Codes, blocking sets and graphs

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Overview

Extremal problems at the intersection of

- Coding theory
- Computer science
- Finite geometry

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Joint work(s) with Noga Alon, Shagnik Das, Dion Gijswijt, Jozefien D'Haeseleer, Alessandro Neri, and Aditya Potukuchi.

Trifferent codes

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$$
\begin{array}{ccccc}
0 & 0 & 0 & 0 & \leftarrow \\
1 & 0 & 1 & 2 & \\
2 & 0 & 2 & 1 & \\
0 & 2 & 2 & 2 & \leftarrow \\
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\end{array}
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A trifferent code of size 9 and length 4.

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Theorem (Körner 1973)

$$T(n) \leq 2\,(1.5)^n$$

Theorem (Körner-Marton 1984)

$$T(n) \geq (9/5)^{n/4} \simeq (1.158)^n$$

Linear trifferent codes

Identify $\{0, 1, 2\}$ with the finite field $\mathbb{F}_3 \cong \mathbb{Z}/3\mathbb{Z}$.

> *What is the largest size $T_L(n)$ of a trifferent code $C \subseteq \mathbb{F}_3^n$ which is also a* **linear subspace***?*

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then, $C = \langle \{(0, 1, 1, 1), (1, 0, 1, 2)\} \rangle$.

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- Further motivations coming up soon ...

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Theorem (Wang-Xing 2001)

There are (explicit) linear trifferent codes of length n and dimension $\frac{n}{112}$, and thus

$$T_L(n) \geq (1.0098)^n.$$

Our results

Bishnoi, D'Haeseleer, Gijswijt, Potukuchi, *Blocking sets, minimal codes and trifferent codes* arXiv:2301.09457

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Every linear trifferent code of length n has dimension at most $\frac{n}{4.55}$, and thus

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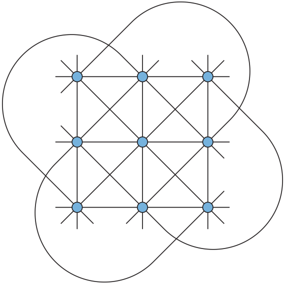
Theorem

An explicit construction of linear trifferent codes of length n and dimension $\frac{n}{13.5}$.

Finite affine spaces



The Affine plane $\mathbb{F}_3^2$

Points: $\mathbb{F}_q^2$
Lines: translates of 1-dimensional vector subspaces

Smallest affine blocking sets

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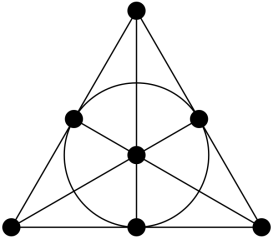
Theorem (Lovász 1975)

$$b_q(k, s) \leq q^s \left(1 + \ln \begin{bmatrix} k \\ s \end{bmatrix}_q \right) \approx (q^s \ln q)(s(k - s)).$$

Projective blocking sets



The projective plane over $\mathbb{F}_2^3$.

How many 1-dimensional subspaces of $\mathbb{F}_q^k$ do we need to block every hyperplane?

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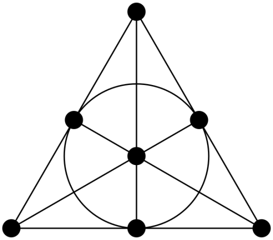

The projective plane over $\mathbb{F}_2^3$.

How many 1-dimensional subspaces of $\mathbb{F}_q^k$ do we need to block every hyperplane?

Answer: $q+1$ subspaces spanning a plane.

Strong blocking sets

How many 1-dimensional linear subspaces of $\mathbb{F}_q^k$ do we need to meet every $(k-1)$-dimensional linear subspace in a spanning set?

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How many 1-dimensional linear subspaces of $\mathbb{F}_q^k$ do we need to meet every $(k-1)$-dimensional linear subspace in a spanning set?

Let $\mathcal{S}_i$ be the set of i-dimensional linear subspaces of $\mathbb{F}_q^k$.

$$m_q(k) := \min\{|B| : B \subseteq \mathcal{S}_1, \langle B \cap H \rangle = H, \forall H \in \mathcal{S}_{k-1}\}.$$

Smallest strong blocking sets

Let $m_q(k)$ be the smallest size of a strong blocking set in $\mathbb{F}_q^k$.

