_1}\psi^2\dot{\psi}\right. \\
& \left.\left. + 3\psi\dot{\psi}^2 + \frac{c_1}{\beta_1\mathcal{U}}\dot{\psi}^3\right)\right),
\end{aligned}
\tag{2.17}
$$

$$
\ddot{w} + R\ddot{\psi} + Q^2 w + \beta_2\dot{w} \;=\; w\dot{\psi}^2 - \lambda_5\dot{\psi}^2 + \lambda_6\psi^2.
\tag{2.18}
$$

Introducing a final scaling to equations (2.17 - 2.18) through $\bar{w} = \frac{w}{d}$ and $\bar{\psi} = \frac{R}{d}\psi$ then after neglecting the "bar" one gets

$$
\begin{aligned}
\ddot{\psi} + \psi \;=\; & \epsilon\left(-\eta_3\ddot{w} - \beta_1(1-\mathcal{U})\dot{\psi} + \frac{\beta_1^2}{c_1}\mathcal{U}^2\psi - \eta_1 w^2\ddot{\psi} \right. \\
& + \eta_2 w\ddot{\psi} - 2\eta_1 w\dot{w}\dot{\psi} + \eta_2\dot{w}\dot{\psi} + \eta_4 w\psi \\
& + \tilde{c}_3\left(\frac{\beta_1^2\mathcal{U}^2}{c_1^2}\psi^3 + \frac{3\beta_1\mathcal{U}}{c_1}\psi^2\dot{\psi}\right. \\
& \left.\left. + 3\psi\dot{\psi}^2 + \frac{c_1}{\beta_1\mathcal{U}}\dot{\psi}^3\right)\right),
\end{aligned}
\tag{2.19}
$$

$$
\ddot{w} + \ddot{\psi} + Q^2 w + \beta_2\dot{w} \;=\; -\eta_5\dot{\psi}^2 + \eta_6\psi^2 + \eta_7 w\dot{\psi}^2,
\tag{2.20}
$$

where $\eta_1 = d^2\lambda_1$, $\eta_2 = d\lambda_2$, $\eta_3 = R\lambda_3$, $\eta_4 = d\lambda_4$, $\eta_5 = \frac{d}{R^2}\lambda_5$, $\eta_6 = \frac{d}{R^2}\lambda_6$, $\eta_7 = \frac{d^2}{R^2}$ and $\tilde{c}_3 = \frac{c_3 d^2}{R^2}$. Note that $\mathcal{U} = 1$ corresponds with the critical flow velocity for the seesaw oscillator without absorber.

3 Analysis of the equation

In this section the equations (2.19 - 2.20) are analyzed. Firstly, the harmonic balance method is applied for the linear analysis and then the two time scales method is applied for the nonlinear analysis.

3.1 Linear analysis

In this section a linear analysis for equations (2.19 - 2.20) around the equilibrium position is presented. One gets the following linearized equations from (2.19 - 2.20)

$$
\ddot{\psi} + \left(1 - \frac{\epsilon\beta_1^2}{c_1}\mathcal{U}^2\right)\psi + \epsilon\beta_1\left(1-\mathcal{U}\right)\dot{\psi} + \epsilon\eta_3\ddot{w} = 0,
\tag{3.21}
$$

$$\ddot{w} + \ddot{\psi} + Q^2 w + \beta_2 \dot{w} = 0. \tag{3.22}$$

To apply the harmonic balance method one sets $\psi = cos(\Omega\tau)$ and $w = Acos(\Omega\tau) + Bsin(\Omega\tau)$ and substituting into equations (3.21 - 3.22) then one finds

$$(1 - \Omega^2) - \frac{\epsilon\beta_1^2\mathcal{U}^2}{c_1} - \epsilon\eta_3\Omega^2 A = 0, \tag{3.23}$$

$$\beta_1(1 - \mathcal{U}) + \eta_3\Omega B = 0, \tag{3.24}$$

$$(Q^2 - \Omega^2)A + \beta_2\Omega B = \Omega^2, \tag{3.25}$$

$$-\beta_2\Omega A + (Q^2 - \Omega^2)B = 0. \tag{3.26}$$

Note that in this analysis no assumptions have to be made with respect to the size of ϵ. From equations (3.25 - 3.26) one finds $A = \frac{\Omega^2(Q^2 - \Omega^2)}{(Q^2 - \Omega^2)^2 + \beta_2^2\Omega^2}$ and $B = \frac{\beta_2\Omega^3}{(Q^2 - \Omega^2)^2 + \beta_2^2\Omega^2}$. After substituting B into equation (3.24) one gets

$$\mathcal{U} = 1 + \frac{\beta_2\eta_3\Omega^4}{\beta_1\left((Q^2 - \Omega^2)^2 + \beta_2^2\Omega^2\right)}. \tag{3.27}$$

After substituting A and $\mathcal{U}$ into equation (3.23) then one gets

$$\left((1 - \Omega^2) - \frac{\epsilon\beta_1^2}{c_1}\left(1 + \frac{\beta_2\eta_3\Omega^4}{\beta_1\left((Q^2 - \Omega^2)^2 + \beta_2^2\Omega^2\right)}\right)^2\right) \times$$
$$\left((Q^2 - \Omega^2)^2 + \beta_2^2\Omega^2\right) - \epsilon\eta_3\Omega^4(Q^2 - \Omega^2) = 0. \tag{3.28}$$

