How to improve fidelity between two entangled qubits using distillation

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***Abstract*—In quantum computing the communication between two systems using a high fidelity entangled pair of qubits is crucial. To improve the fidelity between two entangled qubits a technique called distillation is used. There are many types of distillation protocols. In 1995 Bennet et al. published a paper [1] in which a protocol called BBPSSW was presented. This protocol was further improved upon by a new paper published in 1996 by Deutsch et al. in which a new protocol called DEJMPS was presented. Finally, in 2008, a paper published by Campbell and Benjamin presented a final new way of distilling entangled qubit pairs using photon loss. In order to compare each protocol with another an explanation will be given of each protocol. Then a comparison will be made between the three protocols comparing the protocols by four different points. These are the input flexibility[1], probability of successful distillation, fidelity improvement per iteration and efficiency[2]. After comparison the conclusion of DEJMPS being the overall best protocol can be drawn, due to it ranking overall the highest between the four points stated above.**

Keywords: *Entanglement, Distillation protocols, Bell states, Quantum gates, Fidelity, Quantum computation*

I. Introduction

In quantum computation and communication entanglement plays an crucial roll, as it plays a key part in information processing, which uses entangled particles to travel information between parts of quantum systems. This gives motivation to make sure these entangled qubits are as close to theoretically ideal as possible in the experiments. This, however, is never the case as the quantum system in which the qubits are present will interact with the surroundings, without control over it. This causes the system to have some imperfections and go from a theoretically ideal state to a state which has experienced noise, and thus has drifted away from the theoretically ideal state. Thus, very precise control over the qubits is a necessity for a reliable quantum system. To make sure the qubits behave as expected, the states of the qubits should be as close to the theoretically predetermined ideal as possible. This is, however, no easy task. The qubits experience many uncertainties and imperfections in the quantum gates are almost a given. The extent to which these uncertainties and imperfections affect the state of the entangled pair of qubits can be very harmful to the performance of the quantum computer or experimental setup.

In order to quantify how similar the theoretical qubits, denoted by $|\Psi_t\rangle$, are to the experimental qubits, denoted by ρ_e, scientists use the concept of fidelity, which is a measure of accuracy when comparing entangled pairs with each other. The fidelity between these two states can be found using a formula. Different versions of this formula can be used, depending on the situation. These may include: pure state with a pure state, pure state with a mixed state, mixed state with a mixed state. The case discussed in this paper will be the pure state with a mixed state since in the real world there are no theoretically ideal state. One can, however, shrink the margin of error to become practically indistinguishable. This means the following: the goal state one is comparing to is a state which has noise in it, among other imperfections. These can be represented as follows: $|\Psi_t\rangle$ will be compared to $\rho_e = p|\Psi_t\rangle\langle\Psi_t| + (1-p)|\Phi_{noise}\rangle\langle\Phi_{noise}|$. The fidelity between $|\Psi_t\rangle$ and ρ_e is given by the following, Eq. 1 [2]:

$$F = \sqrt{\langle\Psi_t|\,\rho_e\,|\Psi_t\rangle} \geq 1 - \epsilon. \tag{1}$$

There are multiple methods to improve the fidelity of the entangled qubits, these may include: Quantum Optimal Control theory (QOC for short), dynamical decoupling and distillation. Though papers have been published discussing the latter of the three, few have compared the three protocols in the same paper. This paper will fill the gap and answer the question: ”What is the optimal two-to-one distillation method for improving fidelity of entangled qubit pairs?”

II. Method

In this literature review the following published articles will be discussed:

1) ”Measurement based entanglement under conditions of extreme photon loss”, by E.T. Campbell and S.C. Benjamin [1].

2) ”Purification of Noisy Entanglement and Faithful Teleportation via Noisy Channels”, by C.H. Bennet et al. [4].

3) ”Quantum privacy amplification and the security of quantum cryptography over noisy channels”, by D. Deutsch et al. [5].

These articles have been chosen as the methods described in these papers form a basis for the distillation techniques, and many more recent methods are based on these protocols.

These articles were suggested by a study colleague. If one were to use search terms to find the same articles, fitting search terms would include: ”Entanglement purification protocols”,

[1]This denotes how many types of initial states are allowed as inputs. For instance, if protocol A says it is designed for initial states in the form $\alpha_{A1}|\Psi\rangle + \alpha_{A2}|\epsilon_{noise}\rangle$, while protocol B is designed for initial states of form $\alpha_{B1}|\Psi\rangle + \alpha_{B2}|\Phi\rangle + \alpha_{B3}|\epsilon_{noise}\rangle$. Then clearly protocol B is more flexible for initial states as it has a wider range of allowed initial states.

