Black-Scholes Option Pricing Using Three Volatility Models: Moving Average, GARCH(1, 1), and Adaptive GARCH

Bachelor Thesis

by
Heru Sataputera Na
260330
Informatics & Economics

Erasmus University
Rotterdam, Netherlands
August 2003

Acknowledgements

I thank God for He has given me wisdom, strength, and time to be able to write this bachelor thesis.

I am also very grateful to my supervisor, Dr. Ir. Associate Professor Jan van den Berg, for giving me the opportunity to evaluate this topic and especially for his time, inputs and very helpful comments during the writing process.

Finally, I am really thankful for my family and friends who always supported and encouraged me along the way.

Table of Contents

Abstract

This article evaluates the empirical performance of three different models to estimate the parameter volatility in the Black-Scholes option pricing model. The three models are: (a) Moving Average model, (b) *GARCH(1,1)* model, and (c) Adaptive GARCH model. I examine these models using two datasets: AEX index and ING GROEP CERTS asset. Performance is measured by minimizing the error between the model-determined price and the real market price. The Adaptive GARCH model and the Moving Average model outperform the *GARCH(1,1)* model. This might be due to the ability of the Adaptive GARCH model and the Moving Average model to update the information used in the maximization process daily.

The Adaptive GARCH model outperforms the Moving Average model in the AEX data. Conversely, the Moving Average model does better when using the ING GROEP CERTS data. The underperformance of the Adaptive GARCH model via the ING data might be because of the extra volatility of the ING asset compared to the AEX index.

1. Introduction

When Fischer Black and Myron Scholes developed the Black-Scholes model in the early 1970's [1], it soon became a major breakthrough. Since then many traders use the Black-Scholes model as their premier model for pricing and hedging options. An important property of the Black-Scholes model is that all variables in the equation are not influenced by the risk preferences of investors. In particular, the analysis is based on a risk-neutral pricing approach[1], which in return simplifies the analysis of derivatives.

In the classic Black-Scholes model, the volatility is assumed to be constant. However, empirical research shows that the volatility of financial asset prices is following a stochastic process and varies through time. It means that while other properties of an option- such as exercise price, time to maturity, current price of underlying asset- can be observed directly from the market, the return volatility is the uncertainty factor in the Black-Scholes model.

As volatility increases, the probability that stock price will raise or fall increases, which in response will also increase the value of both call and put options. Return volatility thus plays a major role in option pricing. Therefore, accurate measures and good forecasts of volatility are critical for option pricing theories as well as trading strategies. At present, there have been many models developed to determine volatility, and some of them act as alternatives or improvement from earlier models. The family of GARCH models is an example, starting from the autoregressive conditional heteroskedasticity (ARCH) model of Engle (1982) [4].

This paper will analyze the performance of option pricing by using three different models to estimate one-day-ahead volatility in the Black-Scholes model. The estimated volatility of each model is used as an input in the Black-Scholes option pricing formula to price 3-months-options daily during the lifetime of the options. The errors between the model-determined prices and the real price will then be computed. Smaller errors will denote a better performance. Call options from two underlying assets, namely the AEX

[1] Explanation over the risk-neutral valuation is given in Hull [8], chapter 12.7.

index and the ING GROEP CERTS asset - one of the assets that construct the AEX index-, are chosen for this purpose.

The first volatility model is the *Moving Average* model. In this model, daily volatility is determined by taking the mean of historical returns of underlying asset within a certain time horizon. The time horizon is set to be fixed. It is called the Moving Average volatility model because the average-based determined volatility moves with time. An assumption lying behind this model is that expected volatility today is given by the average of the returns in the past period. Estimated volatility is updated daily by recalculating the average of past returns within the same time horizon.

The second model is chosen from the family of GARCH models, the *GARCH(1,1)* model. Although there are many more complicated GARCH models, *GARCH(1,1)* often performs as well as others. A study of comparing volatility models done by Hansen and Lunde [7], using *GARCH(1,1)* as benchmark, had as result that the best models do not provide a significantly better forecast than the *GARCH(1,1)* model. For this reason *GARCH(1,1)* is preferred here above all family of GARCH models. The volatility is computed from the observations of historical daily asset prices, taking both the conditional and unconditional variance into account in the estimation process.

An alternative to the *GARCH(1,1)* model is also proposed to improve the performance of estimating volatility, which serves as our third model. Because the parameters of the GARCH model are maximized by means of historical data and then held constant afterward to forecast future volatility, it is possible that the information contained in the parameters differs from what will occur in the future. To avoid this, the parameters of the GARCH model are adjusted over time and updated daily through the option's lifetime. Using this approach, the parameters are re-maximized on daily basis and expected to show an improvement in the performance of the model.

The remainder of the paper is organized as follow: Section 2 explains the Black-Scholes model and the three models to forecast volatility. Section 3 describes the experimental setup. Section 4 gives the results obtained from the experiment and Section 5 discusses the experiment's outcomes. Finally, Section 6 concludes this thesis.

2. Option pricing and volatility models

2.1. Black-Scholes option pricing model

A basic assumption of the Black-Scholes model is that the stock price is log-normally distributed. One of the attributes the lognormal distribution has is that stock price can never fall by more than 100 percent, but there is some small chance that it could raise by much more than 100 percent [3]. Several assumptions that lie behind the Black and Scholes model are [1]:

a) Markets are efficient, which implies that people are unable to consistently predict the direction of the market or an individual asset. Stock prices are supposed to follow the continuous Itô process. To understand the Itô process, we have to know what a Markov process is. A Markov process is "a process where the observation at time *t* depends only on the previous observation." An Itô process is simply a Markov process in continuous time[2].
b) Trading on securities is continuous and short selling is allowed.
c) No commissions are charged to buy and sell options (no transaction costs or taxes). All securities are perfectly divisible.
d) The stock pays no dividends during option's life. European exercise terms are used; where the option can only be exercised on the expiration date.
e) Risk free arbitrage opportunities are not available.
f) The risk free interest rate (r) is known and remains constant for all maturities.

Option prices can be determined by a risk neutral pricing approach. This valuation serves as the most important tool for the analysis of derivatives, because pricing a derivative that provides a payoff at one exact time in the future simply becomes taking the discounted value of the expected payoff from the option at its maturity. Using the risk-neutral valuation, the expected return (μ) from the underlying asset and the discount value is the risk-free interest rate (r). The Black-Scholes formulas for the prices of a

[2] Further details over the Itô process are given in Hull [8], chapter 11.

European call option on a non-dividend paying stock and a European put option on a non-dividend paying stock are given in the following equations [8]:

$$c = S_o N(d_1) - Ke^{-rT} N(d_2)$$
$$p = Ke^{-rT} N(-d_2) - S_o N(-d_1) \tag{1}$$

The variables d_1 and d_2 are calculated by:

$$d_1 = \frac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{\ln(S_0 / K) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{2}$$

The variables c and p are the European call and European put price, S_o is the stock price at time zero, K is the strike price of the option, r is the continuously compounded risk-free rate, σ is the stock price standard deviation, and T is the time to maturity of the option. $N(x)$ is the cumulative probability distribution function for a standardized normal distribution, or the probability that a variable with a standard normal distribution, with zero mean and standard deviation is one, will be less than x.

In the above formulas, standard deviation per annum is used instead of daily standard deviation as an input for the variable σ. Because we forecast the daily volatility, standard deviation per annum is obtained using:

$$\sigma_{per.annum} = \sigma_{daily} \times \sqrt{trading.days.per.year} \tag{4}$$

The normal assumption in equity markets is that there are 252 trading days per year [8].

Nevertheless, recent findings show that volatility tends to vary over time and thus the assumption of constant volatility is unrealistic. Many models have been developed to relax the constant volatility assumption. One of them is GARCH option model, which assumes that the conditional volatility of stock prices depends on the past pricing errors.

2.2. Volatility models

2.2.1. Moving Average model

The first model to estimate volatility[3] is based on historical returns from stock prices. Suppose that the value of the asset at the end of day i is S_i. Define u_i as the percentage change of asset price between the end of day $i\text{-}1$ and the end of day i, so that:

$$u_i = \frac{S_i - S_{i-1}}{S_{i-1}} \tag{5}$$

We can also define u_i as the continuously compounded return during day i (between the end of day $i\text{-}1$ and the end of day i) by taking the logarithm of current asset value divided by value of the day before:

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \tag{6}$$

The unbiased estimate of one-day volatility, using u_i of m days before today, is

$$\sigma_n^2 = \frac{1}{m-1}\sum_{i=1}^{m}(u_{n-i} - \bar{u})^2 \tag{7}$$

In the equation above, $\bar{u}$ is the mean of the u_i's:

$$\bar{u} = \frac{1}{m}\sum_{i=1}^{m} u_{n-i} \tag{8}$$

A simplified approach to estimate volatility is given by [8]:

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}^2 \tag{9}$$

Variable u_i is estimated using the equation (5). Two modifications takes place in this equation compared to the equation (7): $\bar{u}$ is assumed to be zero and $m\text{-}1$ is replaced by m. The assumption of zero mean of u_i could be taken if the actual mean does give a value of zero or almost zero. We will use the equation (9) to estimate volatility for the Moving Average model.

