

TU DELFT
FACULTY OF MECHANICAL, MARITIME AND
MATERIALS ENGINEERING (3ME):

**Review of Background Knowledge
for the Course “Advanced Mechanics”**

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Foreword

This document is meant to serve as part of the preparation for the course “Advanced Mechanics” at the faculty of 3ME at TU Delft. It highlights several mathematical concepts that are critical background knowledge and that we have found in the past to be particularly hard for many students, hindering their ability to adequately participate in the course. Therefore, this document does not claim completeness and is merely meant as additional support during the preparation phase.

First, please consult a linear algebra book to familiarize yourself with the definition of a vector, the graphical and numerical calculation of its magnitude, and addition of vectors. Also, make sure you are fully aware of the meaning and applications of dot and cross product. For vector definitions and operations as addition, inner product, vector product in a Cartesian coordinate system space of up to 3 dimensions, an overview is found in Chapter 12 and appendix B of [1] or Chapter 1 of [2, 3]. For vector operations in an N -dimensional space including coordinate transformation, see Sections 4.1-4.7 of [4] or Sections 3.5, 3.6, 6.1, and 6.2 of [2, 3]. Furthermore, a combination of matrix and vector operations in an N -dimensional space can be found, for example, in Chapter 1 and Sections 6.1-6.3 of [4] or Chapter 2 and Section 5.1 of [2, 3]. Basic matrix properties and operations such as transpose, diagonality, inverse, singularity, and structure are introduced in Sections 2.1-2.4 of [4] or Sections 3.1-3.3 of [2, 3], while the determinant and its relation to the inverse are discussed further in Chapter 3 of [4] or Section 4.2 of [2, 3]. Eigenvalues and eigenvectors are, for example, discussed in Sections 5.1, 5.2, and 5.5 of [4] or Sections 4.1, 4.3, and 4.3 of [2, 3], while symmetry and (semi-)positive/negative definiteness are presented in Sections 7.1 and 7.2 of [4] or Section 5.5 of [2, 3]. A brief overview of some concepts is found in Section 0.2 of this document.

Second, please consult a book on mathematical analysis to review differentiation and integration rules for scalars and vectors and make sure you know the difference between taking the time derivative and the partial derivative. Calculus of (common) linear and nonlinear functions of one variable (including interpolation and extrapolation) can be found for example in Chapter 1 of [1]. Moreover, calculus of limits and differentiation (including standard derivatives of common functions) is presented in, for example, Chapters 2 and 3 of [1]. Functions with multiple variables and their partial derivatives are discussed in Sections 14.1 and 14.3 of [1], with the gradient operator further discussed in 14.6

of [1]. Furthermore, Taylor series are presented in Section 11.10 of [1], and integration is covered in Chapter 5 of that same book with line, surface, and volume integrals in Sections 16.2, 16.7, and 6.2-6.3 respectively. A brief overview of some concepts is found in Section 0.3 of this document.

Third, the course will rely on background knowledge in mechanics, for example in statics. A brief overview of some critical generic concepts is found in Section 0.4. Also, because this is a very frequent source of errors, the document contains some remarks on units in Section 0.1.4.

Finally, this course will make extensive use of the summation symbol and index notation. More information on this is presented in appendix E of [1] or Appendix A of [2, 3]

Be aware that the summary contained in this short document cannot be exhaustive.

0 Background Knowledge

0.1 Notation

0.1.1 Typesetting of Scalars, Vectors, and Matrices

Scalars, vectors, unit vectors and matrices in this book are typeset as depicted in Table 0.1.

Table 0.1: Notation of scalars, vectors, unit vectors and matrices

	This book	Handwriting
Scalars	F	F
Vectors	\mathbf{F}	\mathbf{F}
Unit vectors	\hat{e}	\hat{e}
Matrices	\mathbf{R}	\mathcal{R}

Note that we often use the same letter when we refer to a vector's magnitude, so for example a could be used to refer to the magnitude $|\mathbf{a}|$ of a vector \mathbf{a} . However, such a relationship must still always be explicitly defined.

0.1.2 Indices

A position vector that points to a point A from a point P will be denoted as $\mathbf{r}_{A/P}$ (which reads: "position of A with respect to P "). Analogous notation will be used for other relative quantities, such as linear or angular velocity.

To indicate components of m -dimensional vectors, we use the index notation:

$$\mathbf{r} = (r_1 \quad r_2 \quad \dots \quad r_m)^T. \quad (0.1)$$

To indicate components of $m \times n$ -dimensional matrices, we use the index notation:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}. \quad (0.2)$$

In case we operate in \mathbb{R}^3 and with Cartesian coordinate systems, we also often use the names of the axes as indices (for example x , y , z) instead of the numerical values 1, 2, 3, to allow quick association.

0.1.3 Drawing Vectors

Concerning vector notation in drawings: If we label an arrow with a vector symbol (so a boldface variable), for example \mathbf{F} , then this label alone already refers to the definition of the vector including its magnitude *and* direction. The direction drawn for the arrow is merely an illustration. This notation is particularly used if the direction of the vector is unknown or changes.

In contrast, if we label the same arrow with a scalar variable name F (so typeset regular), this scalar value is the magnitude of the vector, and the direction of the vector is defined by the drawn direction of the arrow. We do allow this scalar value F to take on a negative value (even though the magnitude of a vector is strictly seen always positive), in order to enable inversion of the vector's direction later on: This way, when drawing Free-Body Diagrams, we do not need to know yet in which direction a vector (component) points, we only need to know its line of action. This also implies that an arrow labeled with $-F$ is equivalent to an arrow pointing in the opposite direction and labeled F .

○ **Example 0.1** The same forces of a system are drawn in three different ways in Figure 0.1. If a force vector arrow points upwards and is labeled by \mathbf{N} , then the reaction force (following Newton's third law) to this vector, drawn as pointing downward, *must* be labeled with a $-\mathbf{N}$ (note the minus sign), to make clear that the direction is opposite (Figure 0.1, left). If the same upwards-pointing vector is labeled by a scalar N , then the reaction force to this vector should still be drawn as pointing downward, but it *must* then be labeled by N as well, not by $-\mathbf{N}$ (Figure 0.1, center). Alternatively, and still formally correct, one could draw the same reaction force vector as pointing upward and label it by $-\mathbf{N}$. However, this is discouraged as it can be confusing (Figure 0.1 right).

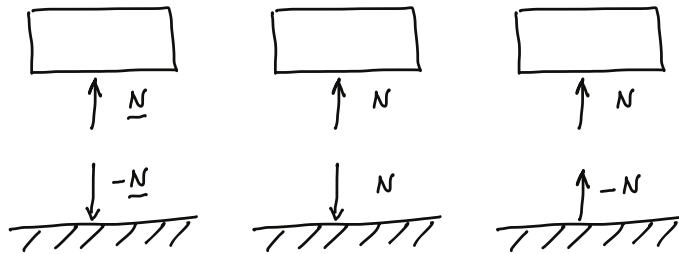


Figure 0.1: Force vectors, symbols, scalars, and arrows. All drawings are correct, but the one on the right is confusing.

Particularly when a vector is split into components along specific axis directions, scalar labeling will be used for these components.

In order to indicate a vector direction when it is pointing inside or outside of the paper plane (Figure 0.2, top row), the vector is drawn as a circle with a cross or a dot inside, following the analogy of a dart. More specifically, we indicate vector components pointing into the plane by \otimes and those pointing out by \odot .

Further, the right-hand rule states that the curled fingers of one hand indicate the positive direction of an angle or a moment, when the thumb points into positive axis direction (Figure 0.2, center). Finally, vectors that encode rotational information, such as moments, will be drawn as double arrows (Figure 0.2, bottom row).