[2]Efficiency denotes the fraction of pairs of entangled qubits with the desired fidelity to the amount of pairs of entangled qubits in total needed to achieve such fidelity for the purified qubits.

”Entanglement distillation protocols” and ”Entanglement fidelity purification”. A different method of finding the articles is searching up ”BBPSSW” on Google. The first result will a BSc Thesis, by T. Schiet [3]. In the list of references one may find the three articles.

III. Distillation

A very simple and easy way to improve fidelity in certain systems is distillation. Distillation uses other entangled pairs in order to increase the fidelity of a single pair and by performing certain types of gates on them, has a chance of improving the fidelity each time a sample has been measured. The last part plays a important role. One first measures the output of the gates. After this, pairs which whose measurements imply distillation failed are discarded. This causes the entangled pairs which are left to have their fidelity increased. Distillation consists of multiple methods, each better than another in a certain situation. The methods discussed in this paper will be: Extreme Photon Loss (EPL) [3] [1], BBPSSW[3] [3] [4] and DEJMPS [3] [5] [3], as those are the most basic methods, easy to implement and also easy to understand.

A. Method of distillation: EPL

The first method to be discussed will be EPL, which stands for Extreme Photon Loss, as it is very similar to BBPSSW and DEJMPS. EPL was developed by a paper [1] written by Campbell and Benjamin. In the EPL-protocol the scientists take two different entangled qubit pairs, $|\Psi_{0A}\rangle$, $|\Psi_{0B}\rangle$, $|\Psi_{1A}\rangle$ and $|\Psi_{1B}\rangle$, where $|\Psi_{0A}\rangle$ and $|\Psi_{0B}\rangle$, $|\Psi_{1A}\rangle$ and $|\Psi_{1B}\rangle$ form two entangled pairs of qubits which have experienced noise. From this on there are another two sub-methods of EPL which are very similar but differ in the amount of operations on the qubits.

1) EPL protocol described in original paper: In this method the scientists start off with systems whose state is of form $|\theta\rangle\,|\theta\rangle$. The scientists stimulate the two systems with lasers to make them emit photons, as when photons are emitted, the system’s internal state and certain characteristics of the photon are correlated. Once a detector detects a lone emitted photon, the system’s state can be written as the following density operators:

$$\rho_A = (1-\eta_A)\left|\Psi^+\right\rangle\left\langle\Psi^+\right| + \eta_A\,|11\rangle\,\langle 11| \tag{2}$$

$$\rho_B = (1-\eta_B)\left|\Psi^+\right\rangle\left\langle\Psi^+\right| + \eta_B\,|11\rangle\,\langle 11|\,, \tag{3}$$

where η_A denotes the amplitude of the noise-component of the state ρ_A (same holds for η_B with ρ_B). Then the scientists rotate each qubit which makes up the system B by $\frac{I_2+i\sigma_Y}{\sqrt{2}}$, which changes the density operator of system B to:

$$\rho_B = (1-\eta_B)\,|\Psi^-\rangle\,\langle\Psi^-| + \eta_B\left|\Psi^+\right\rangle\left\langle\Psi^+\right|.$$

With these two pairs, the scientist twice performs a CPHASE[4] gate on $|\Psi_{0A}\rangle$ and $|\Psi_{0B}\rangle$, and on $|\Psi_{1A}\rangle$ and $|\Psi_{1B}\rangle$. The scientist then measures the output of qubits $|\Psi_{0B}\rangle$ and $|\Psi_{1B}\rangle$ in the X-basis, preserving the entanglement between the two pairs. Depending on the outcome the scientist either keeps the qubits due to the fidelity having gone up, or discards the qubits due to the fidelity having gone down.

2) Simplified EPL protocol: The scientists have density operators ρ_0 and ρ_1, describing the density matrices of entangled pairs 0 and 1. These pairs each consist of qubit $|\Psi i A\rangle$ and $|\Psi_{iB}\rangle$. First a local CNOT operation is performed between qubits $|\Psi_{0A}\rangle$ and $|\Psi_{0B}\rangle$, and $|\Psi_{1A}\rangle$ and $|\Psi_{1B}\rangle$. Then measurements are taken in the Z-basis on qubits $|\Psi_{0A}\rangle$ and $|\Psi_{0B}\rangle$. If both measurement results are ’1’, the distillation is a success. In Fig. 1 one can see a schematic of the setup of this protocol.