[3] In this paper, we will use the term 'volatility' also for the variance rate.

The amount of historical days used to estimate volatility should be determined carefully. Some expect that more data would lead to better precision, but data that are too old might be unrelated to predict the future. Closing prices from daily data over the last 90, 180 or even 252 days are often used. Another option is setting the amount of days, variable *m* in the equation (9), to be equal to the number of days to which the volatility is to be applied [8]. In another word, to value for instance a three month option, daily data for the last three months are used. We will also use this method in this paper.

2.2.2. GARCH(1,1) model

Many models are developed that correspond to stochastic volatility process characteristic. One widely known model is the ARCH model, introduced by Engle (1982) [4]. This model is setting unconditional volatility constant, while allowing the conditional volatility to change over time. This conditional volatility is subsequently altered by past returns. The ARCH (*q*) model is specified as:

$$\sigma_n^2 = \omega + \sum_{i=1}^{q} \alpha_i u_{n-i}^2 \tag{10}$$

The variable ω is:

$$\omega = \gamma . V_L \tag{11}$$

In this equation, $\omega > 0$, and $\alpha_i \geq 0$ hold. V_L is the long-run average variance rate and γ is the weight assigned to V_L. Daily return (u_i) is calculated using the equation (5). The variable *q* is the order of dependency to past returns. This model differs with the Moving Average model in that it assigns more weight to recent data, and also a weight is assigned for the long-average variance rate (the unconditional volatility).

An assumption underlying this model is that volatility is changing over time and there is tendency that a large error will be likely followed by a large error and a small error followed by a small error. The variable *q* is the period the conditional variance depends on. The larger the variable *q*, the longer is the period of volatility clustering. This characteristic coincides with the findings of Fama [6] and Mandelbrot [9] that the volatilities of financial series cluster.

A generalized approach of ARCH models was proposed by Bollerslev (1986) [2], and as well known as GARCH model. The GARCH (p, q) model is specified as:

$$\sigma_n^2 = \omega + \sum_{i=1}^{q} \alpha_i u_{n-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{n-i}^2 \tag{12}$$

In this equation, $\omega > 0$, $\alpha_i \geq 0$, and $\beta_i \geq 0$ hold.

As in ARCH models, variables p and q are the order of dependency. The distinction of GARCH model is that the conditional variance is specified not only as a linear function of past sample variances, but also including lagged conditional variances to enter the equation as well. This corresponds to some sort of adaptive learning mechanism.

The simplest GARCH model is *GARCH(1,1)* model, which is expressed as:

$$\sigma_i^2 = \omega + \alpha \cdot u_{i-1}^2 + \beta \cdot \sigma_{i-1}^2 \tag{13}$$

Although there are many GARCH models developed, *GARCH(1,1)* often performs as well as others[4]. *GARCH(1,1)* model is also the most popular among GARCH models.

The weights assigned to both conditional and unconditional volatility - γ, α, and β - must sum to one. For a stable *GARCH(1,1)* process, $\alpha + \beta < 1$ is required, otherwise the weight applied to the long-term variance is negative. An interesting empirical finding is that in financial series, particularly in daily series, $\alpha + \beta$ is often close to one. Engle and Bollerslev (1986) [5] introduced the IGARCH (Integrated GARCH) model in which the sum of α and β equals to 1.

The parameters in the equation of the models are estimated using the maximum likelihood method. This method will choose the values of parameters by maximizing the probability of a set of observations occurring. Define $v_i = \sigma^2$, which is the estimated variance for day i. We assume that there are m observations, consisting of $u_1, u_2 \ldots u_m$ and

[4] For further details, see Hansen and Lunde [7].

that the probability distribution of u_i is normal. The likelihood of the m observations is [8]:

$$\prod_{i=1}^{m}\left[\frac{1}{\sqrt{2\pi v_i}}\exp\left(\frac{-u_i^2}{2v_i}\right)\right] \tag{14}$$

The best parameters are the ones that maximize this expression. Fortunately, maximizing an expression and maximizing the logarithm of the expression is equivalent. Taking logarithms of the expression in the equation (14), the log-likelihood is given by:

$$\sum_{i=1}^{m}\left[c+\frac{1}{2}\left\{-\ln(v_i)-\frac{u_i^2}{v_i}\right\}\right]$$

Ignoring constant multiplicative factors c and ½, we accordingly want to maximize:

$$\sum_{i=1}^{m}\left(-\ln(v_i)-\frac{u_i^2}{v_i}\right) \tag{15}$$

To find the maximum, we differentiate this formula to variable v_i and set the equation result to zero. The maximum likelihood estimator of v_i is:

$$v_i=\frac{1}{m}\sum_{i=1}^{m}u_i^2$$

2.2.3. Adaptive GARCH model

The original *GARCH(1,1)* model estimates the parameters using maximum likelihood estimation given historical data. These parameters are then held constant and used to forecast future volatility by the equation (13) above. Hence, if historical returns of the last three months, for example, have a different structure compared to the returns in the next three months, then the maximized parameters might not be able to perform well to estimate daily volatility for the next three months.

As an alternative, we can also define the parameters in GARCH as a function of time. It means that the parameters at day i are estimated by means of maximum likelihood just like the original *GARCH(1,1)* model, and subsequently these parameters are used to forecast volatility at day i. The difference is the parameters' maximization is repeated daily during the lifetime of the options. The equation (13) is then adapted to:

$$\sigma_i^2 = \omega_i + \alpha_i \cdot u_{i-1}^2 + \beta_i \cdot \sigma_{i-1}^2 \qquad (17)$$

Parameters ω, α, β are changing over time. Assumption made is that the contributions of V_L, u^2_{i-1} and σ^2_{i-1} change over time and that they can be estimated over a fixed time window. Because of daily maximization of the parameters, it is expected to obtain better maximized parameters which will be used to estimate future volatility on a daily basis. This model will be labeled 'the Adaptive GARCH model'.

3. Experimental setup

3.1. Data description

Data used for implementation consists of daily close prices from AEX index and ING GROEP CERTS from August 27, 2001 to February 15, 2002, which are collected for the purpose of estimating volatility; while daily close prices of AEX index and ING GROEP CERTS call options between November 19, 2001 and February 15 2002 are obtained to determine the performance result.

We use this data based on the following consideration. First, the data is freely available at Erasmus University Rotterdam, acquired from DataStream. Second, AEX index is actively traded in the Euronext Amsterdam Exchange. AEX index is made up of the 25 most active securities in the Netherlands. While many studies have used indexes such as S&P 500 or Dow Jones as the experiment data because they are actively traded, we therefore also choose an index, AEX, to test the performance of the models. Third, the ING GROEP CERTS asset is one of the stocks included in the AEX index, which we shall use as our second underlying asset. A change in the return of the ING asset will have a fraction effect to the return of the AEX index. Hence, we are interested to see whether the performance result of each model is still the same when we use a stock in place of a market index. We choose for ING GROEP CERTS stock because the evolution of the price is rather similar with the AEX price during the period we are interested in.

Figure 1A plots the movement of the AEX index levels in our sample, and Figure 1B plots the daily return for the period of 6 months. We can notice from Figure 1A that in the first 30 days, the index prices fall substantially, and afterward rise to in the neighborhood of € 500, and eventually vary within this region. The price of the index at the end of the period is *498.1*. From Figure 1B, we can see that the volatility of AEX index changes over time. Meanwhile, Table 1 demonstrates that the returns have an average of 0.0011.

The evolution of ING GROEP CERTS prices and the daily return are plotted in Figure 2A respectively Figure 2B. One can notice that the prices also exhibit a decreasing movement in the first 30 days, just like in the AEX index levels, except that these prices quickly increase to subsequently vary in the region of € 28. The price of the ING asset at the end of the period is *28.15*. The daily return in Figure 2B shows that the volatility change is higher over time than the AEX volatility. The mean of return sequences is 0.0008, as we can read from Table 1.

It is interesting to verify whether our implementation using the ING GROEP data will give the same result as when using the AEX index data. Therefore these two data are chosen to test the empirical performance of our models above. The options data have several attributes. In the first place, the time to expiration is 3 months. Second, we consider 5 options with different exercise prices for each underlying assets, whereas in-the-money, near at-the-money, and out of-the-money options are all included. For AEX index, we use options with exercise price 460, 480, 500, 520, and 540; while for ING asset, options with exercise price 26, 28, 30, 32, 34 are selected. We would like to see whether the performance of the option-pricing is constant on the options mentioned above. In Table 1, we give the actual means of the returns during the lifetime of the options. Because the mean values are always slightly above or below 0 and the average is approximately zero, we can then safely assume that the mean is zero. It is possible therefore to use the equation (9) above to estimate volatility.

We also obtain the daily yield rates of Netherlands CBS Government Bond that matures within 9-10 years, during the life of the options. While there is no such thing as risk-free interest rate in the real world, we assume that the yield rates of long-term government bond would serve as a good representative to fill in this parameter[5]. The interest rates of financial institutions are less appropriate because they are not risk-free rates. The reason is that financial institutions bear the risk to default. Even the most established institution bears this risk, though the amount is rather small. In contrast, government is very unlikely to default. The daily rates are then converted to continuously compounded rates.