Without a dynamical absorber the critical wind velocity for the onset of galloping is $\mathcal{U} = 1$. From (3.27) one finds that after applying the dynamical absorber the critical value of the velocity is shifted to a higher value. A numerical result is presented in Figure 2 showing the minimum values for the wind velocity $\mathcal{U}$ in dependence on the tuning coefficient Q for different values of the coefficient β_2 of the absorber damping. (The function $\mathcal{U}(Q)$ is arranged by solving equation (3.28) for Ω then substituting the obtained Ω into equation (3.27)). For the calculation the dynamic absorber is located in the center of the cylinder, i.e. $l_0 = 0$. The values of some parameters are chosen from wind tunnel measurements on an actual seesaw oscillator in [4]. Figure 2(a) shows the critical wind velocity, $\mathcal{U}$, depending on tuning Q for different values of the absorber damping coefficient β_2 with $\eta_3 = 0.33, \beta_1 = 1.285, \epsilon = 1.33 \times 10^{-3}$ and $c_1 = -3$. For smaller values of β_2, $\mathcal{U}(Q)$ consists of two branches reaching its optimal value near $Q = 1$. The lower branch (for $Q < 1$) corresponds to higher frequencies Ω, the lower branch (for $Q > 1$) to the lower frequencies Ω. Figure 2(b) shows the projection of Figure 2(a) on the $(\beta_2, \mathcal{U})$-plane. One finds that $\mathcal{U}$ reaches its optimal value at $\mathcal{U} = 8.375$, near $\beta_2 = 0.028$ and tuning $Q = 1.021$, with resulting oscillation frequency $\Omega = 1.029$. Figure 3(a) shows the oscillation frequency of the obtained periodic solution as a function of tuning Q for $\beta_2 = 0.028$. From this figure, one can see that there are some values of Q having three corresponding values of Ω^2. This means that three periodic solutions may exist for some (β_2, Q) combinations,
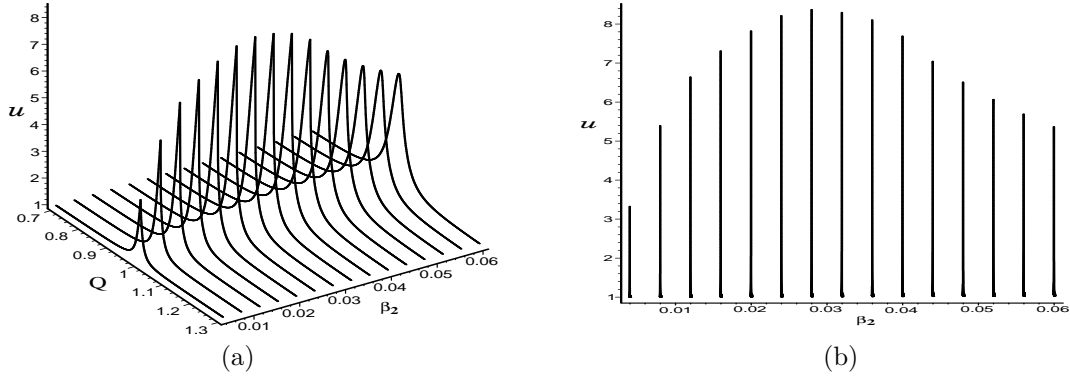
Figure 2: Critical wind velocity $\mathcal{U}$ in dependence on Q for different values of damping coefficient β_2.

each corresponding with a distinct critical flow velocity. Now considering the stability of the trivial solution, one obtains a diagram as given in Figure 3(b). The trivial solution is stable for $0 < \mathcal{U} < \mathcal{U}_1$ and for $\mathcal{U}_2 < \mathcal{U} < \mathcal{U}_3$ and is unstable for $\mathcal{U}_1 < \mathcal{U} < \mathcal{U}_2$ and for $\mathcal{U} > \mathcal{U}_3$. So, after a first instability at $\mathcal{U} = \mathcal{U}_1$, re-stabilization occurs for $\mathcal{U} = \mathcal{U}_2$. The final instability occurs for $\mathcal{U} = \mathcal{U}_3$. For example, for the optimal damping, i.e. $\beta_2 = 0.028$ at $Q = 1.022$ one gets three periodic solutions with frequencies $\Omega_1 = 1.005$, $\Omega_2 = 1.018$ and $\Omega_3 = 1.06$, respectively. For each frequency, one obtains the critical flow velocities $\mathcal{U}_1 = 4.048$, $\mathcal{U}_2 = 5.953$ and $\mathcal{U}_3 = 8.507$, respectively. The diagram of the stability of the trivial solution is that of Figure 3(b).