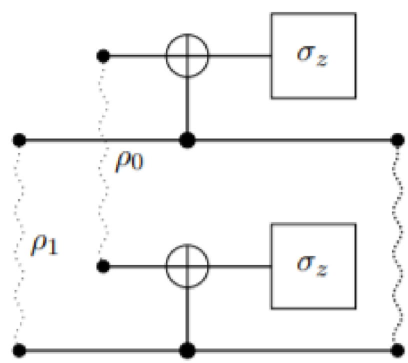

Figure 1. Part of the experimental setup of EPL entanglement purification, [3]. In this figure ρ_i, for i = $\{0, 1\}$, denotes the density matrix representation a entangled qubit pair. The squiggly lines denote the entanglement between the two qubits, which are denoted by nodes. Note how after the measurement has been taken, one can see the entanglement between the qubits which first made up ρ_0 has been broken, while the entanglement between the qubits which first made up ρ_1 does remain intact.

According to [3] [1], the optimal type of input state $|\Psi_i\rangle$ which results in maximally entangled states is of form:

$$\rho_i = p\,|\Phi\rangle\,\langle\Phi| + (1-p)\,|11\rangle\,\langle 11|\,, \tag{4}$$

and in standard ket notation:

$$|\Psi_i\rangle = \frac{1}{\sqrt{2}}(|01\rangle + e^{i\phi}\,|10\rangle), \tag{5}$$

where ϕ denotes any arbitrary angles in range $[-\pi, \pi]$. Furthermore, (1-p) has the same function as $\eta_X, X \in \{A, B\}$ in Eq. 2 and Eq. 3. After some calculation one can see that after the EPL-protocol the final state $\left|\Psi_f\right\rangle$ of the entangled qubit is:

$$\left|\Psi_f\right\rangle = \frac{1}{\sqrt{(2)}}(|01\rangle + |10\rangle), \tag{6}$$

where the global phase has been neglected.
According to [3] the probability of success with these conditions is:

$$p_{succ} = \frac{1}{2}p^2. \tag{7}$$

[3]For both BBPSSW and DEJMPS, the acronyms stand for each of the authors surnames of each paper, in which the protocol was introduced, in alphabetical order

[4]For more information on CPHASE, see [6]. For more information on CNOT, see [2] (p. 177-180)

B. Method of distillation: BBPSSW

In the BBPSSW-protocol [4] the scientists perform a few more operations on the qubits. The BBPSSW-protocol was developed by the scientists Bennet; Brassard; Popescu; Schumacher; Smolin; Wootters, whose first letter of the last name make up the protocol, in 1995. There are multiple ways of performing the DEJMPS protocol, in this paper there will be explained two.

1) BBPSSW protocol described in original paper: In the protocol a bilateral CNOT gate is used, BXOR. This gate has the following table, see Fig. 2.

Before		After	(n.c. = no change)
Source	Target	Source	Target
$\Phi^{\pm}$	Φ^{+}	n.c.	n.c.
$\Psi^{\pm}$	Φ^{+}	n.c.	Ψ^{+}
$\Psi^{\pm}$	Ψ^{+}	n.c.	Φ^{+}
$\Phi^{\pm}$	Ψ^{+}	n.c.	n.c.
$\Phi^{\pm}$	Φ^{-}	$\Phi^{\mp}$	n.c.
$\Psi^{\pm}$	Φ^{-}	$\Psi^{\mp}$	Ψ^{-}
$\Psi^{\pm}$	Ψ^{-}	$\Psi^{\mp}$	Φ^{-}
$\Phi^{\pm}$	Ψ^{-}	$\Phi^{\mp}$	n.c.