3.2. Calculating option price

In computing the spot volatility on a given day, the daily prices of each asset within 3-months prior to this day are used. Consequently, 6-months asset prices are obtained for this purpose, starting from August 27, 2001 until February 15, 2002. For the Moving Average model, we calculate the average of the last 3 months daily returns and apply this average of past squared returns of underlying asset as the obtained volatility to price an option. The average is updated daily and serves as the estimated volatility on the given day. This process is done through the options' lifetime.

For GARCH model, we use the daily returns of the first 3 months to estimate the parameters of the *GARCH(1,1)* equation, and afterward by means of these parameters to forecast volatility for the next 3 months. We make use of the maximum likelihood method to estimate the parameters that maximize the equation (15) above. We choose the Solver program in Microsoft Excel to implement the maximization process.

Eventually for the Adaptive GARCH model, we estimate parameters and volatility each day through the lifetime of the options by means of assets' daily returns of the last 3 months. Analogous to the Moving Average model, this process is also carried out on a daily basis starting from the first day the options come out to the market until the options expire. The next step to be taken is calculating the option price by means of the obtained volatility above. This would be a straightforward effort, since all inputs needed

[5] We thank Mr. Rob Stevense for giving the advice that the long-term Netherlands Government Bond is usually used as input for the parameter 'risk-free interest rate' in the Black-Scholes model.

for the parameters in the Black-Scholes formulas are already available within the data acquired.

3.3. Performance Measure

The pricing error of the model for each option is expressed as:

$$\eta_t = O_t - q_t \tag{18}$$

Parameter η_t denotes the pricing error for an option at time t, *Ot* is the real market price of this option, and q_t is the model-determined option price.

We implement this formula to all daily option prices calculated above. Because the error could be positive or negative, we make use of the square amount of the pricing errors. The Root Mean Squared Error (RMSE) is then the square root of the average of squared pricing errors of options in the whole sample, and given by:

$$RMSE = \sqrt{\frac{\sum_{i=1}^{m} \eta_i^2}{m}} \tag{19}$$

Variable m is the lifetime of the options in the sample, or 3 months.

The smaller the RMSE, the smaller pricing errors are, which subsequently means better price forecasting. Another possible measure is MAE (Mean Absolute Error), which stands for average of the absolute values of difference between the market option prices and model determined prices. In contrast with RMSE, MAE returns the mean from pricing errors that are given in absolute term. We will implement RMSE as our performance measure in this paper.

4. Results

The parameters estimates of the *GARCH(1,1)* model appear in Table 2. Parameter α is 0.0346, β is 0.7664, and γ is 0.1990 for AEX, while the parameters α, β, and γ of the ING data are respectively 0, 0.8337, and 0.1663. Because the parameter α of ING is 0, it means that the past returns do not provide any weight to estimate the volatility in this model.

The parameters of the Adaptive GARCH are given in Table 3. For AEX, the weights α, β, and γ begin by 0.0382, 0.7653, and 0.1965, respectively; and end by 0, 0.9938, and 0.0062, correspondingly. For ING, the weights α, β, and γ start by 0, 0.8288, and 0.1712, respectively; and finish by 0.0125, 0.9602, and 0.0273. The sum of α and β in the first ten days for AEX and the first twenty-one days for ING asset are less than 0.900.

The evolution of the parameters is given in Figure 3A for the AEX index and Figure 3B for the ING GROEP CERTS asset. The sum of α, β, and γ is always 1. From Figure 3A we can see that the parameter α is going up first, and afterward decreasing to zero value at the end of the period. The parameter β declines in the beginning of the period and afterward rises gradually to almost reach value one, while the parameter γ rises in the first week and subsequently diminishes to slightly below the start value.

Examining Figure 3B, we could notice a similar progress for the three parameters of the ING asset. The difference lies in that the parameter α is beginning with zero value, persisting in the first 10 days before rising and then decreasing to a value of 0.0125. As for the parameter β, it declines further than AEX in the first two weeks of December 2001, and suddenly increases significantly at 18^{th} of December 2001, followed by a steady increase until the end of the period. The significant increase of the parameter β is matched by a significant decrease of the parameter γ at the same day, after stepping forward in the first month, and later on parameter γ evolves in the value range between 0.01 and 0.04, as we can read from Table 3. A worthy of note outcome is for the ING asset, in the first ten days, the Adaptive GARCH model estimates the parameter α as 0, which means that return from the preceding day during this period is not taken into account when calculating the volatility.

The total squared differences between the real market prices and the model-determined prices and the RMSE (root mean squared error) are reported in Table 4 for the AEX index and Table 5 for the ING asset. For all the five options of the AEX index considered, we could identify that the RMSE of the *GARCH(1,1)* model is larger than the RMSE of the Moving Average model, while the Adaptive GARCH model produces the smallest RMSE compared to the other two models. The RMSE of the Moving Average model is also better than the RMSE of the *GARCH(1,1)* model for the options of the ING asset; but we obtain a contrast result for the Adaptive GARCH model, where the RMSE is no longer superior. The Moving Average model generates the smallest RMSE, followed by the Adaptive GARCH model and the latest is *GARCH(1,1)* model.

5. Discussion

In the last chapter we have seen that the Adaptive GARCH model gives the best estimation of the AEX option prices measured up to the Moving Average model, in the second place, and the *GARCH(1,1)* model in the third place. We obtain similar result for the *GARCH(1,1)* model using the ING options, creating the largest RMSE than the other two models. A possible explanation of the underperformance of the *GARCH(1,1)* model is that the historical returns used to maximize the parameters α, β, and γ differ a lot from the volatilities throughout the lifetime of the options. These parameters do not act as the appropriate weights for the equation of volatility estimates because we hold the parameters constant during the options' lifetime.

In the Moving Average model, volatility is calculated by the mean of the last 3-months returns, and this calculation is updated daily. The oldest return is removed and replaced with the newest return, where new information is taken into account in the volatility estimates. The Adaptive GARCH model follows this process by removing the earliest return with the latest return and uses this daily-updated-set of returns to re-maximize its parameters. This process is also repeated daily. We have obtained contrast results for the Moving Average model and the Adaptive GARCH model. The Adaptive GARCH is superior to the Moving Average model in estimating the AEX option prices, but the other way around in estimating the ING option prices.

There are three outcomes that might explain why the Adaptive GARCH model does not perform well in estimating the volatility of the ING asset. First, an empirical finding in financial series (as said before), particularly in daily series, $\alpha + \beta$ is often close to one. For AEX index, this characteristic happens after the 11^{th} day of the options' lifetime, where the sum of the parameters α and β rises from 0.8254 to 0.9826 after formerly generates value only in the range of 0.7663 and 0.8254. The sum then diverges itself to a value close to one, between 0.9695 and 0.9938. In contrast, the sum of α and β of the ING asset fluctuates in lower values than the sum of α and β of the AEX index. The sum alters in the first 21 days between 0.6112 and 0.8288 along with a rising in the 21^{st} day (at December 18, 2001) from 0.6359 to 0.9875 and varies close to one, between 0.9534 and 0.9886. This means that the duration of the sum of $\alpha + \beta$ not close to one is longer in the volatility estimates of ING than AEX, which in turns might affect the performance of the model.

Second, the parameter α of the AEX is estimated as zero for the last 3 days of the period. On the contrary, the parameter α of the ING is estimated as zero for a longer period, explicitly the first 10 days. The zero values of the parameter α mean that the previous-day return is not taken into account when estimating volatility. This condition might also affect the performance result of the model since the effect of this condition is heavier for the ING option prices estimates because the period is longer.

Third, we can compare the estimated volatility of the three models with the implied volatility (volatility acquired from option real prices). These data are given in Figure 4A for the AEX index and Figure 4B for the ING asset. The implied volatility is obtained by taking the average of the implied volatilities of the individual call options each day throughout the lifetime of the options. We can notice from Figure 4A that the estimated volatility of the *GARCH(1,1)* model is far from the implied volatility. The volatility of the Moving Average model evolves progressively while the volatility of the Adaptive GARCH model responds quickly after 3 weeks and then evolves near by the implied volatility. In the last 2 weeks the implied volatility rises again and this is the only period the volatility of the Moving Average model is closer to the implied volatility than the volatility of the Adaptive GARCH model. Generally, during the 3-months period, the forecasted volatility of the Adaptive GARCH is closer to the implied volatility than the other two models. It clarifies why the Adaptive GARCH model provides the best volatility estimates in the AEX index.

In Figure 4B, we can see that the estimated volatility of the *GARCH(1,1)* model is also far from the implied volatility. The volatility of the Moving Average model is closer to the implied volatility than the volatility of the Adaptive GARCH model in the first month, but in the second month the volatility of the Adaptive GARCH model is closer. In the third month, the first 20 days the volatilities of both models are more or less near by the implied volatility. In the following 10 days, the implied volatility rises significantly to a value of 0.0033 (above the volatility estimates of the *GARCH(1,1)* model – 0.0030), and the volatility of the Moving Average is closer to the implied volatility matched up to the volatility of the Adaptive GARCH.. The volatility estimates of the Moving Average model is in total amount of days closer to the implied volatility than the volatility estimates of the Adaptive GARCH model. This condition also clarifies why the Moving Average model has a better (smaller) RMSE than the Adaptive GARCH model, though the difference is small.