3.2 Nonlinear analysis

In this section the equations (2.19 - 2.20) are considered again. In the analysis the nonlinear terms in the second equation are neglected because the strong linear coupling term is the most important. Note that the coefficients of the nonlinear terms contain $\frac{d}{R^2} \ll 1$. From equations (2.19 - 2.20) one gets

$$
\begin{aligned}
\ddot{\psi} + \psi &= \epsilon \left(-\eta_3 \ddot{w} - \beta_1(1-\mathcal{U})\dot{\psi} + \frac{\beta_1^2}{c_1}\mathcal{U}^2\psi - \eta_1 w^2\ddot{\psi} \right. \\
&\quad + \eta_2 w\ddot{\psi} - 2\eta_1 w\dot{w}\dot{\psi} + \eta_2 \dot{w}\dot{\psi} + \eta_4 w\psi \\
&\quad + \tilde{c}_3 \left(\frac{\beta_1^2\mathcal{U}^2}{c_1^2}\psi^3 + \frac{3\beta_1\mathcal{U}}{c_1}\psi^2\dot{\psi} \right. \\
&\quad \left. \left. + 3\psi\dot{\psi}^2 + \frac{c_1}{\beta_1\mathcal{U}}\dot{\psi}^3 \right) \right), \quad (3.29) \\
\ddot{w} + \ddot{\psi} + Q^2 w + \beta_2 \dot{w} &= 0. \quad (3.30)
\end{aligned}
$$

The two time scales method is applied by introducing new time variables $\xi = \tau$ and $\eta = \epsilon\tau$ [21] and assuming $\psi = \psi(\xi, \eta)$ and $w = w(\xi, \eta)$ then one gets
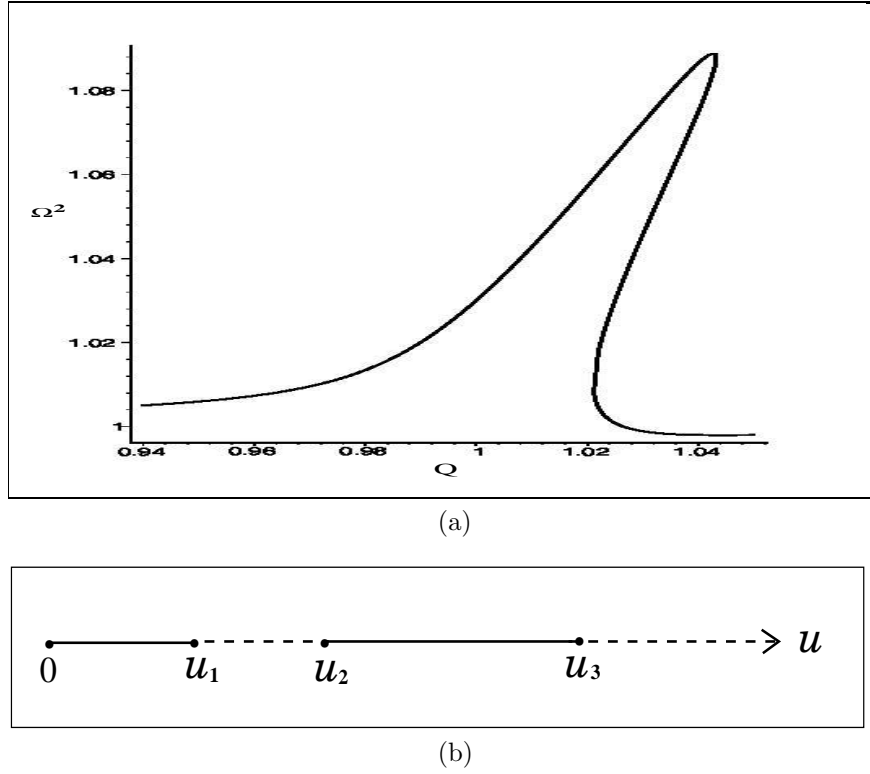
(a)

(b)

Figure 3: (a): Quadratic frequency of periodic solution as a function of tuning Q for $\beta_2 = 0.028$. (b): Stability diagram of the trivial solution depending on the wind velocity for the optimal absorber tuning, i.e. $(\beta_2, Q) = (0.028, 1.022)$.

$$\frac{d\psi(\xi,\eta)}{d\tau} = \frac{\partial\psi(\xi,\eta)}{\partial\xi} + \epsilon\frac{\partial\psi(\xi,\eta)}{\partial\eta},$$

$$\frac{d^2\psi(\xi,\eta)}{d\tau^2} = \frac{\partial^2\psi(\xi,\eta)}{\partial\xi^2} + 2\epsilon\frac{\partial^2\psi(\xi,\eta)}{\partial\xi\partial\eta} + \epsilon^2\frac{\partial^2\psi(\xi,\eta)}{\partial\eta^2},$$

$$\frac{dw(\xi,\eta)}{d\tau} = \frac{\partial w(\xi,\eta)}{\partial\xi} + \epsilon\frac{\partial w(\xi,\eta)}{\partial\eta},$$

$$\frac{d^2w(\xi,\eta)}{d\tau^2} = \frac{\partial^2w(\xi,\eta)}{\partial\xi^2} + 2\epsilon\frac{\partial^2w(\xi,\eta)}{\partial\xi\partial\eta} + \epsilon^2\frac{\partial^2w(\xi,\eta)}{\partial\eta^2}.$$