Figure 2. Table which describes the BXOR gate operations, [4]

The protocol [4](p. 3) starts with "...n impure systems of qubits described by any Bell-diagonal density matrix W with S(W) < 1, and n(S(W)+δ) prepurified Φ^+ states." With Φ^+ as target states, [4] states that the scientists "... perform BXOR tests on sufficiently many random subsets of the impure pairs to locate all Ψ states". Then the scientists perform a unilateral σ_Y rotation on each of the pairs to change the $\Psi^{\pm}$ states to the $\Phi^{\mp}$ states. After this has been done the Φ^- states will be converted to Ψ^+ states using the bilateral B_y rotation, while leaving the other Φ^+ states invariant. Finally another set of BXOR tests will be performed on the remaining states to find many new Ψ^+ states with high probability. Finally these Ψ^+ states are converted to the target Φ^+ states.

2) Simplified BBPSSW protocol: There does exist a simpler version of the BBPSSW protocol which assumes a specific type of initial state, called the Werner state [7]. If a state is not in the Werner state, one first converts it to a Werner state using depolarization, which in turn decreases the efficiency. According to [3] Werner state has the following density matrix, ρ_W:

$$\rho_W = p\left|\Phi^+\right\rangle\left\langle\Phi^+\right| + (1-p)\frac{I_4}{4}. \tag{8}$$

Then the scientists take the two pairs of entangled qubits $\Psi_0 A$ and $\Psi_1 A$, $\Psi_0 B$ and $\Psi_1 B$, and perform a CNOT operation on $\Psi_0 A$ and $\Psi_0 B$, and on $\Psi_1 A$ and $\Psi_1 B$. Finally a measurement will be taken of the qubits $\Psi_0 A$ and $\Psi_0 A$. If the measurements of the spins are equal, the fidelity of the system will have gone up, else the pair is thrown away. In Fig. 3 one can see a schematic of the circuit of the protocol.

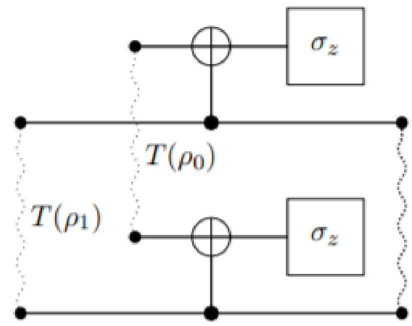

Figure 3. Schematic of circuit of simplified BBPSSW protocol, where T(ρ_i) is the i^{th} pair of entangled qubits Ψ_{iA} and Ψ_{iB}, [3]. The squiggly lines denote the entanglement between the two qubits, which are denoted by nodes. After the measurement has been taken, one can see the entanglement between the qubits which first made up $T(\rho_0)$ has been broken, while the entanglement between the qubits which first made up Tρ_1) does remain intact.

The fidelity of the initial state is given by:

$$F = tr(\rho\left|\Phi^+\right\rangle\left\langle\Phi^+\right|), \tag{9}$$

where $|\Phi^+\rangle$ is the target state. Then according to [3][4] the fidelity of the system after the protocol is:

$$F' = \frac{F^2 + \frac{1}{9}(1-F)^2}{F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2}, \tag{10}$$

with probability of success:

$$p_{succ} = F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2. \tag{11}$$

C. Method of distillation: DEJMPS

Following the BBPSSW protocol, Deutsch et al. proposed a new protocol, the DEJMPS protocol [5]. This protocol uses fewer unitary gates than the BBPSSW and has looser conditions on the initial state, which in turn makes it more efficient. The protocol makes use of the following simple unitary transformation:

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$$
$$|1\rangle \rightarrow \frac{1}{\sqrt{2}}(|1\rangle - i\,|0\rangle)$$

This can be represented in the following matrix form:

$$U_A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}.$$

The protocol makes use of two entangled qubit pairs in any two states ρ_0 and ρ_1 with basis Bell states: $B = \{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}$. There are however not just a few qubit pairs, but many. In order to then represent each state an average of the whole pool is taken. Each entangled pair is set to this average. Thus each state can be represented as follows:

$$\rho_0 = \begin{bmatrix} A_0 \\ B_0 \\ C_0 \\ D_0 \end{bmatrix}, \rho_1 = \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{bmatrix}.$$

With

$$A_i = \langle \Phi^+ | \rho_i | \Phi^+ \rangle ,$$
$$B_i = \langle \Phi^- | \rho_i | \Phi^- \rangle ,$$
$$C_i = \langle \Psi^+ | \rho_i | \Psi^+ \rangle ,$$
$$D_i = \langle \Psi^- | \rho_i | \Psi^- \rangle .$$