Identical to what happens in the parameter estimates, the volatility of the Adaptive GARCH model is suddenly improved at 18th of December 2001. An explanation of this outcome is that the returns of the first month (August 27th, 2001 to September 27th, 2001) have left the basket of returns used to maximize the parameters. These returns do not represent as the proper estimation of the volatility for the rest of the period, because the movement is quite different with the rest. This leads to an improvement in the maximized parameters which consecutively improves the estimated volatility. In the AEX data, the returns of the first month have a similar shape as the returns of the ING data, except the volatility of the AEX returns is less than the volatility of the ING returns. As we can see from Figure 4A, the estimated volatility of the AEX data also improves significantly in the first period, though the improvement is smoother.

6. Conclusion

The empirical performance of the Black-Scholes option pricing model on the AEX options and the ING GROEP CERTS options, using three models to estimate volatility, has been evaluated in this paper. The three models chosen to forecast volatility are the Moving Average model, the *GARCH(1,1)* model and the Adaptive GARCH model.

The Adaptive GARCH model outperforms the Moving Average model, in the second place, and the *GARCH(1,1)* model while pricing the AEX options. We obtain a different result when examining the ING option prices estimates. The *GARCH(1,1)* still underperforms the other two models, but the Moving Average model is giving a better result compared to the Adaptive GARCH model.

We have found from this experiment that the Adaptive GARCH model and the Moving Average model have performed better than the *GARCH(1,1)* model to estimate daily volatility during the lifetime of the 3-months option. The performance result of the *GARCH(1,1)* model depends on the past returns used to maximize the parameters in the equation of volatility estimates. The poor performance of the *GARCH(1,1)* model is caused by the training data (in this case the first 3-months period of returns used to maximize the parameters) having a different evolution compared to the test data (the next 3-months period – the options' lifetime). Because we hold the parameter constant during the volatility estimates, these parameters do not represent as the suitable weights for the equation and therefore do not able to forecast volatility accurately.

Nevertheless, based on this experiment we are not able to conclude whether the Adaptive GARCH model is better in estimating volatility than the Moving Average model. We have indeed found that the underperformance of the Adaptive GARCH model in estimating the volatility of the ING asset might be due to the extra volatility this asset has contrast to the AEX index. This extra volatility can be identified by comparing the returns of the AEX index and the ING asset in Figure 1B and Figure 2B.

Consequently, several adjustments could be made to improve the performance of the Adaptive GARCH model for the ING data. In the first instance, historical data with a longer period could be used to maximize the parameters of the equation, for example 6 months, 1 year or even later. Due to possible shocks in a short-period history data that in response might not represent the characteristic of the development of asset prices as a whole, the use of a longer-term data might result in a better analysis. Other possible adjustment is by implementing an alternative algorithm to calculate the maximum likelihood function, for example the Levenberg-Marquardt algorithm. In this paper we have used Solver algorithm in Microsoft Excel to compute the likelihood function. Finally, it might be necessary to do this experiment on the other 24 assets that compose the AEX index in order to be able to take a valid conclusion. We leave this for future research.

References

1. Black, F., and Scholes, M. (1973): "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637-659.
2. Bollerslev, T. (1986): "Generalized Autoregressive Conditional Heteroscedasticity," *Journal of Econometrics*, 31, 307–327.
3. Brealey, R. A., and Myers, S. C. (2003). Principles of Corporate Finance. *Seventh Edition.* Chapter 21.
4. Engle, R. F. (1982): "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of UK Inflation," *Econometrica*, 50, 987-1008.
5. Engle, R. F. and Bollerslev, T. (1986): "Modelling the Persistence of Conditional Variances", Econometrics Reviews, 5, 1-50.
6. Fama, E. (1965): "The behavior of stock market prices," *Journal of Business*, 38, 34-105.
7. Hansen, P.R., and Lunde, A. (2001): "A Forecast Comparison of Volatility Models: Does Anything Beat a *GARCH(1,1)*?" *Working Paper No. 01-04, Brown University, Department of Economics.*
8. Hull, J.C. (2003). Options, Futures, and Other Derivatives. *Fifth Edition.* Chapter 5, 11, 12, 17.
9. Mandelbrot, B. (1966): "Forecast of future prices, unbiased markets, and martingale models," *Journal of Business*, 39, 241-255.

Appendix A – Tables

Table 1

Actual Means of Returns (u_i) of the AEX Index and the ING GROEP CERTS Asset

The actual means of the returns of the last 3 months, calculated daily during the options' lifetime are given below.

DATE	AEX	ING
19/11/2001	-0.0009	0.0008
20/11/2001	-0.0004	0.0008
21/11/2001	-0.0007	0.0008
22/11/2001	-0.0006	0.0009
23/11/2001	-0.0002	0.0009
26/11/2001	0.0000	0.0009
27/11/2001	-0.0001	0.0009
28/11/2001	-0.0002	0.0009
29/11/2001	0.0000	0.0009
30/11/2001	0.0005	0.0009
3/12/2001	0.0007	0.0009
4/12/2001	0.0017	0.0009
5/12/2001	0.0017	0.0009
6/12/2001	0.0019	0.0009
7/12/2001	0.0031	0.0009
10/12/2001	0.0025	0.0009
11/12/2001	0.0021	0.0009
12/12/2001	0.0025	0.0009
13/12/2001	0.0031	0.0009
14/12/2001	0.0033	0.0009
17/12/2001	0.0023	0.0009
18/12/2001	0.0024	0.0009
19/12/2001	0.0022	0.0009
20/12/2001	0.0022	0.0010
21/12/2001	0.0012	0.0010
24/12/2001	0.0023	0.0010
27/12/2001	0.0020	0.0010
28/12/2001	0.0025	0.0010
2/1/2002	0.0018	0.0010
3/1/2002	0.0016	0.0010
4/1/2002	0.0018	0.0010

DATE	AEX	ING
7/1/2002	0.0016	0.0010
8/1/2002	0.0011	0.0010
9/1/2002	0.0007	0.0010
10/1/2002	0.0010	0.0010
11/1/2002	0.0016	0.0010
14/1/2002	0.0018	0.0010
15/1/2002	0.0010	0.0010
16/1/2002	0.0016	0.0010
17/1/2002	0.0017	0.0010
18/1/2002	0.0016	0.0010
21/1/2002	0.0009	0.0010
22/1/2002	0.0010	0.0010
23/1/2002	0.0013	0.0010
24/1/2002	0.0008	0.0009
25/1/2002	0.0015	0.0009
28/1/2002	0.0019	0.0009
29/1/2002	0.0017	0.0008
30/1/2002	0.0014	0.0008
31/1/2002	0.0010	0.0007
1/2/2002	0.0007	0.0007
4/2/2002	0.0008	0.0006
5/2/2002	0.0003	0.0006
6/2/2002	-0.0003	0.0005
7/2/2002	-0.0002	0.0005
8/2/2002	0.0004	0.0005
11/2/2002	-0.0004	0.0004
12/2/2002	-0.0001	0.0004
13/2/2002	-0.0002	0.0003
14/2/2002	-0.0003	0.0003
15/2/2002	-0.0002	0.0003
Average	**0.0011**	**0.0008**

Table 2

Maximum Likelihood Estimation of the *GARCH(1,1)* model for the Whole Sample

The estimated parameters of the *GARCH(1,1)* model and the long-term variance rates of the AEX Index and the ING GROEP CERTS asset are reported.

- AEX index

VL	ω	γ	α	β	α + β
7.7646E-04	1.5450E-04	0.1990	0.0346	0.7664	0.8010

- ING GROEP CERTS asset

VL	ω	γ	α	β	α + β
3.0308E-03	5.0404E-04	0.1663	0.0000	0.8337	0.8337

Table 3

Maximum Likelihood Estimation of the Adaptive GARCH model during the lifetime of the options

The estimated parameters of the Adaptive GARCH model and the long-term variance rates of the AEX Index and the ING GROEP CERTS asset on each day during the lifetime of the options are displayed.