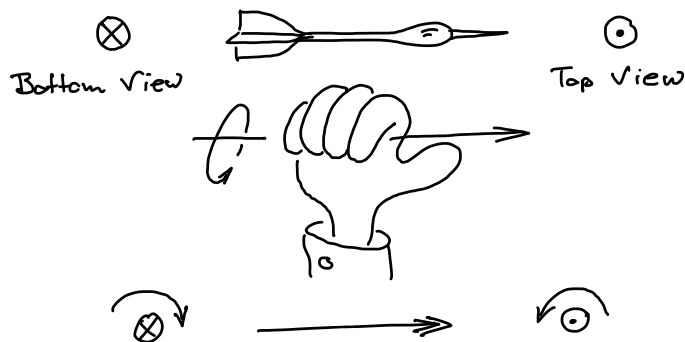


Figure 0.2: Notation for a vector that is perpendicular to the plane of drawing imagined as a dart (top), the right-hand rule to indicate positive direction of a rotation/moment about an axis (center and bottom).

0.1.4 Units

In mechanics, we often deal with quantities that have a measurement unit, such as forces, velocities, or positions. Each such quantity is in fact the *product* of a number and a measurement unit. In vectors or matrices, each component has a quantity value. Note that generally the letters of units are written in roman (upright), while those of scalar physical quantities (variables) are italic.

○ **Example 0.2** Force can be expressed as the product of a numerical value and the measurement unit Newton: $F = 3 \cdot \text{N} = 3 \text{ N}$

Expressions may contain mixtures of quantities that are given with their numerical value and quantities that are given only as variables. Also in those cases, one must pay attention to keep in mind both factors (numerical value and unit) that compose each quantity, and *not* to insert or omit units.

○ **Example 0.3** Given a particle of mass $m = 3 \text{ kg}$ and having acceleration vector \mathbf{a} , the resultant force vector \mathbf{F} on the particle can be calculated as:

Correct is: $\mathbf{F} = m \cdot \mathbf{a} = 3 \text{ kg} \cdot \mathbf{a}$.

Incorrect is: ~~$\mathbf{F} = 3 \cdot \mathbf{a} \text{ N}$~~ .

○ **Example 0.4** In 2D, a particle rotates with an angular velocity of magnitude $\omega = 2 \text{ rad/s}$ about point P . Its angle θ depends on time t :

$$\theta = \omega \cdot t = 2 \text{ rad/s} \cdot t. \quad (0.3)$$

At $t = 3 \text{ s}$, the particle's angle is calculated as:

$$\theta = \omega \cdot t = 2 \text{ rad/s} \cdot 3 \text{ s} = 2 \cdot 3 \text{ rad/s} \cdot \text{s} = 6 \text{ rad}. \quad (0.4)$$

Following the ISO norm [5], for quantities that have a measurement unit, the numerical value of a quantity Q is denoted by $\{Q\}$, and the measurement unit is denoted by $[Q]$. Therefore, the numerical value can also be denoted as $\{Q\} = Q/[Q]$.

○ **Example 0.5** Consider a quantity value of mass $m = 3.5 \text{ kg}$. Its numerical quantity value is $\{m\} = 3.5$, and its measurement unit is $[m] = \text{kg}$. Note that the same quantity value can be expressed as 3500 g . Then, the numerical quantity value $\{m\} = 3500$, and $[m] = \text{g}$.

A common mistake is inconsistent use of square brackets. Such brackets only have meaning when placed around the physical quantity. They should *not* be placed around the unit.

○ **Example 0.6** Consider time t :

Good use is: $t/s = 4$, or $[t] = s$.

Not good use is: ~~$t \gg 4 [s]$~~ , or ~~$t [s]$~~ .

Incorrect square brackets are often found in axes labels of plots. Following the SI [6], a correct label is (for the example of time): t/s . One can also use round brackets if needed, like “time t (in s)”.

○ **Problem 0.1** Which of these expressions make correct use of units, for mass m , time t , force F , angular velocity ω , and angle θ ? Correct the mistakes.

- $m = 20$
- $m = 3 \text{ N}$
- $F = 3 \cdot m [\text{N}]$
- $F = 20 \text{ N}$
- $[m] = \text{kg}$
- $m/[m] = 20$
- $t = 3 \text{ s}$
- $\omega = 2 \text{ rad/s}$
- $\theta = \omega \cdot t = 2 \cdot t [\text{s}]$
- $\theta = \omega \cdot t, [\omega] = \text{rad/s}, [t] = \text{s}, [\theta] = \text{rad}$
- $\{m\} = 20$
- $\omega = t^2$
- $\omega = 1 \text{ rad/s} \cdot t^2$
- $\omega = 1 \text{ rad/s}^3 \cdot t^2$
- $f(t) = \cos(1 \text{ rad/s} \cdot t)$
- $g(t) = e^{-t}$
- $h(t) = \sin(\omega t) = \sin(2t)$

A general advice for solving problems is to work as long as possible with variables, and only to substitute these variables with given numbers and units at the very end of a problem, in the final solution equation. Also, it is almost always helpful to use units as part of plausibility checks for a particular calculation.

0.2 Vector and Matrix Algebra

0.2.1 Vector Addition

Vectors are added (Figure 0.3) by adding their individual components. For the addition of vectors, the same rules apply as for the summation of scalars.

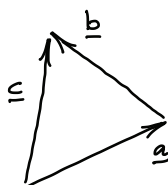


Figure 0.3: Vector addition: The vector \underline{c} is found by adding \underline{a} and \underline{b} .

○ **Example 0.7** The meaning of vector addition is particularly intuitive for position vectors: Assume for example that the vector $\underline{r}_{A/O}$ specifies the location of a point A with respect to another point O , and that the vector $\underline{r}_{C/A}$ specifies the location of a point C with respect to A . Then, the position vector $\underline{r}_{C/O}$, which points from O to C , is given by $\underline{r}_{C/O} = \underline{r}_{A/O} + \underline{r}_{C/A}$.

|| ○ **Problem 0.2** In Figure 0.3, assume the components of \underline{c} and \underline{a} are known. How do you calculate \underline{b} ?

|| ○ **Problem 0.3** If you know $\underline{r}_{C/O}$, how do you calculate $\underline{r}_{O/C}$?

0.2.2 Multiplication of a Vector with a Scalar

Multiplication of a vector with a scalar means multiplication of each single component of the vector by this scalar. This operation changes the vector's magnitude, but it does *not* change its direction. So, multiplication with a scalar only “scales” a vector.

Accordingly, one can also represent each vector by a multiplication of a scalar value (its magnitude) and a unit direction vector. This is particularly helpful when solving problems where the direction of a vector is known, but its magnitude is not. This situation occurs frequently in mechanical systems.

○ **Problem 0.4** Consider again Figure 0.3, and assume we are in a 2-dimensional space (so in \mathbb{R}^2). Also assume that we know the vector \mathbf{a} including its magnitude and direction, but for the vectors \mathbf{b} and \mathbf{c} we only know their directions, given by the unit vectors $\hat{\mathbf{e}}_b$ and $\hat{\mathbf{e}}_c$, respectively. We are now interested in setting up the equations that allow calculating the respective magnitudes b and c of the vectors \mathbf{b} and \mathbf{c} .

○ **Problem 0.5** Consider Problem 0.4, now assume the vectors are defined in \mathbb{R}^3 . Will there always be a solution for b and c ? If not, what is/are the condition(s) on the given variables such that a solution exists?

0.2.3 Cross Product

The *cross product* $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to both vectors \mathbf{a} and \mathbf{b} . The cross product of two vectors is the zero vector if both vectors have the same or the exact opposite direction (i.e. the vectors are linearly dependent). The direction of the cross product is determined by the right-hand rule (Figure 0.4).