Furthermore should the following condition hold:

$$\max_{\Phi \in B} \langle \Phi | \rho | \Phi \rangle > \frac{1}{2}. \tag{12}$$

This condition must hold for both states ρ_0 and ρ_1. When both conditions hold, the following quantum circuit is exercised, where $U_B = (U_A)^{-1}$, see Fig. 4. First two unitary transformations denoted by U_A and U_B are performed. One qubit of each pair goes through a U_A transformation, which are denoted by ρ_{0A} and ρ_{0B}, and one of each pair goes through a U_B transformation, which are denoted by ρ_{1A} and ρ_{1B}. After this a CNOT gate is performed on both inputs ρ_{0A} and ρ_{0B}, and ρ_{1A} and ρ_{1B}, with ρ_{0A} and ρ_{1A} being the target states. Finally a measurement in computational Z-basis will be done on the target qubits after the CNOT gate. If both measurement outcomes coincide, the distillation is a success and only ρ_0 is discarded, else a failure and both pairs of entangled qubits are discarded.

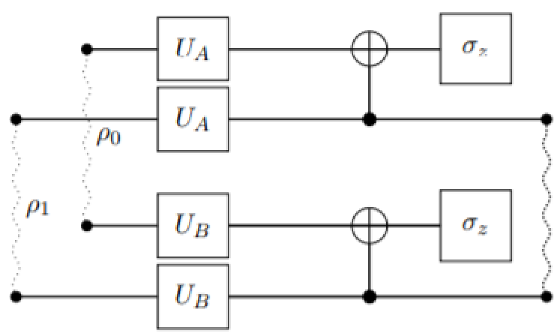

Figure 4. Schematic of circuit of DEJMPS protocol. In this figure ρ_i, for i = $\{0, 1\}$, denotes the density matrix representation a entangled qubit pair. The squiggly lines denote the entanglement between the two qubits, which are denoted by nodes. The σ_Z box denotes a measurement taken in the computational Z-basis. [3]. After the measurement has been taken, one can see the entanglement between the qubits which first made up ρ_0 has been broken, while the entanglement between the qubits which first made up ρ_1 does remain intact.

The probability of success can be shown to be:

$$p_{succ} = (A_0 + A_1)(B_0 + B_1) + (C_0 + C_1)(D_0 + D_1). \tag{13}$$

If the distillation process is declared a success, the fidelity of ρ_1 is given by:

$$F' = \frac{A_0 A_1 + B_0 B_1}{p_{succ}}. \tag{14}$$

If both qubits are Werner states, the probability of success and the fidelity improvement are:

$$F' = \frac{F^2 + \frac{1}{9}(1-F)^2}{F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2}, \tag{15}$$

$$p_{succ} = F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2. \tag{16}$$

The power of the DEJMPS protocol comes up in the fact that each iteration the B, C and D are averaged to become approximately the same. After each iteration each coefficient is given by:

$$\widetilde{A}_1 = \frac{A_0 A_1 + B_0 B_1}{(A_0 + B_0)(A_1 + B_1) + (C_0 + D_0)(C_1 + D_1)},$$
$$\widetilde{B}_1 = \frac{C_0 D_1 + D_0 C_1}{(A_0 + B_0)(A_1 + B_1) + (C_0 + D_0)(C_1 + D_1)},$$
$$\widetilde{C}_1 = \frac{C_0 C_1 + D_0 D_1}{(A_0 + B_0)(A_1 + B_1) + (C_0 + D_0)(C_1 + D_1)},$$
$$\widetilde{D}_1 = \frac{A_0 B_1 + B_0 A_1}{(A_0 + B_0)(A_1 + B_1) + (C_0 + D_0)(C_1 + D_1)}.$$

If $\rho_0 = \rho_1$, then $\widetilde{A}$ is given by Eq. 15, $\widetilde{B} = \widetilde{C}$, and $\widetilde{D}$ is given by:

$$\widetilde{D} = \frac{2F(\frac{(1-F)}{3}}{p_{succ}}.$$

IV. Comparing the distillation protocols

Comparing the different protocols will be done by comparing the following elements of each protocol:

1) Flexibility of the initial state
2) Probability of success (per iteration).
3) Fidelity improvement (per iteration).
4) Efficiency of each protocol. Fraction of purified entangled qubit pairs to starting capacity.