- **AEX Index**

AEX	VL	ω	γ	α	β	α + β
19/11/2001	7.6572E-04	1.5047E-04	0.1965	0.0382	0.7653	0.8035
20/11/2001	7.5269E-04	1.4504E-04	0.1927	0.0436	0.7637	0.8073
21/11/2001	7.4367E-04	1.4357E-04	0.1931	0.0475	0.7595	0.8069
22/11/2001	7.3544E-04	1.5290E-04	0.2079	0.0487	0.7434	0.7921
23/11/2001	7.3044E-04	1.6132E-04	0.2209	0.0493	0.7298	0.7791
26/11/2001	7.2475E-04	1.6936E-04	0.2337	0.0510	0.7154	0.7663
27/11/2001	7.1768E-04	1.6351E-04	0.2278	0.0596	0.7126	0.7722
28/11/2001	7.1191E-04	1.4873E-04	0.2089	0.0701	0.7210	0.7911
29/11/2001	7.0048E-04	1.2809E-04	0.1829	0.0826	0.7345	0.8171
30/11/2001	7.0390E-04	1.2291E-04	0.1746	0.0949	0.7305	0.8254
3/12/2001	5.2335E-05	9.0877E-07	0.0174	0.1373	0.8453	0.9826
4/12/2001	2.6679E-05	4.0485E-07	0.0152	0.1556	0.8292	0.9848
5/12/2001	2.5461E-05	3.5066E-07	0.0138	0.1708	0.8155	0.9862
6/12/2001	1.5075E-04	4.3045E-06	0.0286	0.0727	0.8988	0.9714
7/12/2001	6.9137E-06	1.4938E-07	0.0216	0.0699	0.9085	0.9784
10/12/2001	6.9137E-06	1.4938E-07	0.0216	0.0699	0.9085	0.9784
11/12/2001	5.0340E-05	1.0642E-06	0.0211	0.0729	0.9059	0.9789
12/12/2001	3.8771E-05	9.7116E-07	0.0250	0.0650	0.9100	0.9750
13/12/2001	3.8771E-05	9.7116E-07	0.0250	0.0650	0.9100	0.9750
14/12/2001	3.0084E-05	6.7542E-07	0.0225	0.0371	0.9404	0.9775
17/12/2001	2.2280E-05	5.4494E-07	0.0245	0.0448	0.9308	0.9755
18/12/2001	1.6783E-04	5.1120E-06	0.0305	0.0394	0.9301	0.9695
19/12/2001	1.3368E-04	3.7917E-06	0.0284	0.0439	0.9277	0.9716
20/12/2001	1.3962E-05	2.8609E-07	0.0205	0.0350	0.9445	0.9795
21/12/2001	6.8300E-05	1.7017E-06	0.0249	0.0325	0.9425	0.9751
24/12/2001	1.1761E-04	2.7991E-06	0.0238	0.0278	0.9484	0.9762
27/12/2001	5.2499E-05	1.0728E-06	0.0204	0.0303	0.9493	0.9796
28/12/2001	1.8172E-05	2.6434E-07	0.0145	0.0181	0.9673	0.9855
2/1/2002	3.2457E-05	6.0595E-07	0.0187	0.0258	0.9555	0.9813
3/1/2002	6.2938E-06	1.0082E-07	0.0160	0.0251	0.9588	0.9840
4/1/2002	7.7324E-05	1.6322E-06	0.0211	0.0294	0.9495	0.9789
7/1/2002	3.9489E-05	7.5187E-07	0.0190	0.0277	0.9533	0.9810
8/1/2002	1.5848E-05	3.0416E-07	0.0192	0.0336	0.9472	0.9808
9/1/2002	1.3592E-05	2.6971E-07	0.0198	0.0433	0.9368	0.9802
10/1/2002	4.0607E-05	9.1357E-07	0.0225	0.0270	0.9505	0.9775

11/1/2002	3.6309E-05	8.0311E-07	0.0221	0.0296	0.9483	0.9779
14/1/2002	2.9931E-05	6.3371E-07	0.0212	0.0276	0.9512	0.9788
15/1/2002	1.1140E-04	2.9430E-06	0.0264	0.0236	0.9500	0.9736
16/1/2002	1.3165E-05	1.4317E-07	0.0109	0.0099	0.9792	0.9891
17/1/2002	1.3525E-05	2.3601E-07	0.0175	0.0234	0.9591	0.9825
18/1/2002	1.6733E-05	2.5775E-07	0.0154	0.0156	0.9690	0.9846
21/1/2002	9.9341E-06	1.7549E-07	0.0177	0.0238	0.9586	0.9823
22/1/2002	1.9035E-05	3.8018E-07	0.0200	0.0304	0.9497	0.9800
23/1/2002	6.6753E-06	1.3178E-07	0.0197	0.0256	0.9546	0.9803
24/1/2002	4.0142E-06	8.5878E-08	0.0214	0.0351	0.9435	0.9786
25/1/2002	3.6379E-05	5.7551E-07	0.0158	0.0119	0.9723	0.9842
28/1/2002	8.5827E-06	1.5381E-07	0.0179	0.0216	0.9605	0.9821
29/1/2002	3.4045E-05	6.9711E-07	0.0205	0.0245	0.9551	0.9795
30/1/2002	1.6203E-05	3.1182E-07	0.0192	0.0258	0.9550	0.9808
31/1/2002	2.9254E-05	5.2999E-07	0.0181	0.0169	0.9650	0.9819
1/2/2002	8.5301E-06	1.5142E-07	0.0178	0.0233	0.9590	0.9822
4/2/2002	3.0246E-05	6.1760E-07	0.0204	0.0252	0.9544	0.9796
5/2/2002	2.9284E-05	6.1327E-07	0.0209	0.0268	0.9523	0.9791
6/2/2002	5.6134E-05	9.8579E-07	0.0176	0.0107	0.9718	0.9824
7/2/2002	7.5533E-06	1.1075E-07	0.0147	0.0122	0.9731	0.9853
8/2/2002	3.0494E-05	5.3839E-07	0.0177	0.0123	0.9700	0.9823
11/2/2002	6.8832E-05	6.2688E-07	0.0091	0.0001	0.9908	0.9909
12/2/2002	7.3231E-05	6.5836E-07	0.0090	0.0001	0.9909	0.9910
13/2/2002	1.3375E-05	8.2965E-08	0.0062	0.0000	0.9938	0.9938
14/2/2002	3.3805E-05	2.1983E-07	0.0065	0.0000	0.9935	0.9935
15/2/2002	1.6514E-05	1.0314E-07	0.0062	0.0000	0.9938	0.9938

- **ING GROEP CERTS**

ING	VL	ω	γ	α	β	α + β
19/11/2001	2.9694E-03	5.0834E-04	0.1712	0.0000	0.8288	0.8288
20/11/2001	2.8944E-03	5.1306E-04	0.1773	0.0000	0.8227	0.8227
21/11/2001	2.8532E-03	5.0667E-04	0.1776	0.0000	0.8224	0.8224
22/11/2001	2.7828E-03	5.4961E-04	0.1975	0.0000	0.8025	0.8025
23/11/2001	2.7349E-03	5.8794E-04	0.2150	0.0000	0.7850	0.7850
26/11/2001	2.7501E-03	6.4055E-04	0.2329	0.0000	0.7671	0.7671
27/11/2001	2.7356E-03	6.7651E-04	0.2473	0.0000	0.7527	0.7527
28/11/2001	2.7389E-03	7.0324E-04	0.2568	0.0000	0.7432	0.7432
29/11/2001	2.7492E-03	7.0048E-04	0.2548	0.0000	0.7452	0.7452
30/11/2001	2.7375E-03	7.3457E-04	0.2683	0.0000	0.7317	0.7317
3/12/2001	2.4642E-03	5.4229E-04	0.2201	0.0135	0.7664	0.7799
4/12/2001	2.4213E-03	5.3604E-04	0.2214	0.0201	0.7585	0.7786
5/12/2001	2.4213E-03	5.3592E-04	0.2213	0.0201	0.7585	0.7787
6/12/2001	2.2784E-03	5.1043E-04	0.2240	0.0293	0.7467	0.7760
7/12/2001	2.2451E-03	5.9016E-04	0.2629	0.0377	0.6995	0.7371
10/12/2001	2.2300E-03	6.7669E-04	0.3035	0.0450	0.6516	0.6965
11/12/2001	2.2274E-03	8.0414E-04	0.3610	0.0518	0.5872	0.6390
12/12/2001	2.2113E-03	8.3735E-04	0.3787	0.0575	0.5638	0.6213