Expanding $\psi(\xi,\eta) = \psi_0(\xi,\eta) + \epsilon\psi_1(\xi,\eta) + \cdots$, and $w(\xi,\eta) = w_0(\xi,\eta) + \epsilon w_1(\xi,\eta) + \cdots$ and substituting these into equations (3.29 - 3.30) then one gets:

- A system for $O(1)$ as follows

$$\psi_{0_{\xi\xi}} + \psi_0 = 0, \tag{3.31}$$
$$w_{0_{\xi\xi}} + \psi_{0_{\xi\xi}} + Q^2 w_0 + \beta_2 w_{0_\xi} = 0. \tag{3.32}$$

- A system for $O(\epsilon)$ as follows

$$\psi_{1_{\xi\xi}} + \psi_1 = -2\psi_{0_{\xi\eta}} - \eta_3 w_{0_{\xi\xi}} - \beta_1(1-\mathcal{U})\psi_{0_\xi} + \frac{\beta_1^2\mathcal{U}^2}{c_1}\psi_0$$

$$- \eta_1 w_0^2 \psi_{0\xi\xi} + \eta_2 w_0 \psi_{0\xi\xi} - 2\eta_1 \psi_{0\xi} w_0 w_{0\xi}$$
$$+ \eta_2 w_{0\xi} \psi_{0\xi} + \eta_4 \psi_0 w_0$$
$$+ \tilde{c}_3 \left(\frac{\beta_1^2 \mathcal{U}^2}{c_1^2} \psi_0^3 + \frac{3\beta_1 \mathcal{U}}{c_1} \psi_0^2 \psi_{0\xi} \right.$$
$$\left. + 3\psi_0 \psi_{0\xi}^2 + \frac{c_1}{\beta_1 \mathcal{U}} \psi_{0\xi}^3 \right), \tag{3.33}$$

$$w_{1\xi\xi} + \psi_{1\xi\xi} \quad + \quad Q^2 w_1 + \beta_2 w_{1\xi} = -2 \left(w_{0\xi\eta} + \psi_{0\xi\eta} \right)$$
$$- \beta_2 w_{0\eta}. \tag{3.34}$$

From equation (3.31) one gets a general solution as follows

$$\psi_0(\xi, \eta) \quad = \quad \mathcal{A}(\eta) \, \cos(\xi + \phi(\eta)). \tag{3.35}$$

After substituting the solution into equation (3.32) then the equation becomes

$$w_{0\xi\xi} + \beta_2 w_{0\xi} + Q^2 w_0 \quad = \quad \mathcal{A}(\eta) \, \cos(\xi + \phi(\eta)). \tag{3.36}$$

The general solution of equation (3.36) is

$$w_0(\xi, \eta) \quad = \quad c_1(\eta) e^{r_1 \xi} + c_2(\eta) e^{r_2 \xi} + f_1(\eta) \cos(\xi + \phi(\eta))$$
$$+ f_2(\eta) \sin(\xi + \phi(\eta)), \tag{3.37}$$

with
$r_{1,2} = \frac{1}{2} \left(-\beta_2 \pm \sqrt{\beta_2^2 - 4Q^2} \right)$,
$f_1(\eta) = \frac{(Q^2-1)\mathcal{A}(\eta)}{(Q^2-1)^2 + \beta_2^2}$,
$f_2(\eta) = \frac{\beta_2 \mathcal{A}(\eta)}{(Q^2-1)^2 + \beta_2^2}$,
where r_1 and r_2 are either real negative or complex with negative real part, $c_1(\eta), c_2(\eta)$ and $\phi(\eta)$ can be determined from initial conditions.
From equations (3.35) and (3.37) one gets

$$\psi_{0\xi} \quad = \quad -\mathcal{A}(\eta) \sin(\xi + \phi(\eta)),$$
$$\psi_{0\xi\eta} \quad = \quad -\frac{d\mathcal{A}(\eta)}{d\eta} \sin(\xi + \phi(\eta)) - \mathcal{A}(\eta) \cos(\xi + \phi(\eta)) \frac{d\,\phi(\eta)}{d\eta},$$
$$\psi_{0\xi\xi} \quad = \quad -\mathcal{A}(\eta) \cos(\xi + \phi(\eta)),$$
$$w_{0\xi} \quad = \quad r_1 c_1(\eta) e^{r_1 \xi} + r_2 c_2(\eta) e^{r_2 \xi} - f_1(\eta) \sin(\xi + \phi(\eta))$$
$$+ f_2(\eta) \cos(\xi + \phi(\eta)),$$
$$w_{0\eta} \quad = \quad e^{r_1 \xi} \frac{d\,c_1(\eta)}{d\eta} + e^{r_2 \xi} \frac{d\,c_2(\eta)}{d\eta} + \cos(\xi + \phi(\eta)) \frac{d\,f_1(\eta)}{d\eta}$$
$$+ \sin(\xi + \phi(\eta)) \frac{d\,f_2(\eta)}{d\eta}$$