A. *Flexibility in requirement on initial state*

The flexibility in the requirement of the initial state has great impact on the efficiency. In EPL the initial state should be of form

$$\rho = p |\Phi\rangle \langle\Phi| + (1-p) |11\rangle \langle 11| ,$$

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + e^{i\phi} |10\rangle),$$

which is quite strict. The BBPSSW protocol is more flexible in the initial state as it allows to also have noise in the form of the remaining Bell-states:

$$\rho = F |\Phi^+\rangle \langle\Phi^+| + \frac{1-F}{3} |\Phi^-\rangle \langle\Phi^-| \tag{17}$$
$$+ \frac{1-F}{3} |\Psi^+\rangle \langle\Psi^+| + \frac{1-F}{3} |\Psi^-\rangle \langle\Psi^-| , \tag{18}$$

where F > $\frac{1}{2}$. This is, however, not as flexible as the DEJMPS protocol's requirement for the initial state:

$$\rho = A |\Phi^+\rangle \langle\Phi^+| + B |\Phi^-\rangle \langle\Phi^-|$$
$$+ C |\Psi^+\rangle \langle\Psi^+| + D |\Psi^-\rangle \langle\Psi^-| ,$$

with at least one of the coefficients A, B, C, D > $\frac{1}{2}$. This is the most flexible initial state requirement.

B. Probability of successful distillation

In EPL the probability of successful distillation depends on the initial state. In this comparison, the optimal initial state is chosen to ensure the most honest comparison. Assuming the optimal initial state was indeed the initial state, the probability of successful distillation is given by:

$$p_{succ} = \frac{1}{2}p^2.$$

For both BBPSSW and DEJMPS one can assume the optimal initial Werner state, denoted by Eq.18, where F is close to $\frac{1}{2}$. The two protocols then both produce the same probability of successful distillation, given by the following:

$$p_{succ} = F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2. \tag{19}$$

Thus when p and F are both chosen to be the same, the probability of successful distillation for the BBPSSW protocol and DEJMPS protocol are higher.

C. Fidelity improvement

In ideal operation of EPL the fidelity will have gone up to 1 after a iteration. This clearly cannot be done in practice as many errors arise in gate operation in the protocol. According to [1] one of these errors appear as drift effects. In both the BBPSSW paper [4] and DEJMPS paper [5] the fidelity improvement can characterized by:

$$F' = \frac{F^2 + \frac{1}{9}(1-F)^2}{F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2}. \tag{20}$$

Thus in fidelity improvement for the initial iteration there will be no difference between the BBPSSW protocol and the DEJMPS protocol.

D. Efficiency of each protocol

In the original EPL paper [1] the efficiency of reaching a certain fidelity has not been given. The efficiency can, however, be set a maximum by looking at the probability of successful distillation per iteration. Taking into account this probability, one can put a hard limit on the maximum efficiency of the protocol, for EPL being a halve. This is the case since the probability of success for EPL at maximally halve is. Thus not taking into account the efficiency loss in preparation of each entangled qubit pair into the optimal initial state. When $p \approx 0.5$ the efficiency is maximally $\frac{1}{8}$, once again not taking into account preparation of qubits.

When comparing the BBPSSW protocol to the DEJMPS protocol, with both initial states are the same Werner state with F close to 0.5, in efficiency the difference becomes clear in the superior flexibility of the initial state of the DEJMPS protocol. The BBPSSW protocol requires the input state to be prepared as a Werner state at the beginning of every iteration. The DEJMPS protocol does not require this, thus not losing much efficiency in the preparation of the states. In the DEJMPS paper [5] the efficiency of the DEJMPS protocol, provided both initial states are Werner states with F close to 0.5, is after 10 iterations approximately 1000 times more efficient than the BBPSSW protocol.

V. Conclusion

The aim of this paper was to determine which of the three presented two-to-one distillation methods of improving the fidelity of entangled qubit pairs was the most optimal. Three protocols have been presented and compared to one another to determine the optimal distillation protocol between the three depending on the situation. These three protocols are the EPL protocol, the BBPSSW protocol and the DEJMPS protocol. Unless the initial input states are precisely the optimal initial state for the EPL protocol, the DEJMPS protocol is the best protocol to choose when using $2 \rightarrow 1$ distillation. The main arguments for this were its flexibility in the initial state and the efficiency of distillation as a result was found to be much better compared to the EPL- and BBPSSW protocols. The efficiency difference was present due to the limitations of the real-world apparatus.

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