13/12/2001	2.2122E-03	8.6015E-04	0.3888	0.0592	0.5520	0.6112
14/12/2001	2.1658E-03	7.8191E-04	0.3610	0.0666	0.5724	0.6390
17/12/2001	2.1694E-03	7.8996E-04	0.3641	0.0757	0.5602	0.6359
18/12/2001	2.4978E-05	3.1262E-07	0.0125	0.1004	0.8870	0.9875
19/12/2001	8.1162E-06	9.2822E-08	0.0114	0.0996	0.8889	0.9886
20/12/2001	2.9026E-05	3.3778E-07	0.0116	0.1051	0.8833	0.9884
21/12/2001	3.6505E-04	6.0762E-06	0.0166	0.1139	0.8694	0.9834
24/12/2001	5.3296E-05	9.7397E-07	0.0183	0.0860	0.8957	0.9817
27/12/2001	3.7635E-05	7.0973E-07	0.0189	0.0893	0.8919	0.9811
28/12/2001	0.0000E+00	0.0000E+00	0.0153	0.0860	0.8986	0.9847
2/1/2002	4.5896E-05	9.8053E-07	0.0214	0.0734	0.9052	0.9786
3/1/2002	6.1976E-05	1.0685E-06	0.0172	0.0543	0.9285	0.9828
4/1/2002	1.9456E-04	5.2955E-06	0.0272	0.0520	0.9208	0.9728
7/1/2002	1.5544E-04	3.9693E-06	0.0255	0.0445	0.9300	0.9745
8/1/2002	4.0130E-04	1.0916E-05	0.0272	0.0390	0.9338	0.9728
9/1/2002	3.1170E-04	7.8780E-06	0.0253	0.0429	0.9319	0.9747
10/1/2002	1.9608E-04	4.6480E-06	0.0237	0.0440	0.9323	0.9763
11/1/2002	5.0940E-05	1.0758E-06	0.0211	0.0428	0.9360	0.9789
14/1/2002	1.9096E-04	4.2955E-06	0.0225	0.0395	0.9380	0.9775
15/1/2002	2.9618E-04	7.4029E-06	0.0250	0.0437	0.9313	0.9750
16/1/2002	6.0571E-04	2.0039E-05	0.0331	0.0415	0.9254	0.9669
17/1/2002	5.4920E-04	1.6532E-05	0.0301	0.0448	0.9251	0.9699
18/1/2002	3.7026E-04	9.4940E-06	0.0256	0.0422	0.9322	0.9744
21/1/2002	5.1295E-04	1.3085E-05	0.0255	0.0524	0.9221	0.9745
22/1/2002	4.2211E-04	1.0128E-05	0.0240	0.0514	0.9246	0.9760
23/1/2002	3.1262E-04	6.5893E-06	0.0211	0.0527	0.9262	0.9789
24/1/2002	3.1262E-04	6.5893E-06	0.0211	0.0527	0.9262	0.9789
25/1/2002	8.6753E-05	1.5001E-06	0.0173	0.0135	0.9692	0.9827
28/1/2002	3.5962E-04	1.3778E-05	0.0383	0.0193	0.9424	0.9617
29/1/2002	2.0949E-04	5.2335E-06	0.0250	0.0156	0.9594	0.9750
30/1/2002	7.2620E-05	1.0919E-06	0.0150	0.0129	0.9721	0.9850
31/1/2002	5.0917E-05	7.6162E-07	0.0150	0.0131	0.9719	0.9850
1/2/2002	5.0917E-05	7.6162E-07	0.0150	0.0131	0.9719	0.9850
4/2/2002	4.8315E-04	2.2533E-05	0.0466	0.0135	0.9399	0.9534
5/2/2002	4.4565E-04	2.0458E-05	0.0459	0.0173	0.9367	0.9541
6/2/2002	3.9532E-04	1.4888E-05	0.0377	0.0152	0.9472	0.9623
7/2/2002	4.8607E-05	6.5621E-07	0.0135	0.0100	0.9765	0.9865
8/2/2002	4.5094E-06	5.9178E-08	0.0131	0.0108	0.9761	0.9869
11/2/2002	3.7603E-04	6.8210E-06	0.0181	0.0068	0.9750	0.9819
12/2/2002	4.4337E-04	1.6935E-05	0.0382	0.0114	0.9504	0.9618
13/2/2002	3.8674E-04	1.2234E-05	0.0316	0.0114	0.9569	0.9684
14/2/2002	3.4448E-04	1.0072E-05	0.0292	0.0135	0.9573	0.9708
15/2/2002	3.2759E-04	8.9389E-06	0.0273	0.0125	0.9602	0.9727

Table 4

Aggregate Squared Loss and Pricing Errors of the Black-Scholes Option Pricing for the AEX Index

The sum of squared differences between the market option prices and the model-determined prices for each option is given under 'Total Squared Loss'. The RMSE (root mean squared error) of the three models is also reported.

Exercise Price	**Moving Average**		**GARCH(1,1)**		**Adaptive GARCH**	
	Total Squared Loss	1122.7235	Total Squared Loss	2684.8085	Total Squared Loss	1107.5599
460	**RMSE**	**4.3257**	**RMSE**	**6.6893**	**RMSE**	**4.2964**
	Total Squared Loss	2239.8875	Total Squared Loss	5576.7700	Total Squared Loss	1470.9104
480	**RMSE**	**6.1099**	**RMSE**	**9.6409**	**RMSE**	**4.9513**
	Total Squared Loss	3369.8176	Total Squared Loss	8120.6981	Total Squared Loss	1648.7049
500	**RMSE**	**7.4942**	**RMSE**	**11.6338**	**RMSE**	**5.2420**
	Total Squared Loss	5114.0224	Total Squared Loss	8478.1884	Total Squared Loss	2979.5121
520	**RMSE**	**9.2322**	**RMSE**	**11.8871**	**RMSE**	**7.0469**
	Total Squared Loss	3915.4690	Total Squared Loss	6992.3757	Total Squared Loss	1865.2816
540	**RMSE**	**8.0782**	**RMSE**	**10.7954**	**RMSE**	**5.5757**

Table 5

Aggregate Squared Loss and Pricing Errors of the Black-Scholes Option Pricing for the ING GROEP CERTS asset

The sum of squared differences between the market option prices and the model-determined prices for each option is given under 'Total Squared Loss'. The RMSE (root mean squared error) of the three models is also reported.

Exercise Price	**Moving Average**		**GARCH(1,1)**		**Adaptive GARCH**	
	Total Squared Loss	30.5696	Total Squared Loss	141.0968	Total Squared Loss	47.9347
26	**RMSE**	**0.7138**	**RMSE**	**1.5335**	**RMSE**	**0.8938**
	Total Squared Loss	54.8849	Total Squared Loss	214.2465	Total Squared Loss	78.3715
28	**RMSE**	**0.9564**	**RMSE**	**1.8896**	**RMSE**	**1.1429**
	Total Squared Loss	72.2765	Total Squared Loss	243.4650	Total Squared Loss	101.0783
30	**RMSE**	**1.0975**	**RMSE**	**2.0144**	**RMSE**	**1.2979**
	Total Squared Loss	73.9135	Total Squared Loss	216.4026	Total Squared Loss	107.7862
32	**RMSE**	**1.1099**	**RMSE**	**1.8991**	**RMSE**	**1.3403**
	Total Squared Loss	61.3883	Total Squared Loss	166.3522	Total Squared Loss	94.3840
34	**RMSE**	**1.0115**	**RMSE**	**1.6651**	**RMSE**	**1.2542**

Appendix B - Figures

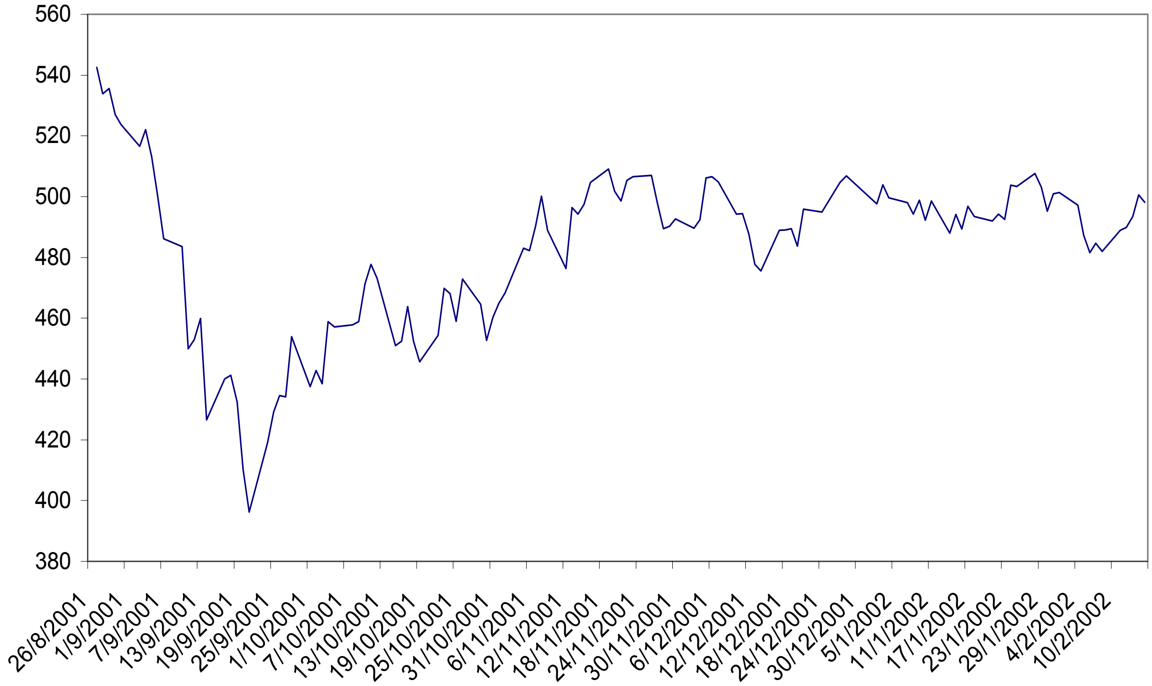

Figure 1A: This figure shows the daily AEX Index levels from August 27, 2001 to February 15, 2002