$$+ \left(-f_1(\eta)\sin(\xi + \phi(\eta)) + f_2(\eta)\cos(\xi + \phi(\eta)) \right) \frac{d\,\phi(\eta)}{d\eta},$$

$$w_{0_{\xi\xi}} = r_1^2 c_1(\eta)e^{r_1\xi} + r_2^2 c_2(\eta)e^{r_2\xi} - f_1(\eta)\cos(\xi + \phi(\eta))$$
$$- f_2(\eta)\sin(\xi + \phi(\eta)),$$

$$w_{0_{\xi\eta}} = r_1 e^{r_1\xi}\frac{d\,c_1(\eta)}{d\eta} + r_2 e^{r_2\xi}\frac{d\,c_2(\eta)}{d\eta} - \sin(\xi + \phi(\eta))\frac{d\,f_1(\eta)}{d\eta}$$
$$+ \cos(\xi + \phi(\eta))\frac{d\,f_2(\eta)}{d\eta}$$
$$- \left(f_1(\eta)\cos(\xi + \phi(\eta)) + f_2(\eta)\sin(\xi + \phi(\eta)) \right)\frac{d\,\phi(\eta)}{d\eta}.$$

After substituting these into the equations of order $O(\epsilon)$, one gets

$$\psi_{1_{\xi\xi}} + \psi_1 = g_1(\xi, \eta), \qquad (3.38)$$
$$w_{1_{\xi\xi}} + Q^2 w_1 + \beta_2 w_{1_\xi} + \psi_{1_{\xi\xi}} = g_2(\xi, \eta), \qquad (3.39)$$

where

$$
\begin{aligned}
g_1(\xi, \eta) =\ & a_0 + a_1 c_1(\eta)e^{r_1\xi} + a_2 c_2(\eta)e^{r_2\xi} \\
& + \eta_1 \mathcal{A}(\eta)\left(c_1(\eta)e^{r_1\xi} + c_2(\eta)e^{r_2\xi}\right)\cos(\xi + \phi(\eta)) \\
& + (\eta_4 - \eta_2)\,\mathcal{A}(\eta)\left(c_1(\eta)e^{r_1\xi} + c_2(\eta)e^{r_2\xi}\right)\cos(\xi + \phi(\eta)) \\
& + 2\eta_1 \mathcal{A}(\eta)\left(r_1 c_1(\eta)^2 e^{2r_1\xi} + r_2 c_2(\eta)^2 e^{2r_2\xi}\right. \\
& +(r_1 + r_2)c_1(\eta)c_2(\eta)e^{(r_1+r_2)\xi} \\
& -\eta_2\left(r_1 c_1(\eta)e^{r_1\xi} + r_2 c_2(\eta)e^{r_2\xi}\right)\Big)\sin\left(\xi + \phi(\eta)\right) \\
& + \left(a_3 c_1(\eta)e^{r_1\xi} + a_4 c_2(\eta)e^{r_2\xi}\right)\cos\left(2\left(\xi + \phi(\eta)\right)\right) \\
& + \left(a_5 c_1(\eta)e^{r_1\xi} + a_6 c_2(\eta)e^{r_2\xi}\right)\sin\left(2\left(\xi + \phi(\eta)\right)\right) \\
& + a_7 \cos(\xi + \phi(\eta)) + a_8 \sin(\xi + \phi(\eta)) \\
& + a_9 \cos(2(\xi + \phi(\eta))) + a_{10}\sin(2(\xi + \phi(\eta))) \\
& + a_{11}\cos(3(\xi + \phi(\eta))) + a_{12}\sin(3(\xi + \phi(\eta))), \\
g_2(\xi, \eta) =\ & b_1 r_1 e^{r_1\xi} + b_2 r_2 e^{r_2\xi} + b_3 \cos(\xi + \phi(\eta)) + b_4 \sin(\xi + \phi(\eta))
\end{aligned}
$$

with

$$a_0 = \frac{(Q^2 - 1)\eta_4\,\mathcal{A}(\eta)^2}{2D0},$$

$$a_1 = \frac{r_1\,\eta_1\,\beta_2\,\mathcal{A}(\eta)^2}{D0} - \eta_3 r_1^2,$$

$$a_2 = \frac{r_2\,\eta_1\,\beta_2\,\mathcal{A}(\eta)^2}{D0} - \eta_3 r_2^2,$$

$$a_3 = \frac{\eta_1}{D_0} \mathcal{A}(\eta)^2 \left(2\left(Q^2 - 1\right) - \beta_2 r_1\right),$$

$$a_4 = \frac{\eta_1}{D_0} \mathcal{A}(\eta)^2 \left(2\left(Q^2 - 1\right) - \beta_2 r_2\right),$$

$$a_5 = \frac{\eta_1}{D_0} \mathcal{A}(\eta)^2 \left(r_1 Q^2 + 2\beta_2 - r_1\right),$$

$$a_6 = \frac{\eta_1}{D_0} \mathcal{A}(\eta)^2 \left(r_2 Q^2 + 2\beta_2 - r_2\right),$$