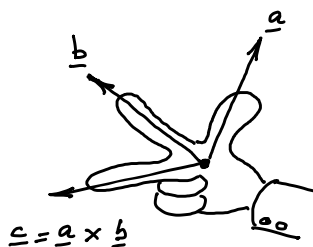


Figure 0.4: Right-hand rule

The magnitude of the cross product equals the area of the parallelogram with the vectors as sides, see Figure 0.5. So, the magnitude of a cross product \mathbf{c} of the vectors \mathbf{a} and \mathbf{b} ,

$$\mathbf{c} = \mathbf{a} \times \mathbf{b}, \quad (0.5)$$

can be calculated using:

$$|\mathbf{c}| = |\mathbf{a}||\mathbf{b}|\sin(\theta), \quad (0.6)$$

whereby $\theta \in [0, \pi]$ is the angle enclosed by the two vectors.

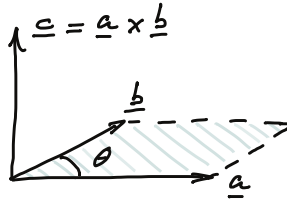


Figure 0.5: The magnitude of the cross product of vectors \mathbf{a} and \mathbf{b} describe the area of the parallelogram, spanned by \mathbf{a} and \mathbf{b} , which are separated by angle θ .

|| ○ **Problem 0.6** What is the cross product of a vector with itself?

The cross product is used very often in dynamic calculations, for example for moments, angular momenta etc..

When using the vector components

$$\mathbf{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix}, \quad (0.7)$$

the cross product is calculated by:

$$c_x = a_y b_z - a_z b_y \quad (0.8)$$

$$c_y = a_z b_x - a_x b_z \quad (0.9)$$

$$c_z = a_x b_y - a_y b_x. \quad (0.10)$$

From its definition, it becomes evident that the cross product of two vectors exists only in three-dimensional spaces. The cross product is *anti-commutative*, meaning that $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.

|| ○ **Problem 0.7** Use the definition of the cross product to show that $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.

0.2.4 Dot Product

The *dot product* of vectors \mathbf{a} and \mathbf{b} , as illustrated in Figure 0.6, is defined as

$$c = \mathbf{a} \cdot \mathbf{b} := |\mathbf{a}| |\mathbf{b}| \cos(\theta), \quad (0.11)$$

where θ is the angle between \mathbf{a} and \mathbf{b} . Note that the dot product results in a scalar value c , so it is also called the *scalar product*.

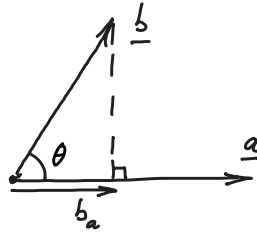


Figure 0.6: Dot product

From this formulation and Figure 0.6, a very important property can be seen: The scalar product can be regarded as the scalar projection b_a of vector \mathbf{b} onto \mathbf{a} , multiplied by the magnitude a of \mathbf{a} :

$$\mathbf{a} \cdot \mathbf{b} = ab_a. \quad (0.12)$$

So, to calculate the component of a vector \mathbf{b} in a particular direction, one takes the scalar product of \mathbf{b} with a unit vector (so a vector having a length equal to 1) in that particular direction. It also implies that if two vectors are orthogonal (perpendicular), their dot product is zero.

⊙ **Problem 0.8** Consider a vector that points in the direction of the x axis. Use the scalar product to calculate the projection of this vector onto the y axis.

When using the same components as in (0.7), the dot product resolves to

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z. \quad (0.13)$$

So, another way to write the dot product is via matrix notation:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}. \quad (0.14)$$

Note that the dot product does not only exist in \mathbb{R}^3 , but also in any other dimension. More broadly in \mathbb{R}^n , it is defined as

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n a_i b_i. \quad (0.15)$$

In contrast to the cross product, the dot product is commutative (which can be seen directly from (0.13)):

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}. \quad (0.16)$$

|| ○ **Problem 0.9** Try calculating dot products and cross products by using MATLAB™.

0.2.5 Basic Matrix Definitions

Among other interpretations, matrices can be seen as *linear transformations* from one space to another: When a vector $\mathbf{x} \in \mathbb{R}^n$ is pre-multiplied by an $m \times n$ -matrix \mathbf{A} , the result is another vector $\mathbf{y} \in \mathbb{R}^m$, mapped to the n -dimensional *image* space of \mathbf{A} . This space is spanned by the column vectors of \mathbf{A} .

The *rank* of \mathbf{A} is the same as the dimension of the column space or image. It is important to note that the rank can be lower than n , because it could be that some columns are *linearly dependent*, meaning that they can be constructed by linear combinations of other columns. In that case, the columns cannot span an n -dimensional space.

|| ○ **Problem 0.10** What is the rank of an $n \times n$ identity matrix?

|| ○ **Problem 0.11** What is the maximum rank that a 3×4 -matrix can have? Provide a reasoning for your answer.

There are some matrices with special structure, for example *square* matrices, where $m = n$. Within this subgroup, there are for example *diagonal* matrices, where all values except for those on the diagonal are zero, or *upper triangular* matrices (also called *right triangular* matrices), where all values below the diagonal are zero.

The *transpose* \mathbf{A}^T of an arbitrary matrix \mathbf{A} is another matrix that contains the columns of \mathbf{A} as rows. For *symmetric* matrices, $\mathbf{A}^T = \mathbf{A}$, which means that all entries are mirror-symmetric with respect to the diagonal.

|| ○ **Problem 0.12** Can non-square matrices be symmetric?

The *trace* of a square matrix is defined as the sum of all diagonal elements of this matrix.

A critical characteristic of a square matrix \mathbf{A} is its *determinant* $|\mathbf{A}|$. The determinant is a scalar value and can be calculated and interpreted as described in Chapter 3 of [4].

|| ○ **Problem 0.13** Calculate the determinant of a diagonal matrix.

|| ○ **Problem 0.14** Calculate the determinant of an upper triangular matrix.

|| ○ **Problem 0.15** If you know the determinant of a matrix \mathbf{A} , how can you find the determinant of \mathbf{A}^T ?

Using its determinant, one can also calculate the *inverse* of a square matrix, as described in Sections 2 and 3 of [4]. When pre- or post-multiplying a matrix \mathbf{A} with its inverse \mathbf{A}^{-1} , one obtains the identity matrix. However, not all square matrices can be inverted. Matrices that have an inverse are called *invertible* or *nonsingular* matrices, while those without an inverse are called *singular* matrices.

Consider a square matrix that has *deficient* rank, so where the rank is lower than the number of columns. In that case, mapping a vector \mathbf{x} to its counterpart \mathbf{y} in the image space, via $\mathbf{A}\mathbf{x} = \mathbf{y}$ means compressing the original dimension of \mathbf{x} to a lower dimensionality.

○ **Example 0.8** Consider a vector $\mathbf{x} = (1, 2, 3)^T$ in 3D that is mapped onto a paper plane via pre-multiplication with a matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

We notice that this matrix only has rank 2. So, even though it also has three components, the vector $\mathbf{y} = \mathbf{A}\mathbf{x} = (1, 2, 0)^T$ is merely a two-dimensional “shadow” of the original vector \mathbf{x} . Now, if we try to reconstruct the original vector from its shadow, we will not succeed, because there is an infinite number of possible original vectors, for example $(1, 2, 0)^T$ or $(1, 2, 325)^T$. That means there cannot be a unique inverse \mathbf{A}^{-1} such that one can be sure that $\mathbf{A}^{-1}\mathbf{y} = \mathbf{x}$.

|| ○ **Problem 0.16** What can you say about the inverse of a matrix that has a determinant of 0?