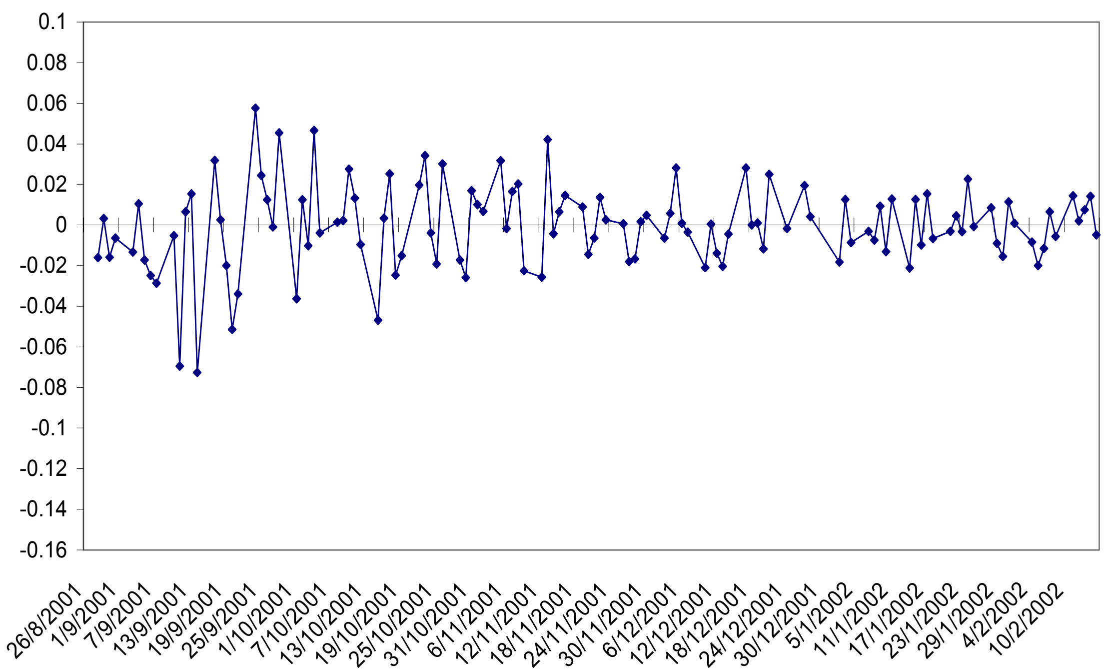

Figure 1B: This figure shows the daily return of the AEX Index from August 27, 2001 to February 15, 2002

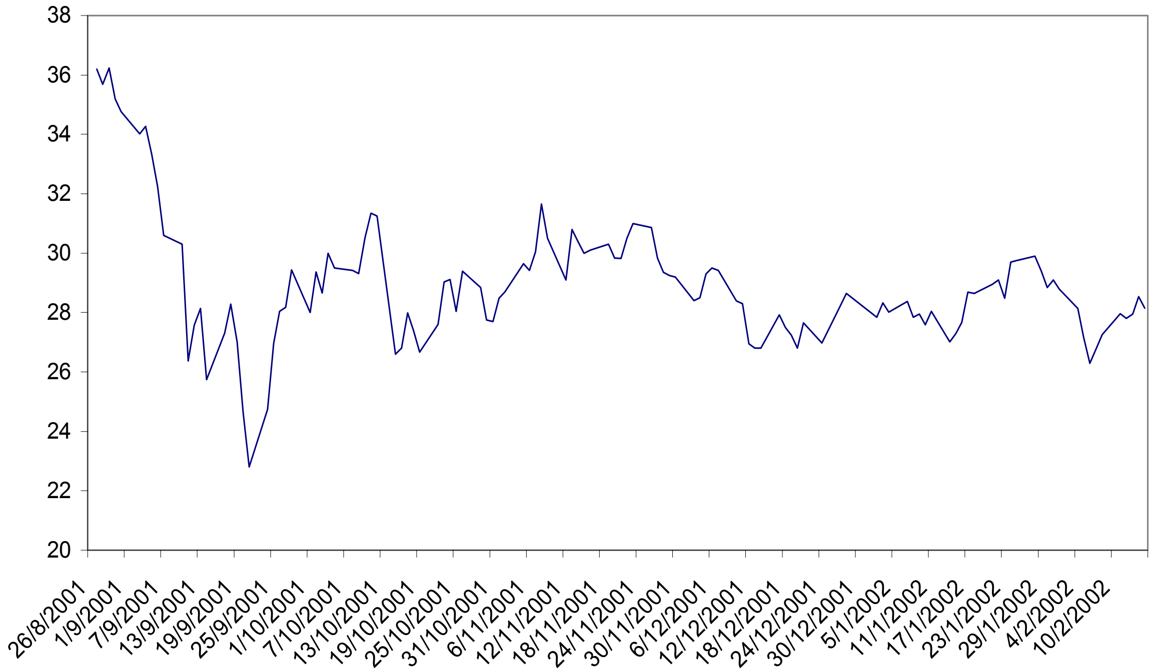

Figure 2A: This figure shows the daily ING GROEP CERTS asset from August 27, 2001 to February 15, 2002

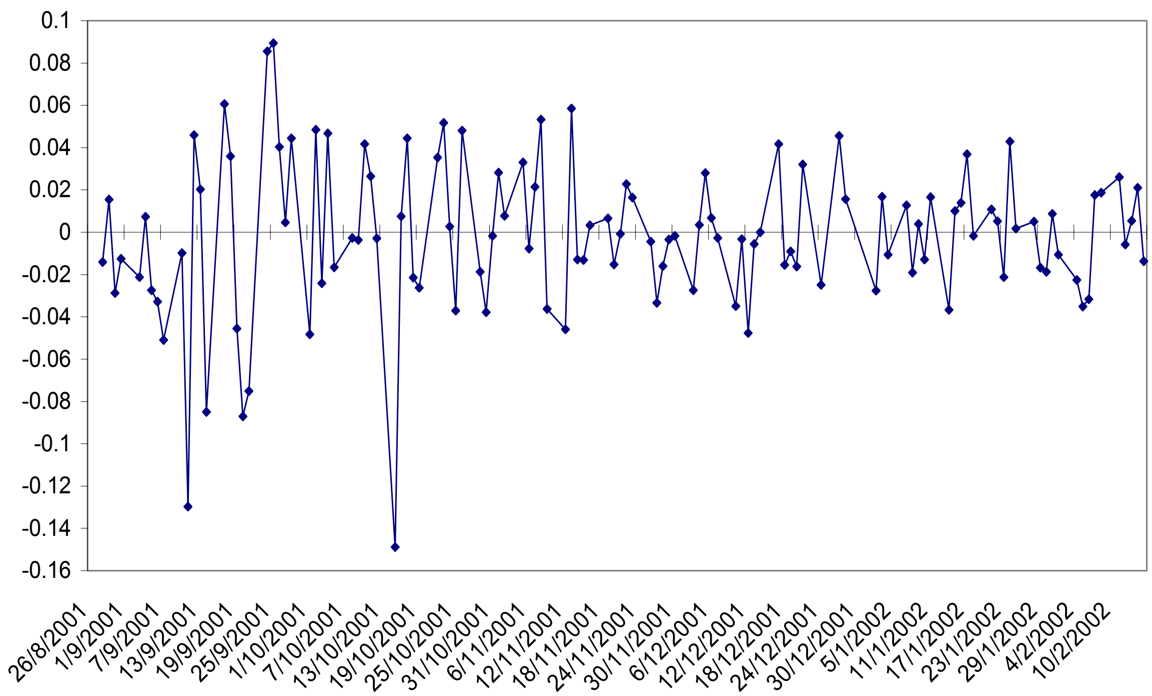

Figure 2B: This figure shows the daily return of the ING GROEP CERTS asset from August 27, 2001 to February 15, 2002

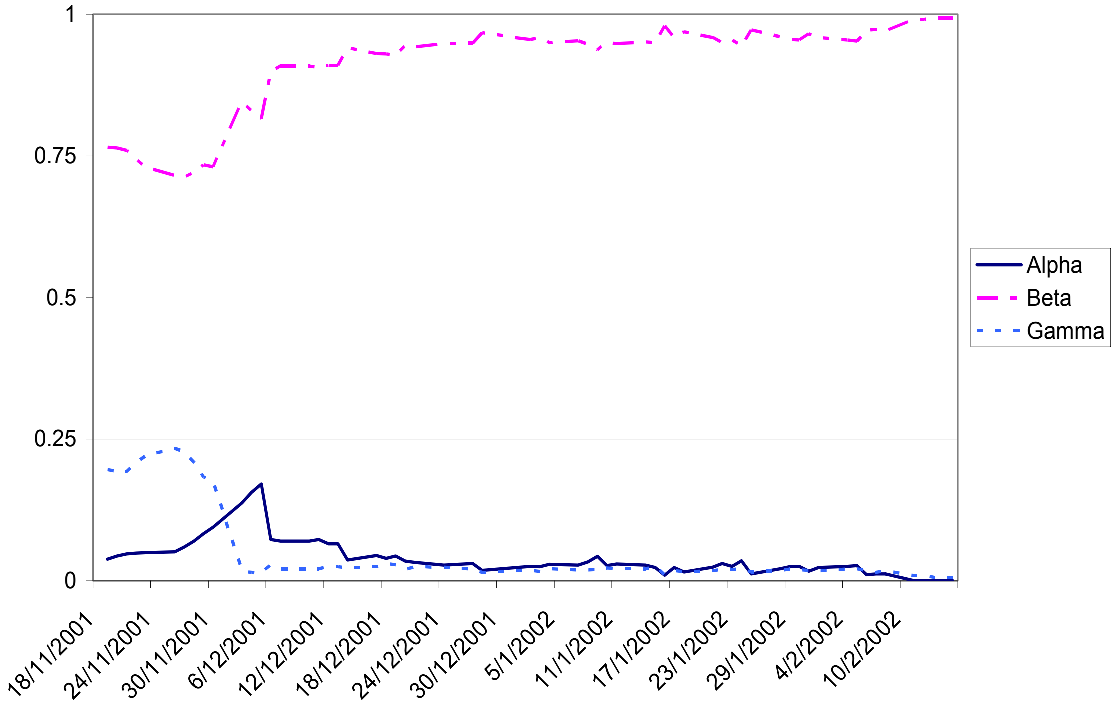

Figure 3A: This figure shows the evolution of the estimated parameters α, β, and γ for the AEX Index during the options' lifetime.