Motivation: minimal error-correcting codes, digital fingerprinting, code-based cryptography, circuits in matroids, ...

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Theorem (Héger and Nagy 2021)

$$m_q(k) \leq 2(q+1)(k-1).$$

Various new equivalences

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\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\text{$\mathcal{L}$ is a strong blocking set in $\mathbb{F}_q^k$}} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \boxed{\text{$B = \bigcup_{\ell \in \mathcal{L}} \ell$ is an affine 2-blocking set in $\mathbb{F}_q^k$}} \arrow[d, leftrightarrow, "\text{for } q = 3"] \\
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Equivalences between blocking sets and codes.

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Equivalences between blocking sets and codes.

New bounds and constructions for all of these objects!

Lower bounds on strong blocking sets

Theorem (B., D'Haeseleer, Gijswijt, Potukuchi 2023)

For any prime power q, there is a constant $c_q > 1$ such that

$$m_q(k) \geq (c_q - o(1))(q+1)(k-1).$$

The constant c_q is the unique solution $x \geq 1$ to the equation

$$M_q\left(\frac{q-1}{x(q+1)}\right) = \frac{1}{x(q+1)},$$

where M_q is the function appearing in the LP bound on asymptotic rate of a code.

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Corollary

$$T_L(n) \leq \frac{n}{4c_3} + 1 \leq \frac{n}{4.55}$$

Upper bounds on blocking sets

Theorem (B., D'Haeseleer, Gijswijt, Potukuchi 2023)

$$b_q(k,s) \leq (q^s - 1) \cdot \frac{s(k-s)+s+2}{\log_q\left(\frac{q^4}{q^3-q+1}\right)} + 1.$$

Proof.

Pick random s-dimensional subspaces through the origin. □

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Proof.

Pick random s-dimensional subspaces through the origin. $\square$

Corollary

$$m_q(k) \leq (q+1)\frac{2k}{\log_q\left(\frac{q^4}{q^3-q+1}\right)}.$$

Corollary

$$T(n) \geq T_L(n) \geq \frac{1}{3}\left(\frac{9}{5}\right)^{n/4}$$

Explicit Constructions

Big open problem: construct small strong blocking sets explicitly.

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Theorem (Alon, B. , Das and Neri, 2023+)

An explicit construction of a strong blocking set in $\mathbb{F}_q^k$ of size $c(q+1)(k-1)$.

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An explicit construction of affine-2 blocking sets in $\mathbb{F}_q^k$ of size $c(q^2-1)(k-1)+1$.

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Corollary

An explicit construction of trifferent codes of size $3^{\frac{n}{4c}}$.

Integrity of a graph

Definition

For a graph G, let $\iota(G) = \min\{|S| + \kappa(G - S)\}$, where $\kappa(G - S)$ is the largest size of a connected component in $G - S$.

Examples: $\iota(K_n) = n, \iota(C_n) = 2\lceil\sqrt{n}\rceil - 1$ and $\iota(Q_n) =?$.

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Theorem (Alon, B., Das, Neri, 2023+)

For any (n, d, λ)-graph G,

$$\iota(G) \geq \frac{d - \lambda}{d + \lambda} n.$$

The construction

Let V be a collection of n 1-dim vector subspaces of $\mathbb{F}_q^k$ that meets every hyperplane in at most $n-d$ points.

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For each edge $e=uv$, let $\mathcal{P}_e$ be the collection of 1-dim subspaces contained in the span of u and v. Then the set

$$S=\cup_{e\in E}\mathcal{P}_e,$$

is a strong blocking set of size $n+(q-1)|E|$.

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By using explicit $[n,k,d]_q$ codes with k,d linear in n (algebraic-geometric codes), and constant-degree expanders Ramanujan graphs, we get our explicit construction.

Future work

1. Improve the upper bound on $m_q(k)$, and in particular for $q = 3$.
2. Improve the lower bound $b_q(k, s) \geq (q^s - 1)(k - s + 1) + 1$ for $s > 1$.
3. Further explore the graph theoretic construction, and apply it to other problems in finite geometry/coding theory.

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Thank you!