For a matrix that has a rank lower than its number of columns, there is always a set of vectors that must all map to the same image, namely to the zero vector. The set of these vectors is called the *nullspace* or *kernel* of a matrix.

|| ○ **Problem 0.17** What is the relationship between the number of columns of a matrix, the dimension of its kernel, and the rank of the matrix?

|| ○ **Problem 0.18** Use the MATLABTM command `null` to calculate the nullspace of the matrix \mathbf{A} from Example 0.8. Does the result match your expectations?

Definiteness of a symmetric matrix is an important property that can be employed for example for stability analysis. If a matrix \mathbf{A} is positive definite, it means that for any nonzero vector \mathbf{x} , the scalar $\mathbf{x}^T \mathbf{A} \mathbf{x}$ is positive. If \mathbf{A} is negative definite, the same scalar will be always negative. Semi-definiteness extends to the case where $\mathbf{x}^T \mathbf{A} \mathbf{x}$ may also be zero for some vectors \mathbf{x} .

⊙ **Problem 0.19** Consider the two-dimensional identity matrix. Can you make a statement about its definiteness?

⊙ **Problem 0.20** Consider a general two-dimensional diagonal matrix \mathbf{D} with only negative entries on the diagonal. Can you make a general statement about the definiteness of such a matrix?

0.2.6 Eigenvectors and Eigenvalues

A vector \mathbf{a} that does not change its direction when it is pre-multiplied by a square matrix \mathbf{C} (which is not the identity matrix) is called an *eigenvector* of this matrix. The multiplication factor needed to represent the change in length is called the associated *eigenvalue* λ :

$$\mathbf{C}\mathbf{a} = \lambda\mathbf{a}. \quad (0.17)$$

If we are interested in finding an eigenvalue for a particular matrix, we can re-write (0.17) as

$$(\mathbf{C} - \lambda\mathbf{E})\mathbf{a} = \mathbf{0}, \quad (0.18)$$

where \mathbf{E} is the identity matrix of the same dimension as \mathbf{C} .

For example in \mathbb{R}^3 , this equation system, together with for example the condition that $\mathbf{a} = \hat{\mathbf{a}}$ has unit length, provides four equations for four unknowns: The components of $\hat{\mathbf{a}}$ and the scalar λ .

If the matrix \mathbf{C} has full rank, we will find as many independent eigenvectors with nonzero eigenvalues as there are dimensions. Remark: It is possible that some or all vectors are complex, in which case also their associated eigenvalues are complex. These solutions always occur in complex conjugate pairs. Algebra of complex numbers is not required in the following; for more information, please consult Appendix H of [1] or Appendix C of [2, 3].

A commonly employed method to solve the equation system is to first establish the determinant of the matrix $\mathbf{C} - \lambda\mathbf{E}$, which delivers the so-called *characteristic polynomial* in λ . Finding the n eigenvalues means

finding the n roots of this n -degree polynomial. Afterwards, for each of these roots, one finds the associated eigenvector from (0.18).

It is possible that some eigenvalues have a *multiplicity* larger than 1. If the characteristic polynomial contains factors $(p - \lambda_i)^{k_i}$, then k_i is called the *algebraic multiplicity* of the eigenvalue λ_i . Note that it could be that there are less than k_i eigenvectors associated with a particular eigenvalue λ_i . In that case, the *geometric* multiplicity of that eigenvalue is lower, in fact it equals the number of independent eigenvectors for the particular eigenvalue. It will always be at least one.

Further useful properties of eigenvalues are that the determinant of a matrix equals the product of all eigenvalues, and the trace of the matrix equals the sum of all eigenvalues.

⊙ **Problem 0.21** Computing eigenvalues by hand can be much work. However, checking if a vector is an eigenvector is usually quicker. Show that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$.

If all its eigenvalues are positive, a symmetric matrix is *positive definite*. If eigenvalues are only non-negative (meaning some could be zero), the matrix is *positive semidefinite*. In analogy, if all eigenvalues are negative (nonpositive), a matrix is *negative (semi-)definite*.

⊙ **Problem 0.22** Calculate the eigenvalues of a general diagonal matrix.

⊙ **Problem 0.23** Calculate the eigenvalues of a general upper triangular matrix.

⊙ **Problem 0.24** Use the MATLABTM command `eig` to calculate the eigenvectors and eigenvalues of the matrices given in Example 0.8 and in Problem 0.21. Do the results match your expectations?

0.3 Differentiation and Integration

0.3.1 Derivatives with Respect to Time

Many variables that describe mechanical systems (such as distances or angles) change over time and so are time-dependent.

To denote an infinitesimally small change in a variable, we use the operator d . For example, an infinitesimal increment in the vector \mathbf{r} would be denoted as $d\mathbf{r}$.

To represent the rate of change of a quantity with respect to time, one calculates the quotient of this infinitesimal increment and an increment in time dt , which is the derivative with respect to time. To denote the derivative of a vector \mathbf{r} or a scalar r with respect to time t , we use the abbreviated notations:

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}, \text{ and } \dot{r} = \frac{dr}{dt} = \lim_{\Delta t \rightarrow 0} \frac{r(t + \Delta t) - r(t)}{\Delta t}, \quad (0.19)$$

respectively. Note that the time derivative of constants, so quantities that do not vary over time, is zero.

0.3.2 Product Rule

The product rule of differentiation is used when an expression contains a multiplication of several functions. To take a derivative of the scalar function $h(t) = f(t) \cdot g(t)$ with respect to time, one can apply the following:

$$\dot{h} = \frac{dh}{dt} = \frac{d}{dt} (f(t)g(t)) = \frac{df(t)}{dt}g(t) + f(t)\frac{dg(t)}{dt} \quad (0.20)$$

⊙ **Problem 0.25** Take the time derivative of $h(t) = t^2$ by rewriting h as $h(t) = f(t)g(t)$, where $f(t) = g(t) = t$ and applying the product rule.

⊙ **Problem 0.26** Take the time derivative of $h(t) = t^3$ by rewriting h as $h(t) = f(t)g(t)$, where $f(t) = t^2$, $g(t) = t$ and applying the product rule.

⊙ **Problem 0.27** Show (0.20) from the definition of the time derivative in (0.19).

For the scalar product of two vector functions, so $h_1(t) = \mathbf{f}(t)^T \mathbf{g}(t)$, the rule can equally be applied:

$$\dot{h}_1 = \frac{dh_1}{dt} = \frac{d}{dt} (\mathbf{f}(t)^T \mathbf{g}(t)) = \dot{\mathbf{f}}(t)^T \mathbf{g}(t) + \mathbf{f}(t)^T \dot{\mathbf{g}}(t) \quad (0.21)$$

as well as for the cross product $\mathbf{h}_2(t) = \mathbf{f}(t) \times \mathbf{g}(t)$:

$$\dot{\mathbf{h}}_2 = \frac{d\mathbf{h}_2}{dt} = \frac{d}{dt} (\mathbf{f}(t) \times \mathbf{g}(t)) = \dot{\mathbf{f}}(t) \times \mathbf{g}(t) + \mathbf{f}(t) \times \dot{\mathbf{g}}(t) \quad (0.22)$$

⊙ **Problem 0.28** Use the product rule of differentiation to show that $\int \dot{\mathbf{r}}^T \dot{\mathbf{r}} dt = \frac{1}{2} \dot{\mathbf{r}}^T \dot{\mathbf{r}}$.

⊙ **Problem 0.29** Use the scalar formulation (0.20) of the product rule of differentiation, in combination with the definitions of scalar and cross product, to prove (0.21) and (0.22).