$$a_7 = \left(\frac{\beta_1^2 \mathcal{U}^2}{c_1} + 2\frac{d\phi(\eta)}{d\eta} + \frac{\eta_3}{D_0}(1 - Q^2)\right) \mathcal{A}(\eta)$$
$$+ \left(\frac{3\tilde{c}_3}{4}\left(1 + \frac{\beta_1^2 \mathcal{U}^2}{c_1^2}\right) + \frac{\eta_1}{4D_0^2}\left(\left(Q^2 - 1\right)^2 + 3\beta_2^2\right)\right) \mathcal{A}(\eta)^3,$$

$$a_8 = 2\frac{d\mathcal{A}(\eta)}{d\eta} + \left(\beta_1\left(1 - \mathcal{U}\right) + \frac{\beta_2 \eta_3}{D_0}\right) \mathcal{A}(\eta)$$
$$- \left(\frac{3\tilde{c}_3}{4}\left(\frac{c_1}{\beta_1 \mathcal{U}} + \frac{\beta_1 \mathcal{U}}{c_1}\right) + \frac{\beta_2 \eta_1}{2D_0}\left(Q^2 - 1\right)\right) \mathcal{A}(\eta)^3,$$

$$a_9 = \frac{\left(Q^2 - 1\right)}{D_0}\left(\frac{\eta_4}{2} - \eta_2\right) \mathcal{A}(\eta)^2,$$

$$a_{10} = \left(\frac{\eta_4 - 2\eta_2}{2D_0}\right) \beta_2 \mathcal{A}(\eta)^2,$$

$$a_{11} = \left(-\frac{3\tilde{c}_3}{4}\left(\frac{c_1}{\mathcal{U}} - \frac{\beta_1 \mathcal{U}}{c_1}\right) + \frac{\eta_1 \beta_2 \left(1 - Q^2\right)}{2D_0^2}\right) \mathcal{A}(\eta)^3,$$

$$a_{12} = \left(\frac{\tilde{c}_3}{4}\left(\frac{c_1}{\beta_1 \mathcal{U}} - \frac{3\beta_1 \mathcal{U}}{c_1}\right) + \frac{3\beta_2 \eta_1}{2D_0^2}\left(Q^2 - 1\right)\right) \mathcal{A}(\eta).$$

$$b_1 = -\left(2\frac{d\,c_1(\eta)}{d\eta} + \beta_2 c_1(\eta)\right),$$

$$b_2 = -\left(2\frac{d\,c_2(\eta)}{d\eta} + \beta_2 c_2(\eta)\right)$$

$$b_3 = \frac{df_2(\eta)}{d\eta} - f_1(\eta)\frac{d\phi(\eta)}{d\eta} - \mathcal{A}(\eta)\frac{d\,\phi(\eta)}{d\eta} - \beta_2 f_2(\eta),$$

$$b_4 = -\frac{d\,f_1(\eta)}{d\eta} - f_2(\eta)\frac{d\,\phi(\eta)}{d\eta} - \frac{d\,\mathcal{A}(\eta)}{d\eta}\frac{d\,\phi(\eta)}{d\eta} + \beta_2 f_1(\eta),$$

$$D_0 = \left(Q^2 - 1\right)^2 + \beta_2^2.$$

From equation (3.38) one knows that the secular terms are the terms containing $\cos(\xi + \phi(\eta))$ and $\sin(\xi + \phi(\eta))$. After equating to zero the coefficient of the secular terms, one gets

$$\left(\frac{\beta_1^2 \mathcal{U}^2}{c_1} + 2\frac{d\phi(\eta)}{d\eta} + \frac{\eta_3(Q^2 - 1)}{D_0}\right) \mathcal{A}(\eta)$$
$$+ \left(\frac{\eta_1}{4D_0^2}\left(\left(Q^2 - 1\right)^2 + 3\beta_2^2\right) + \frac{3}{4}\tilde{c}_3\left(1 + \frac{\beta_1^2 \mathcal{U}^2}{c_1^2}\right)\right) \mathcal{A}(\eta)^3 = 0, \tag{3.40}$$

$$\frac{d\mathcal{A}(\eta)}{d\eta} + \frac{1}{2}\left(\beta_1 - \beta_1 \mathcal{U} + \frac{\beta_2 \eta_3}{D_0}\right) \mathcal{A}(\eta)$$
$$- \frac{1}{2}\left(\frac{3\tilde{c}_3(c_1^2 + \beta_1^2 \mathcal{U}^2)}{4\beta_1 c_1 \mathcal{U}} + \frac{\beta_2 \eta_1}{2D_0^2}(Q^2 - 1)\right) \mathcal{A}(\eta)^3 = 0. \tag{3.41}$$

From equation (3.41) one obtains

$$\frac{d\mathcal{A}(\eta)}{d\eta} = -\frac{1}{2}\left(\beta_1 - \beta_1\mathcal{U} + \frac{\beta_2\eta_3}{D_0}\right)\mathcal{A}(\eta)$$
$$+ \frac{1}{2}\left(\frac{3\tilde{c}_3(c_1^2 + \beta_1^2\mathcal{U}^2)}{4\beta_1 c_1\mathcal{U}} + \frac{\beta_2\eta_1}{2D_0^2}(Q^2 - 1)\right)\mathcal{A}(\eta)^3, \tag{3.42}$$

from which the critical flow velocity follows as

$$\mathcal{U} = 1 + \frac{\beta_2\eta_3}{\beta_1\left((Q^2 - 1)^2 + \beta_2^2\right)}.$$