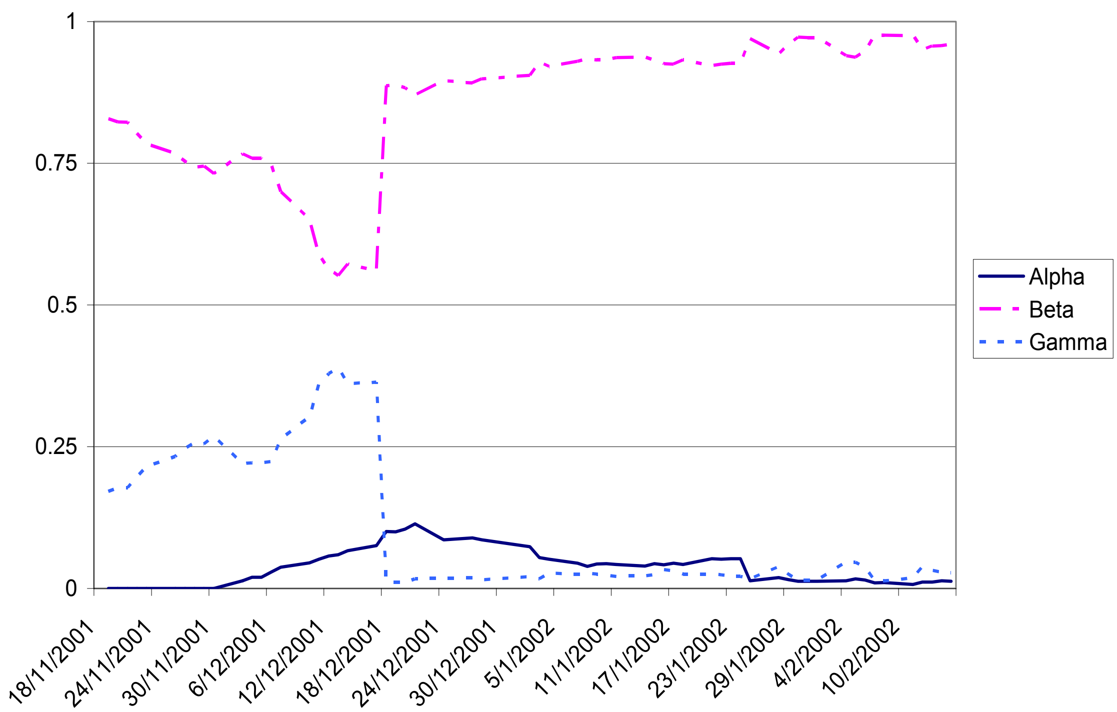

Figure 3B: This figure shows the evolution of the estimated parameters α, β, and γ for the ING GROEP CERTS asset during the options' lifetime.

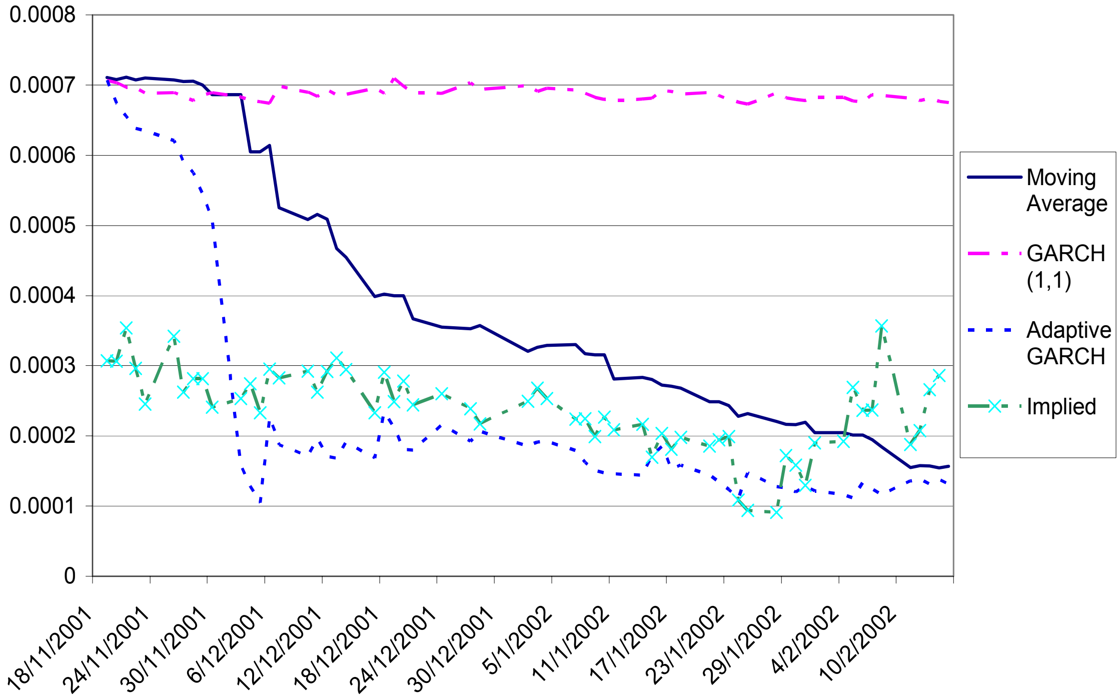

Figure 4A: This figure shows the estimated volatility of the AEX Index obtained from the three models and the implied volatility during the lifetime of the options.

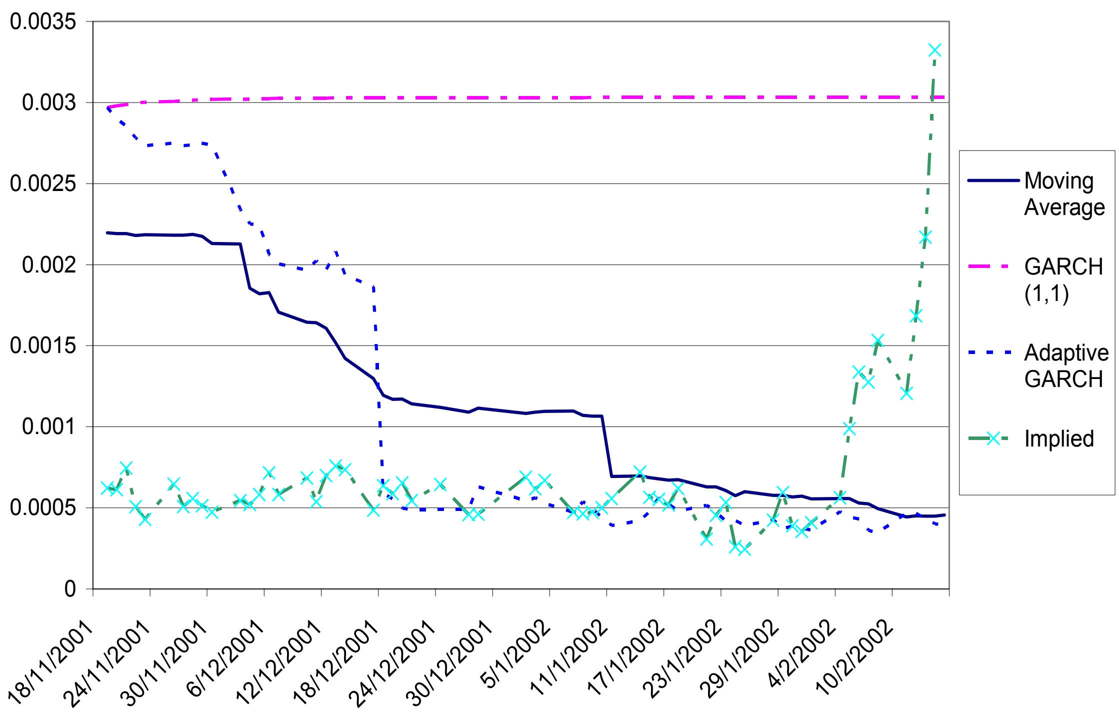

Figure 4B: This figure shows the estimated volatility of the ING GROEP CERTS asset obtained from the three models and the implied volatility during the lifetime of the options.