0.3.3 Chain Rule

The chain rule is used when the expression is the nested composition of two or more functions. To calculate the time derivative of a single nested function $h(g(t))$, the chain rule yields:

$$\dot{h} = \frac{dh}{dt} = \frac{d}{dt} (h(g(t))) = \frac{dh}{dg} \frac{dg}{dt} \quad (0.23)$$

⊙ **Problem 0.30** Calculate the time derivative \dot{h} of $h = \sin(\theta)$ for which you know that θ is time-dependent, using the chain rule.

⊙ **Problem 0.31** Calculate the time derivative of $h = 3\theta \sin(\theta)$ for which you know that θ is time-dependent, using both the product rule and the chain rule.

For a more deeply nested function, the chain rule expands to:

$$\dot{h} = \frac{dh}{dt} = \frac{d}{dt} (h(g_1(g_2(g_3(t)))))) = \frac{dh}{dg_1} \frac{dg_1}{dg_2} \frac{dg_2}{dg_3} \frac{dg_3}{dt}. \quad (0.24)$$

⊙ **Example 0.9** To calculate $\dot{h} = \frac{d}{dt} \left(e^{\cos(\sqrt{x(t)})} \right)$, we decompose h as a function of g_1 first. That is:

1. $h(g_1) = e^{g_1}$. In its turn, g_1 is a function of g_2 . That is
2. $g_1(g_2) = \cos(g_2)$. Then, g_2 is a function of g_3 , namely
3. $g_2(g_3) = \sqrt{g_3}$. Finally, we realise that g_3 is the time-dependent variable:
4. $g_3(t) = x(t)$.

Now we can do the separate derivations:


1. $\frac{dh(g_1)}{dg_1} = e^{g_1}$
2. $\frac{dg_1(g_2)}{dg_2} = -\sin(g_2)$

$$3. \frac{dg_2(g_3)}{dg_3} = \frac{1}{2\sqrt{g_3}}$$


$$4. \frac{dg_3(t)}{dt} = \dot{x}$$

This way we find

$$\dot{h} = -\frac{\dot{x}e^{\cos(\sqrt{x(t)})} \sin(\sqrt{x(t)})}{2\sqrt{x(t)}} \quad (0.25)$$


||  **Problem 0.32** Calculate $\dot{h} = \frac{d}{dt}(\cos(x(t)^2))$.

||  **Problem 0.33** Show (0.23) from the definition of the time derivative in (0.19).


||  **Problem 0.34** Use the Symbolic Toolbox in MATLABTM to calculate $\frac{dh}{dx}$ from Example 0.9 and from Problem 0.32 and compare the results with your manual calculations.

0.3.4 Partial Derivatives

Now, we consider a function f that depends on multiple variables. A *partial derivative* of a function f is the derivative of this function with respect to a single variable. Taking a partial derivative, one is only interested in the influence of the variation of one of the variables on f , while all the other variables are held constant.

 **Example 0.10** The elevation at a particular location in a mountain landscape can be given as a function of two variables: latitude and longitude. We could be interested in the influence of the variation of latitude, while the longitude is kept constant. To obtain this information, we calculate the partial derivative of elevation with respect to latitude. This provides the slope of the landscape in the requested direction.

So, a partial derivative represents the dependency of an expression on an isolated variable. The partial derivative of a function f with respect to a variable x is denoted by $\frac{\partial f}{\partial x}$.

 **Example 0.11**

$$\frac{\partial}{\partial x}(ax^2 + 3yx + cx^2) = 2ax + 3y \quad (0.26)$$

○ **Example 0.12**

$$\frac{\partial}{\partial x} (ax^2 + 3yx + cx^2) = 2cx \quad (0.27)$$

○ **Example 0.13**

$$\frac{\partial}{\partial y} (ax^2 + 3yx + cx^2) = 3x \quad (0.28)$$

We can use partial derivatives to calculate increments of a function f that depends on multiple variables x_i , with $i = 1 \dots N$:

$$df = \sum_{i=1}^N \frac{\partial f}{\partial x_i} dx_i. \quad (0.29)$$

○ **Example 0.14** Consider a particle moving in three-dimensional space, within a temperature field. The particle's coordinates x , y , and z determine the temperature $f(x, y, z)$. We find the infinitesimal temperature increment df that is due to changes in x , in y , and in z as:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz. \quad (0.30)$$

○ **Example 0.15** In Example 0.10 about the mountain landscape, the elevation f is only a function of two variables, namely latitude x_1 and longitude x_2 . The relationship (0.29) indicates how much elevation df is gained as a function of infinitesimal changes in latitude dx_1 and longitude dx_2 .

In case the x_i are all functions of time t , we find the *total derivative* of a function $f(x_1(t), x_2(t), \dots, x_N(t))$ with respect to time as:

$$\frac{df}{dt} = \sum_{i=1}^N \frac{\partial f}{\partial x_i} \frac{dx_i}{dt}. \quad (0.31)$$

○ **Problem 0.35** Consider a function $f(x(t), y(t))$ with

$$f = 3x^2 + 5y, \quad x = 5t, \quad y = t^2. \quad (0.32)$$

Find \dot{f} in two ways: a) by substituting the functions for x and y into f , such that it becomes a function of time only, and then taking its derivative, and b) by using (0.31). You should obtain the same result.

0.3.5 Gradient

For a function f that depends on multiple variables, for example x , y , and z , the partial derivatives of f in all directions, so in the example respectively in x , y and z direction, can be subsumed in a vector, the *gradient* of f :

$$\nabla f := \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} \quad (0.33)$$

The symbol ∇ is called the *nabla* operator; it produces the gradient of a scalar function, like here of $f(x, y, z)$. At any specific location, this vector points in the direction of the steepest ascent.

A gradient of a function will always be orthogonal to the *level curves* (often also denoted *iso lines*) or *level surfaces* of this function, which denotes the manifolds along which the function does not change its value.

○ **Example 0.16** Figure 0.7 shows two 2D examples, namely the gradient of two functions $f(x, y)$ that each specify saturation for a given location. Note how the vectors indicating the gradient are always perpendicular to the iso lines, which in the examples are circles (left) and straight vertical lines (right).

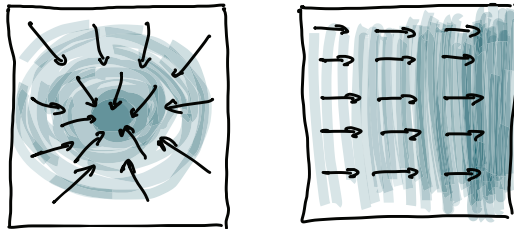


Figure 0.7: A visualization of two gradient fields (Adapted and re-drawn from [7]). The arrows represent the gradient of change in saturation. They point in the direction of the greatest change in saturation at their specific location.


⊙ **Problem 0.36** Show that the gradient of a function $f(x, y)$ of two variables x and y at any particular point is orthogonal to the function's level curve through that point.

Hint: In order to show this, do the following: Consider an infinitesimal vector that points in a direction that is tangential to the level curve and has

the components $(dx \ dy)^T$. Now, express the infinitesimal increment df of f along the level curve, using (0.29), as a function of the individual increments dx and dy . Notice that this increment should be zero in the direction of the curve. Then, re-write df as a scalar product of two vectors. What can you say about these two vectors?

0.3.6 Integration

For single and multiple integrals, please consult for example Chapter 5 of [1], which explains line, surface, and volume integrals in Sections 16.2, 16.7, and 6.2-6.3, respectively. Pay particular attention to the limits of the integrals, they might be functions of other variables.

 **Problem 0.37** Use volume integrals to calculate centroids of volume of several common bodies, such as a cone, a solid hemisphere, and a thin-walled hollow cone.

0.4 Basic Definitions and Tools Used in Mechanics

Readers are expected to have already knowledge on the branch of mechanics that is called *Statics*. This branch deals with mechanical systems that are at rest or move with constant linear velocity. It defines conditions for forces and moments acting on or within these systems such that indeed the systems remain in this state. Statics and other branches of mechanics share many important definitions and tools. In this section, some of these critical concepts will be reviewed.