Note that the highest flow velocity is obtained for $Q = 1$.
The nontrivial critical point of equation (3.42) is

$$\mathcal{A} = \left(\frac{4\beta_1 c_1 D_0\mathcal{U}\left(\beta_1 D_0(1 - \mathcal{U}) + \beta_2\eta_3\right)}{3\tilde{c}_3 D_0^2(c_1^2 + \beta_1^2\mathcal{U}^2) + 2\beta_1\beta_2 c_1\eta_1\mathcal{U}(Q^2 - 1)}\right)^{\frac{1}{2}}. \tag{3.43}$$

In absence of a dynamic absorber one gets the following

$$\mathcal{A}_0 = \left(\frac{4\beta_1^2 c_1\mathcal{U}(1 - \mathcal{U})}{3\tilde{c}_3(c_1^2 + \beta_1^2\mathcal{U}^2)}\right)^{\frac{1}{2}}. \tag{3.44}$$

Comparing $\mathcal{A}$ from (3.43) and with $\mathcal{A}_0$ from (3.44) one obtains

$$\frac{\mathcal{A}}{\mathcal{A}_0} = \left(\frac{1 + \frac{\beta_2\eta_3}{\beta_1 D_0(1 - \mathcal{U})}}{1 + \frac{2\beta_1\beta_2 c_1\eta_1\mathcal{U}(Q^2 - 1)}{3\tilde{c}_3 D_0^2(c_1^2 + \beta_1^2\mathcal{U}^2)}}\right)^{\frac{1}{2}}. \tag{3.45}$$

From equation (3.43) it follows readily that, for case $Q \leq 1$ and $\mathcal{U} > 1 + \frac{\beta_2\eta_3}{\beta_1 D_0}$ one gets

$$\mathcal{A} = \left(\frac{4\beta_1 c_1 D_0\mathcal{U}\left\{\beta_1 D_0(1 - \mathcal{U}) + \beta_2\eta_3\right\}}{3\tilde{c}_3 D_0^2(c_1^2 + \beta_1^2\mathcal{U}^2) + 2\beta_1\beta_2 c_1\eta_1\mathcal{U}(Q^2 - 1)}\right)^{\frac{1}{2}},$$
$$< \left(\frac{4\beta_1^2 c_1\mathcal{U}(1 - \mathcal{U})}{3\tilde{c}_3(c_1^2 + \beta_1^2\mathcal{U}^2)}\right)^{\frac{1}{2}},$$
$$= \mathcal{A}_0. \tag{3.46}$$

From here one concludes that the dynamic absorber is useful to suppress the amplitude of the vibrations. A similar result holds for $Q > 1$. From equation (3.45), it can be shown that the optimal amplitude suppression, i.e. the lowest ratio of $\mathcal{A}$ over $\mathcal{A}_0$, is obtained for $Q = 1$. In Figure 4, the amplitude ratio $\frac{\mathcal{A}}{\mathcal{A}_0}$ as a function of wind velocity $\mathcal{U}$ for various values of tuning parameter Q is shown. The same specific parameter values are used as for the linear example, i.e. one sets $\eta_1 = 0.024$, $\eta_3 = 0.33$, $\beta_1 = 1.285$, $c_1 = -3$, $\tilde{c}_3 = 5.867$ and
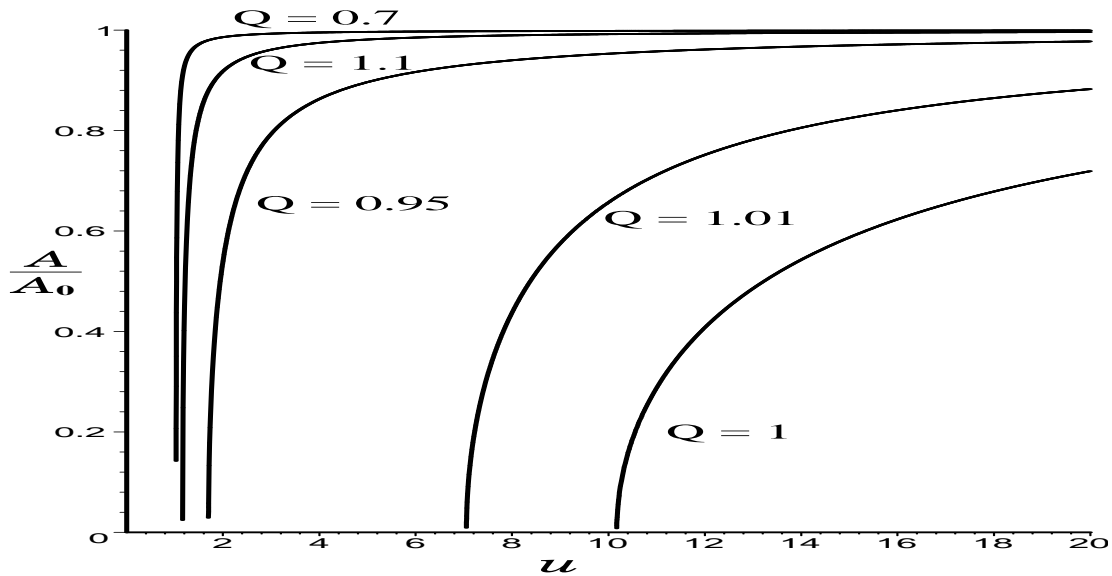
Figure 4: Comparison of amplitude ratios $\frac{A}{A_0}$ versus wind velocity $\mathcal{U}$ for $\beta_2 = 0.028$ and various values of Q

