MSc Project on

Deriving a nonlinear model equations for internal waves in continuously stratfied fluids

supervised by Anna Geyer
(Mathematical Physics)

Background

More than 250 years ago, Leonard Euler derived the equations of hydrodynamics which still present a great challenge to the scientific community: even in the simplest non-trivial setting of physical relevance the exact governing equations are almost intractable, so-called Gerstner's waves being the only known explicit solutions. To make headway, an approach that proved very successful was to use asymptotic expansions of the unknowns in small physical parameters (such as the wave amplitude) to derive simplified model equations such as the well-known Korteweg-de Vries (KdV) equation, and more recently the Camassa-Holm (CH) equation. These model equations describe the evolution of shallow water waves and are accurate approximations of Euler's equations in certain physical regimes, cf. [1]. The shallow water regime of small amplitudes gives rise to the weakly nonlinear KdV equation, whereas the highly nonlinear CH equation is obtained in the regime of moderate amplitudes. While the KdV equation models the evolution of the free surface as well as the horizontal velocity componant at every depth in the fluid, it was shown in [2] that the CH equation actually models the horizontal velocity component *only at a specific fixed depth* within the fluid domain. A corresponding equation for the free surface was also derived and justified as an approximation of Euler's equations in [4], but no model equation is available to describe the dynamics at other depths.

Project

The model equations described above are usually derived under the assumption that the water density is constant. However, density stratification of water is common and occurs due to differences in temperature or salinity, for instance in equatorial oceans or river estuaries. This density stratification gives rise to internal waves. A KdV type equation modeling the propagation of surface and internal waves in a fluid with continuous density stratification was recently derived in [3]. Due to the presence of density dependent coefficients, one actually obtains

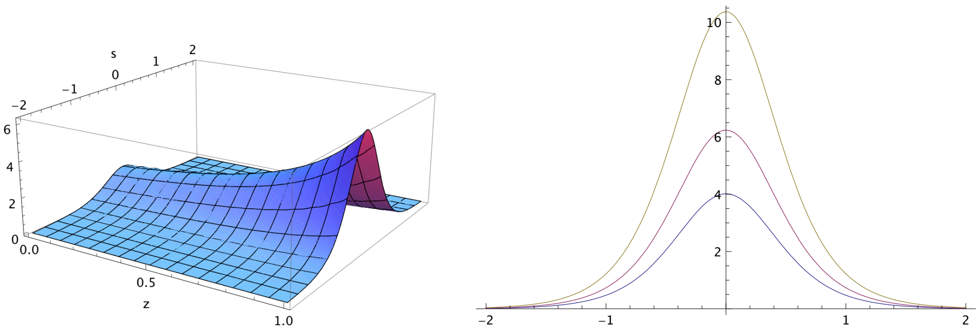

Figure 1: Solitary wave solution of a KdV type equation corresponing to linear density stratification [3].

a different equation at every depth. This project is about extending that approach to the regime of moderate amplitues and to derive CH type equations for every depth within a stratified fluid. The challenge compared to the derivation of KdV type equations is the fact that due to the different regime, asymptotic expansions up to higher orders have to be computed and matched, resulting in higher order nonlinearities in the derived equation. The advantage of the CH type equations over the KdV type equations is that they are capable of modeling highly nonlinear behaviour such as peaking and breaking of waves.

Outline

- Derive CH type equation from the Euler equations with arbitrary depth dependent density distribution
- Possibly reduce the complexity of the general model by focusing on a linear stratification profile
- Find explicit periodic and solitary traveling wave solutions
- Perform qualitative analysis of how the amplitude of the traveling wave solutions depends on the wave speed and water density.

Contact

If you are interested in this project do not hesitate to contact me: a.geyer@tudelft.nl

References

[1] R.M. Johnson, *A Modern Introduction to the Mathematical Theory of Water Waves*, Cambridge University Press, Cambridge, UK, 1997.

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[4] A. Constantin and D. Lannes, *The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations*, Arch. Ration. Mech. Anal. 192 (2009) 165–186.