0.4.1 Moments

The *moment*, often also called torque¹, of a force \mathbf{F} with respect to point P is defined as:

$$\mathbf{M}_P := \mathbf{r}_{A/P} \times \mathbf{F}, \quad (0.34)$$

where the vector $\mathbf{r}_{A/P}$ is a vector from point P to point A , the point of application of the force. Note that from the definition of the cross product, it follows that A may in fact be any *arbitrary* point on the line of action of \mathbf{F} (see Problem 0.39). Also note that the order in which this cross product is written matters, as $\mathbf{F} \times \mathbf{r}_{A/P} = -\mathbf{M}_P$.

¹The term *torque* is preferably used for a torsional moment, so a moment about a longitudinal axis. A specific example is the torque that a motor applies to a shaft.

⊙ **Problem 0.38** Calculate the moment about point P using (0.34), for force $\mathbf{F} = (3, 4, 0)^T \text{ N}$, if the position vector of the point of application of this force, A , with respect to point P is $\mathbf{r}_{A/P} = (2, 3, 0)^T \text{ m}$. Note that both vectors are contained in the XY -plane to check direction of the resulting moment vector. Also, choose another point A' on the line of action of the force to calculate the moment and compare results.

⊙ **Problem 0.39** Use the definition of the cross product to show that when calculating the moment vector \mathbf{M}_P of a force vector \mathbf{F} about point P according to (0.34), it does not matter which vector $\mathbf{r}_{A/P}$ you choose from point P to the line of action of the force.

Hint: There are multiple ways to show this. Try these two: a) Draw the parallelogram spanned by position and force vector and show that neither the magnitude nor the direction of the cross product changes when the force shifts along its line of action. Use (0.6) and the right-hand rule. b) Describe the location of all points on the line of action of a force as a linear combination of a position and a direction vector. Then, calculate the cross product of such an arbitrary point with the force vector.

If two force vectors act on a system and if these two forces have parallel lines of action, equal magnitude, but opposite direction, then the sum of these two forces (also called a *force couple*) is the zero vector. However, they still cause a moment vector. This *couple moment* vector is identical regardless of the point of reference P , such that it is a free vector. This means its effect on the rigid body does not depend on its location.

0.4.2 Force System Resultants: Equipollence

When several forces and moments act on a rigid body, it is always possible to formulate an equivalent system that consists of 1) one single resultant force vector acting at an arbitrary point O and 2) one couple moment vector. Equivalence, or more precisely *equipollence*, means that the effect of the original and of the reduced force/moment system on the rigid body is identical. This requirement leads to the following steps of determining the resultant system:

1. To determine the magnitude and direction of the resultant force \mathbf{F}_r , all N original force vectors \mathbf{F}_i need to be summed.


$$\mathbf{F}_r = \sum_{i=1}^N \mathbf{F}_i. \quad (0.35)$$

2. To determine the associated couple moment vector \mathbf{M}_r , the moment caused by the N original forces and K original moments about the same point O needs to be determined:

$$\mathbf{M}_r = \sum_{j=1}^K \mathbf{M}_j + \sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_i, \quad (0.36)$$

where the \mathbf{r}_i are the position vectors of the points of application of the N original forces with respect to point O . Note that the equation already considers the fact that the resultant force vector does not create a moment about point O , its chosen point of application.

Remark: Simplifying even further, it is also always possible to find an equivalent system that contains only one single force vector and a single couple moment vector *that is parallel to the force vector's line of action*. The exact location of this line of action needs to be determined.

 **Problem 0.40** For an original system of N forces \mathbf{F}_i , $i = 1 \dots N$, set up an algorithm (set of equations) that allows determining the resultant force vector, one point on its line of action, and a resultant couple moment vector that is parallel to this line of action.

Hint: Remember that the resultant force vector already determines the direction of the possible couple moment. Therefore, express the couple moment vector as a product of its magnitude and direction, which contains only one scalar unknown. In order to determine the exact location of the line of action of this force as well as the magnitude of the couple moment, recall that the moment of all original forces and moments about any arbitrary point O needs to be identical to the moment of the resultant force and the couple moment vector. Remember that any point on the line of action of the force is a possible point of application. So, you could choose a position vector that is perpendicular to the line of action and only has unknown length. Test your approach for an example system and verify that the resultant system is indeed equipollent.

0.4.3 Newton's First Law and Static Equilibrium

In 1687, Newton stated his three laws of motion [8]. The first law reads:

1. *“Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.”*

So, a particle that moves at constant linear velocity \mathbf{v} will keep the same velocity unless a force \mathbf{F} acts upon it. Note that a particle

at rest has a constant velocity of zero.

$$\sum \mathbf{F} = 0 \leftrightarrow \dot{\mathbf{v}} = 0. \quad (0.37)$$

Based on this law, static equilibrium of a rigid body was defined as a state where the sum of all forces acting on the system is zero and also the sum of all moments acting on the body about any arbitrary point O is zero.

Remark: Note that the term equilibrium does not necessarily imply *stable* equilibrium. For example, a ball in a convex valley resides in a stable equilibrium, while the same ball balancing on top of a concave mountain peak is in an unstable equilibrium. Stability can be investigated by analyzing the system's response to infinitesimally small perturbation from its state of rest.

0.4.4 Center of Mass

The position vector \mathbf{r}_C from an origin O to the overall *center of mass* of a system composed of N particles having masses m_i is defined by:

$$\mathbf{r}_C := \frac{\sum_{i=1}^N m_i \mathbf{r}_i}{\sum_{i=1}^N m_i}, \quad (0.38)$$

where \mathbf{r}_i is the position vector of the center of mass of the i -th particle with respect to the same origin O .

When calculating the center of mass for a single body, the summation is turned into an integral:

$$\mathbf{r}_C := \frac{\int_m \boldsymbol{\rho} dm}{\int_m dm}, \quad (0.39)$$

where $\boldsymbol{\rho}$ is the position vector of an infinitesimally small volume element inside the body of mass dm . The integral resolves to a triple integral for the case of three-dimensional bodies.

In order to calculate the center of mass for a composite body where the mass and center of mass of each individual element are known, (0.38) can again be used. When dealing with hollow sections in systems with homogeneous mass distribution, the calculation can be simplified by considering first a solid structure and then considering each hollow section as a body with negative mass.

Note that the centroid, the center of mass, and the center of gravity (point of application of the resultant gravitational force) may all be

different points. This is the case if mass distribution is not homogeneous and/or if the gravitational field is not homogeneous.

○ **Example 0.17** Consider three particles of masses m_1 , m_2 and m_3 , at locations with position vectors \mathbf{r}_1 , \mathbf{r}_2 and \mathbf{r}_3 . The values are given as:

$$m_1 = 2 \text{ kg}, m_2 = 3 \text{ kg}, m_3 = 5 \text{ kg}, \text{ and}$$

$$\mathbf{r}_1 = (5 \ 10 \ 1)^T \text{ m}, \mathbf{r}_2 = (-10 \ 0 \ 6)^T \text{ m}, \mathbf{r}_3 = (2 \ 0 \ 4)^T \text{ m}.$$

Their combined center of mass is given by:

$$\begin{aligned} \mathbf{r}_C &= \frac{1}{(2+3+5) \text{ kg}} \left(2 \text{ kg} \begin{pmatrix} 5 \\ 10 \\ 1 \end{pmatrix} \text{ m} + 3 \text{ kg} \begin{pmatrix} -10 \\ 0 \\ 6 \end{pmatrix} \text{ m} + 5 \text{ kg} \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \text{ m} \right) \\ &= \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} \text{ m}. \end{aligned}$$

○ **Problem 0.41** Use integration to calculate the center of mass of several common bodies, such as a cone, a solid hemisphere, and a thin-walled hollow cone. Assume homogeneous mass distribution.