$\beta_2 = 0.028$. The figure shows clearly that the optimal suppression result is obtained for $Q = 1$. In that case the critical flow velocity is the highest and the vibration amplitude is the lowest. Figure 5 shows the comparison of amplitude and critical wind velocity for the seesaw oscillator with and without a dynamic absorber. One can see that the critical velocity is shifted from $\mathcal{U} = 1$ to $\mathcal{U} = 10.17$. Also the amplitude of the vibrations is suppressed to a lower value for $\mathcal{U} > 1$. Note that this critical flow velocity compares with the highest critical flow velocity, $\mathcal{U}$, as obtained from the linear example. The asymptotic analysis does not reproduce the re-stabilization phenomenon as obtained from the linear analysis. This may be understood from the fact that the linear analysis does not assume ϵ to be small. The asymptotic analysis under-estimates the actual critical flow velocity. However, the amplitude occurring after the first critical flow velocity are rather small. The asymptotic analysis adequately calculate the final instability after which oscillations with larger amplitudes are found.

4   Conclusions

The suppression of wind-induced vibrations of a seesaw-type oscillator by means of a dynamic absorber has been considered in this paper. The absorber is placed inside the cylinder of the seesaw oscillator.

The linear analysis shows that an optimal combination of absorber tuning Q and damping coefficient β_2 exists, such that the critical flow velocity is shifted to the highest possible value. This analysis also shows that a re-stabilization phenomenon may occur for a flow velocity above the critical flow velocity. Subsequently, a second critical flow velocity is
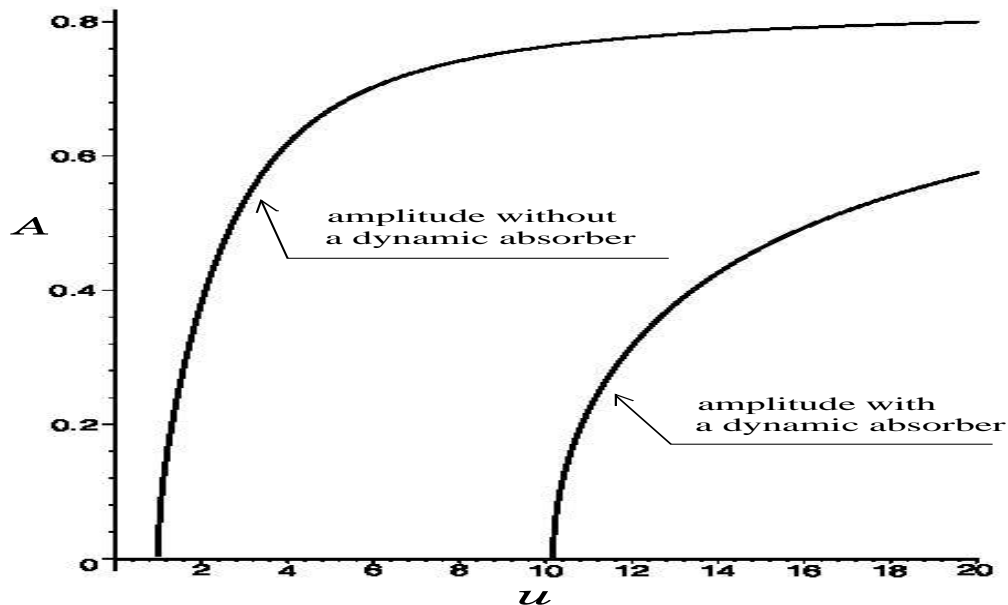
Figure 5: Comparison of amplitude and critical wind velocity for the seesaw oscillator with and without a dynamic absorber for $\beta_2 = 0.028$ and $Q = 1$.

found.

The nonlinear analysis using the two time scales method, shows that the absorber suppresses the vibration amplitudes that exist above the critical flow velocity. The (single) critical flow velocity obtained from the asymptotic analysis corresponds with the highest critical flow velocity obtained from the linear analysis. The asymptotic analysis neglects the first instability and does not reproduce the re-stabilization phenomenon.

One concludes that a small dynamic absorber is capable of suppressing wind-induced vibrations of a seesaw-type oscillator in the following way,

1. The critical flow velocity is shifted to a higher value from $\mathcal{U} = 1$ to $\mathcal{U} = 1 + \frac{\beta_2 \eta_3}{\beta_1 \left((Q^2-1)^2 + \beta_2^2 \right)}$.

2. The amplitudes of the oscillations occurring above the critical flow velocity are reduced.

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