○ **Problem 0.42**

Consider the thin-walled cylindrical rotor depicted in Figure 0.8, of mass m and radius R . The center of mass of the rotor has the position vector $\mathbf{r}_D = R \cdot (\frac{1}{40}, \frac{1}{50}, \frac{1}{50})^T$ in xyz .

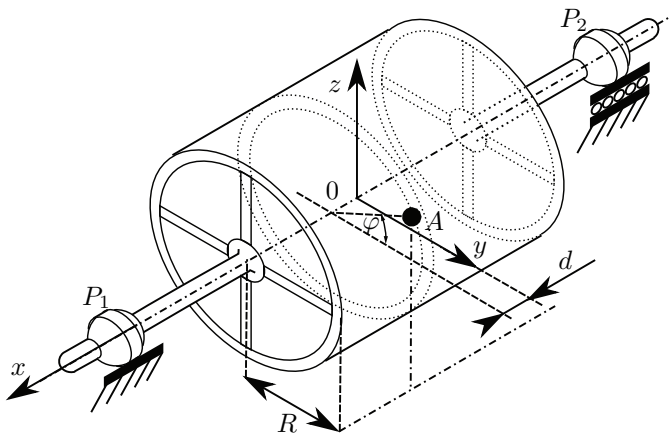


Figure 0.8: Unbalanced rotor.

As the center of mass of the rotor is not on the x -axis, the rotor is said to

be in static unbalance, which can cause undesired vibrations when in motion.

A weight of mass m_a is to be added to the rotor at a strategic place on the shell of the rotor A , described by the angle φ and the distance d . The weight should compensate for static unbalance, such that the center of mass of the combined body (rotor plus mass m_a), with position vector \mathbf{r}_C , is on the x -axis. The weight is considered a point mass that is rigidly attached to the rotor.

This is done so that the bearings P_1 and P_2 do not have to support any net force due to rotation of the rotor.

Determine m_a and *one possible* location A to compensate for static unbalance of the rotor. In your response, provide m_a , φ , and d as functions of R and m (Remark: None of these terms has to necessarily appear in the requested functions).

0.4.5 Newton's Third Law of Motion

Newton's third law of motion [8] states:

3. *"To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts."*

So, if one object A exerts a force \mathbf{F}_A on a second object B , then B simultaneously exerts a force \mathbf{F}_B on A , and the two forces are equal and opposite:

$$\mathbf{F}_A = -\mathbf{F}_B \quad (0.40)$$

The forces share the same line of action. Philosophically speaking: "A force is never alone"; the action-reaction forces form a unit, the force pair.

0.4.6 Internal and External Forces

Two types of forces are distinguished which act on a particle that is part of a system of N particles: *external* forces \mathbf{F}_i and *internal* forces \mathbf{f}_{ij} , such that the sum of all forces acting on a particle is:

$$\mathbf{F}_i + \sum_{j=1}^N \mathbf{f}_{ij} \quad (0.41)$$

The external force vector \mathbf{F}_i is defined as the sum of all forces that act on particle i by sources outside the system.

Internal forces \mathbf{f}_{ij} are the forces that act between particles in a system. For example, if a spring interconnects two particles in one system,

then the spring force is called an internal force. The subscript ij means “acting on particle i , exerted by particle j ”. Note that $\mathbf{f}_{ij} = \mathbf{0}$ for $i = j$. According to Newton’s third law (0.40), there must be an equal and opposite reaction exerted on particle j , namely

$$\mathbf{f}_{ji} = -\mathbf{f}_{ij}. \quad (0.42)$$

Therefore, when summed over the entire system of particles, the sum of all these internal forces is zero:

$$\sum_{i=1}^N \sum_{j=1}^N \mathbf{f}_{ij} = 0. \quad (0.43)$$

Note that the interpretation of a force as being internal or external depends on the chosen definition of system boundaries. We tend to see gravity as an external force, because we consider Earth just in terms of providing us with an inertial coordinate system, but not as being part of our mechanical system. In case we do consider Earth as part of the system (for example simplified as another particle), then gravity suddenly becomes an internal force. This demonstrates that the idea of internal and external is simply a matter of system definition.

0.4.7 Free-Body Diagrams

A Free-Body Diagram (FBD) is useful to understand and visualize a problem and is used to define the system boundaries and personal conventions. It is a simplified drawing of (a part of) a mechanical system with all forces, moments and dimensions. The name is slightly misleading, as FBDs cannot only be drawn for single rigid bodies. They can also be established for parts of bodies or for systems of bodies or particles. Choosing appropriate system boundaries is a critical step towards solving any problem.

These are the steps to draw a FBD:

1. Draw the system in a free state, i.e. “cut” the system at convenient locations and draw outlined shapes of the separate pieces. Note that each cut will introduce new external (formerly internal) action-reaction forces at the system boundaries.

Some helpful guidelines:

- a) Always draw the system in a generic state. So for example if you draw the Free-Body Diagram of a pendulum, draw it

with a generic angle, and not in upright or downright position, as those are special cases.

- b) Whenever possible, choose your system boundaries such that you expose only action-reaction forces that you are actually interested in calculating. Otherwise, extra equations and unknowns are introduced, needlessly complicating the solution process.
2. Establish a coordinate system, choose it smart. That is, choose a convenient position for the origin and look if rotating and/or translating the coordinate system can make things easier.
 3. Indicate all known and unknown external forces and moments that act on the body on the right locations (e.g. external loads, support reactions, weight). Some rules:
 - a) Do not show internal forces or moments.
 - b) If a connection prevents movement of the body in a particular direction, then forces/moments are drawn in that direction, see Figure 0.9a and Figure 0.9b.
 - c) If a segment is split in two, the forces acting on the two segments in the separated FBDs are equal in magnitude and opposite in direction.
 - d) Forces acting on a rigid body may be shifted along their lines of action (sliding vectors), see Figure 0.9c.
 - e) Moments acting on a rigid body may be placed anywhere (free vectors), see Figure 0.9c.
 4. Label all forces and moments with unique names.
 5. Draw all necessary dimensions to calculate moments, i.e. distances and angles.

Make sure that the FBD has all relevant information of the original drawing. Once the FBD is finished, one should not need to look into the original problem drawing anymore to establish the sums of forces and moments.

A very frequent mistake when drawing FBDs for dynamical systems is to already consider the specific movement the system performs. For example, if a particle rotates about one axis, one may be tempted to

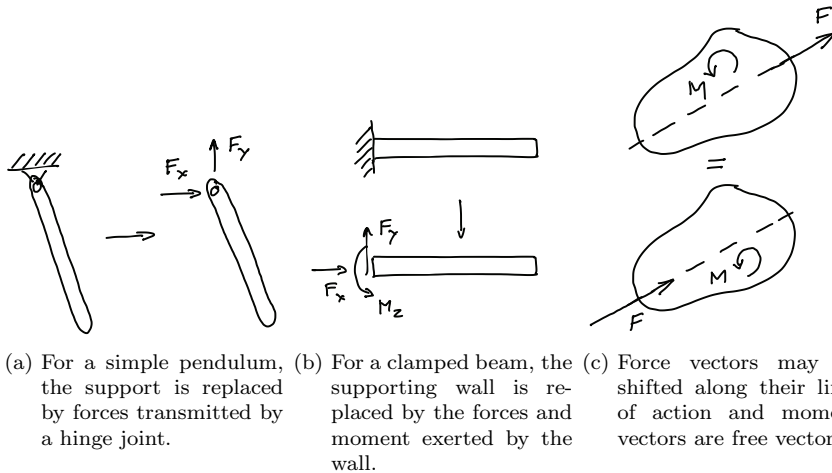


Figure 0.9: Examples for drawing components of FBDs (notice that these are NOT complete FBDs)

draw a force in radial direction based on previous knowledge on dynamics and the intuition that there must be a radial force sustaining this rotation. This is a critical mistake, as it defies the purpose of a FBD, reducing it to a confirmation of one's own intuition. Such a procedure exposes any subsequent calculation to critical omissions. Instead, a FBD must contain all the *possible* reaction forces generated at the system boundaries, only based on the nature of the connection, and disregarding any information that is given on the system's movement. Consequently, an FBD drawn to determine static equilibrium looks in fact identical to one intended to derive a system's equations of motion.

Another very frequent mistake when drawing FBDs is to make unconscious assumptions, for example about the point of application of a distributed load. Whenever the point of application is unclear, variables must be introduced to parameterize its location.

○ **Example 0.18** A Free-Body Diagram for a homogeneous block that slides down a slope (left of Figure 0.10) is depicted on the right of Figure 0.10.

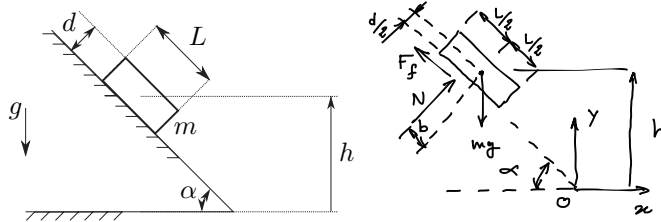


Figure 0.10: Sliding block and corresponding Free-Body Diagram

The friction force F_f and normal force N are resultants of shear stress and pressure distributions, respectively. While the resultant friction force F_f can be placed anywhere along the slope, which equals its line of action, the resultant normal force N has a point of application somewhere at a distance b from the block's center of mass. A common mistake is to make the assumption that b is zero, which is only certain if the dimensions of the block are negligible.

⊙ **Problem 0.43** Use the FBD in Figure 0.10 to determine the signs of b and of F_f in the case of static equilibrium.

Hint: Note that in that case, the block reduces to a so-called “three-force member”, a body with only three forces acting on it.

⊙ **Example 0.19** Consider the cone depicted in Figure 0.11. The cone has mass m and uniform density ρ . Its radius R and height $\sqrt{3}R$ impose a constant angle of $\frac{\pi}{6}$ rad between the cone's contact line with the ground and its central axis. The center of mass C of the cone is located at $1/4$ of the height from its circular base.

The figure indicates an inertial XYZ coordinate system, and a rotating coordinate system $x'y'z'$. The y' -direction is always aligned with the cone's contact line with the ground, and z' is always vertical.

The cone rolls (possibly with slipping) on the XY -plane, and its tip always remains in the origin O . It is subjected to gravity with field strength g in the negative Z -direction.

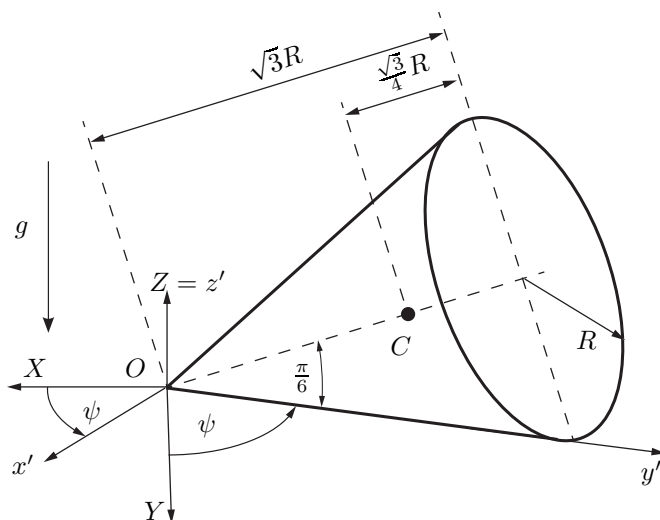


Figure 0.11: Cone

Draw a Free-Body Diagram (FBD) of the cone projected onto the $y'z'$ -plane. Show distributed loads only by their resultant, and leave out all components of these reactions which are zero.

Exemplary solution

A possible FBD for the cone is shown in Figure 0.12.

One may ask why it contains a moment component about the z' -axis, but not about the other axes. To understand this, one has to consider the type of constraint and use equipollence, as treated in statics and reviewed in Section 0.4.2: The resultant vertical ground reaction force $F_{z'}$, representing the centroid of the distributed load acting on the cone, has been drawn at a (yet unknown) distance a from the origin O . This way, there is no additional constraint moment about the x' -axis to be considered. Alternatively, using equipollence one could draw this resultant force acting in the origin, in which case an additional moment component about the x' -axis would appear. A case with the occurrence of a couple moment component about the x' -axis without any resultant vertical force is impossible given the unilateral contact. Drawing the force at the unknown distance a and solving for a has a clear advantage for results checking: A value for a being negative or larger than $2R$ would be implausible, because the resultant ground reaction force can only be applied within the base of support of the cone.

A similar relationship holds for the force component in x' -direction and the

moment about the z' -axis, although here it is possible that a pure (friction) couple moment can be transmitted without any resultant force $F_{x'}$ being present. The resulting redundancy would need to be resolved for a given situation by an additional condition during the calculations. For example, by stating that the couple moment about the z' -axis is only considered in case $F_{x'}$ resolves to be zero, or by forcing b to be zero.

About the y' -axis, no moment can be transmitted as none of the potential distributed loads in the line contact has any lever arm.

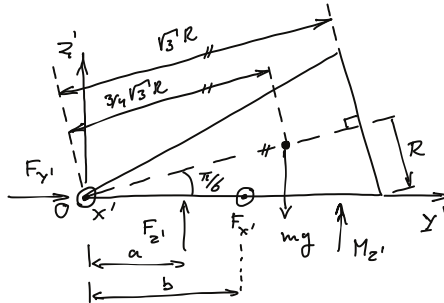


Figure 0.12: Exemplary FBD for the rolling cone.

⊙ **Problem 0.44** Consider again Example 0.19, but now the cone is at rest. Use static equilibrium in that case to calculate the distance a between the cone's tip and the resultant vertical ground reaction force.

Bibliography

- [1] J. Stewart, *Calculus: Early Transcendentals*, 8th ed. Cengage Learning, 2015.
- [2] D. Poole, *Linear Algebra: A Modern Introduction*, 3rd ed. Cengage Learning, 2011.
- [3] —, *Linear Algebra: A Modern Introduction*, 4th ed. Cengage Learning, 2015.
- [4] D. C. Lay, S. R. Lay, and J. J. McDonald, *Linear Algebra and Its Applications*, 5th ed. Pearson Education, 2015.
- [5] *ISO80000-1:2009 Quantities and units - Part 1: General*, International Organization for Standardization (ISO) Std. 1, November 2009. [Online]. Available: <https://www.iso.org/obp/ui/#iso:std:iso:80000:-1:ed-1:v1:en>
- [6] Bureau International des Poids et Mesures, “Le Système international d’unités / The International System of Units,” SI Brochure, 8th edition, 2006, updated in 2014. [Online]. Available: <https://www.bipm.org/en/publications/si-brochure/>
- [7] Cweiske, Sarang, and Kalliauer, “A visualization of gradient,” Wikimedia Commons, 2016. [Online]. Available: <https://commons.wikimedia.org/wiki/File%3AGradient2.svg>
- [8] Sir Isaac Newton, *Newton’s Principia : the mathematical principles of natural philosophy, 1st American Edition.*, N. Chittenden, Ed. Daniel Adee in New-York, 1846.